THE NUMBER OF 1'S IN THE PARTITIONS OF n

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ABSTRACT. Consider the partitions of n. Each partition contains some number of 1's. We study the statistical distribution of the number of 1's across all the partitions of n.

1. Introduction

Consider the partitions of n. Each partition contains some number of 1's. We study the statistical distribution of the number of 1's across all the partitions of n.

We shall see that the distribution is, roughly speaking, a negative exponential, with mean and standard deviation given by

$$\mu \approx \sigma \approx \frac{\sqrt{6n}}{\pi}.$$

2. Exact Calculations

Of the $p(n)$ partitions of n, it is easy to see that there are $p(n-1)$ partitions with at least one 1: simply strip a 1 off those partitions, and we obtain the partitions of $n-1$ (and the process is reversible). In the same way we see that the number of partitions of n with at least two 1's is $p(n-2)$ (strip a 1 off the $p(n-1)$ partitions of n with at least one 1). Continuing this way, we see that the number of partitions of n with at least k 1's is $p(n-k)$.

It follows that the number of partitions of n with exactly k 1's is $p(n-k)-p(n-k-1)$.

If we let X be the number of 1's in a partition of n, and let f_k be the relative frequency with which $X = k$, $k = 0, 1, 2, \ldots$, then

$$f_k = \frac{p(n-k)-p(n-k-1)}{p(n)}.$$

Check that

$$\sum_{k=0}^{n} f_k = 1.$$

Thus, the mean number of 1's is

$$\begin{aligned}
\mu = E(X) &= \sum_{k\geq 0} f_k k \\
&= 0\left(\frac{p(n)-p(n-1)}{p(n)}\right) + 1\left(\frac{p(n-1)-p(n-2)}{p(n)}\right) + 2\left(\frac{p(n-2)-p(n-3)}{p(n)}\right) + \cdots \\
&= \frac{p(n-1)+p(n-2)+p(n-3)+\cdots}{p(n)} .
\end{aligned}$$

Also,

$$\begin{aligned}
E(X^2) &= \sum_{k\geq 0} f_k k^2 \\
&= 0\left(\frac{p(n)-p(n-1)}{p(n)}\right) + 1\left(\frac{p(n-1)-p(n-2)}{p(n)}\right) + 4\left(\frac{p(n-2)-p(n-3)}{p(n)}\right) + \ \cdots \\
&= \frac{p(n-1)+3p(n-2)+5p(n-3)+\ \cdots}{p(n)}.
\end{aligned}$$

And, of course,

$$\sigma^2 = E(X^2) - (E(X))^2.$$

3. Approximate Calculations

We will show that the distribution of X is roughly negative exponential, with

$$f_k \approx f_0(1-f_0)^k \approx \left(1-\exp\left\{-\frac{\pi}{\sqrt{6n}}\right\}\right)\exp\left\{-\frac{\pi k}{\sqrt{6n}}\right\}$$

and both

$$\mu \approx \frac{\sqrt{6n}}{\pi} \text{ and } \ \sigma \approx \frac{\sqrt{6n}}{\pi}.$$

We will see that our approximation works fairly well, even though it is so crude.

In order to approximate μ, σ and f_k, we will make use of the rough approximation

$$p(n) \approx \frac{1}{4n\sqrt{3}}\exp\left\{K\sqrt{n}\right\},$$

where

$$K = \pi\sqrt{\frac{2}{3}}.$$

Thus,

$$\begin{aligned}
\mu &= \frac{1}{p(n)}\sum_{k=0}^{n} p(k) \ - \ 1 \\
&\approx \frac{n}{\exp\{K\sqrt{n}\}}\int_1^n \frac{\exp\{K\sqrt{x}\}}{x}\,dx \ - \ 1 \\
&\approx \frac{n}{\exp\{K\sqrt{n}\}}\int_1^n \frac{1}{\sqrt{x}}\cdot\frac{\exp\{K\sqrt{x}\}}{\sqrt{x}}\,dx \ - \ 1 \\
&\approx \frac{n}{\exp\{K\sqrt{n}\}}\left\{\frac{2\exp\{K\sqrt{n}\}}{K\sqrt{n}} + \frac{1}{K}\int_1^n \frac{\exp\{K\sqrt{x}\}}{x^{\frac{3}{2}}}\,dx\right\} \ - \ 1 \\
&\approx \frac{2\sqrt{n}}{K} \\
&\approx \frac{\sqrt{6n}}{\pi}.
\end{aligned}$$

