SOME IDENTITIES FOR FOUR TERM RECURRENCE RELATIONS

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Abstract. We generalize a result by Voll for three term recurrence relations to four term recurrence relations and apply the result to a class of Tribonacci sequences, the four term Lucas (Trucas) sequence and the Tribonacci polynomials.

1. Introduction

Three term recurrence relations is a well studied mathematical object and as a consequence of this study several identities related to three term recurrence relations have been established. Several such identities are given in [2] and a general class of such identities are also proven in [6]. We will in the following give a class of identities for four term recurrence relations with constant coefficients by a method similar to the one in [6] and apply our result to the Tribonacci and related sequences.

2. Framework

Our object of study will in this paper be the *four term recurrence relations*, defined by

$$X_n = \alpha X_{n-1} + \beta X_{n-2} + \gamma X_{n-3} \tag{2.1}$$

where α, β and γ are (possibly complex) constants. A sequence $\{X_i\}_{i=-2}^{\infty}$ is called a *solution* of (2.1) if all its elements satisfies this equality for all $n \in \mathbb{N}$. As we know from [3], if $\{X_n\}$ is a solution of the recurrence (2.1), then $\{aX_n\}$ for complex a is also a solution of (2.1). In addition, if $\{X_n\}$ and $\{Y_n\}$ are solutions then so is also $\{X_n + Y_n\}$. Furthermore, if $\{X_i\}_{i=-2}^{\infty}$ is a solution of the recurrence, then so is $\{X_{i+l}\}_{i=-2}^{\infty}$ since α, β and γ are constants.

Let us now assume that $\{X_n\}$, $\{Y_n\}$, and $\{Z_n\}$ are all solutions of the recurrence in (2.1) and then define $\Delta_m^{(k)}$ to be given by the determinant

$$\Delta_m^{(k)} = \begin{vmatrix} X_{m+k} & Y_{m+k} & Z_{m+k} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} \tag{2.2}$$

which is valid whenever $k \geq -2$ and $m \geq 0$. We now observe that

$$\Delta_m^{(-2)} = \begin{vmatrix} X_{m-2} & Y_{m-2} & Z_{m-2} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} = 0 \tag{2.3}$$

and that

$$\Delta_m^{(-1)} = \begin{vmatrix} X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} = 0 \tag{2.4}$$

since two rows are equal. We also see that

$$\Delta_m^{(0)} = \begin{vmatrix} X_m & Y_m & Z_m \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} \tag{2.5}$$

$$= \begin{vmatrix} \alpha X_{m-1} + \beta X_{m-2} + \gamma X_{m-3} & \alpha Y_{m-1} + \beta Y_{m-2} + \gamma Y_{m-3} & \alpha Z_{m-1} + \beta Z_{m-2} + \gamma Z_{m-3} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} \tag{2.6}$$

$$= \begin{vmatrix} \alpha X_{m-1} & \alpha Y_{m-1} & \alpha Z_{m-1} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} + \begin{vmatrix} \beta X_{m-2} & \beta Y_{m-2} & \beta Z_{m-2} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} + \begin{vmatrix} \gamma X_{m-3} & \gamma Y_{m-3} & \gamma Z_{m-3} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} \tag{2.7}$$

$$= \gamma \Delta_{m-1}^{(0)} = \gamma^{m-l} \Delta_l^{(0)} \tag{2.8}$$

as long as $0 \le l \le m$. Furthermore, we observe that

$$\Delta_m^{(k)} = \begin{vmatrix} X_{m+k} & Y_{m+k} & Z_{m+k} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} \tag{2.9}$$

$$= \begin{vmatrix} \alpha X_{m+k-1} & \alpha Y_{m+k-1} & \alpha Z_{m+k-1} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} \tag{2.10}$$

$$+ \begin{vmatrix} \beta X_{m+k-2} & \beta Y_{m+k-2} & \beta Z_{m+k-2} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} \tag{2.11}$$

$$+ \begin{vmatrix} \gamma X_{m+k-3} & \gamma Y_{m+k-3} & \gamma Z_{m+k-3} \\ X_{m-1} & Y_{m-1} & Z_{m-1} \\ X_{m-2} & Y_{m-2} & Z_{m-2} \end{vmatrix} \tag{2.12}$$

$$= \alpha \Delta_m^{(k-1)} + \beta \Delta_m^{(k-2)} + \gamma \Delta_m^{(k-3)} \tag{2.13}$$

and hence, $\{\Delta_m^{(k)}\}_{i=-2}^{\infty}$ is also a solution of the recurrence given in (2.1) for each fixed $m \ge 0$. We now calculate

$$\Delta_m^{(1)} = \alpha \Delta_m^{(0)} + \beta \Delta_m^{(-1)} + \gamma \Delta_m^{(-2)} = \alpha \Delta_m^{(0)} \tag{2.14}$$

$$\Delta_m^{(2)} = \alpha \Delta_m^{(1)} + \beta \Delta_m^{(0)} + \gamma \Delta_m^{(-1)} = (\alpha^2 + \beta) \Delta_m^{(0)