Also,

$$
\begin{aligned}
E(X^2) &= \frac{1}{p(n)}\sum_{k=0}^{n}(2n-2k)p(k) - \frac{1}{p(n)}\sum_{k=0}^{n-1}p(k)\\
&= \frac{2}{p(n)}\sum_{k=0}^{n}(n-k)p(k) - \mu\\
&\approx \frac{2n}{\exp\{K\sqrt{n}\}}\int_1^n \frac{(n-x)\exp\{K\sqrt{x}\}}{x}\,dx \ - \ \mu\\
&\approx \frac{2n}{\exp\{K\sqrt{n}\}}\int_1^n \frac{n-x}{\sqrt{x}}\cdot\frac{\exp\{K\sqrt{x}\}}{\sqrt{x}}\,dx \ - \ \mu\\
&\approx \frac{2n}{K\exp\{K\sqrt{n}\}}\int_1^n \frac{n+x}{x}\cdot\frac{\exp\{K\sqrt{x}\}}{\sqrt{x}}\,dx \ - \ \mu\\
&\approx \frac{2n}{K\exp\{K\sqrt{n}\}}\left\{\frac{4\exp\{K\sqrt{n}\}}{K} + \frac{2}{K}\int_1^n \frac{\exp\{K\sqrt{x}\}}{x^2\sqrt{x}}\,dx\right\} \ - \ \mu\\
&\approx \frac{8n}{K^2}\\
&\approx \frac{12n}{\pi^2}.
\end{aligned}
$$

It follows that

$$
\sigma^2 = E(X^2) - E(X)^2 \approx \frac{12n}{\pi^2} - \left(\frac{\sqrt{6n}}{\pi}\right)^2 = \frac{6n}{\pi^2}
$$

and

$$
\sigma \approx \frac{\sqrt{6n}}{\pi}.
$$

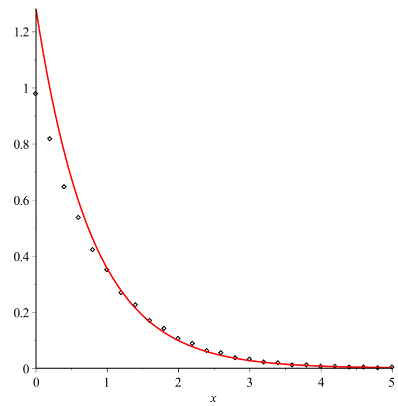
As for f_k, we have

$$
\begin{aligned}
f_k &= \frac{p(n-k) - p(n-k-1)}{p(n)}\\
&= \left(1 - \frac{p(n-k-1)}{p(n-k)}\right)\cdot\frac{p(n-1)}{p(n)}\cdot\frac{p(n-2)}{p(n-1)}\cdot\ \cdots\ \cdot\frac{p(n-k)}{p(n-k+1)}\\
&\approx \left(1 - \exp\left\{-\frac{K}{2\sqrt{n}}\right\}\right)\left(\exp\left\{-\frac{K}{2\sqrt{n}}\right\}\right)^k\\
&\approx \left(1 - \exp\left\{-\frac{\pi}{\sqrt{6n}}\right\}\right)\exp\left\{-\frac{\pi k}{\sqrt{6n}}\right\}.
\end{aligned}
$$

4. An Illustration

Let us choose n large, approximate $\left(1 - \exp\left\{-\frac{\pi}{\sqrt{6n}}\right\}\right)$ by $\frac{\pi}{\sqrt{6n}}$, then scale by plotting $f_k\sqrt{n}$ against $k/\sqrt{n}$ for, say, $k < 5\sqrt{n}$ (that is, for roughly three standard deviations above the mean), and compare it with the negative exponential $\frac{\pi}{\sqrt{6}}\exp\left\{-\frac{\pi}{\sqrt{6}}x\right\}$ for $0 \le x \le 5$.

See figure 1. We see that even for $n = 25$ we get a reasonable fit, but for $n = 625$ a much better fit.



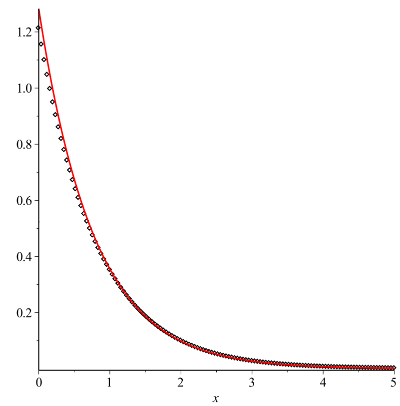


FIGURE 1. Cases $n = 25$ (left) and $n = 625$ (right).

5. Concluding Remark

Whereas in an exact negative exponential distribution

$$\mu f_0 = 1,$$

in this distribution we have

$$\mu f_0 \leq \frac{p(n-1)}{p(n)} < 1$$

for $n > 1$, with strict inequality for $n > 2$. But the proof of this is beyond the scope of this paper.

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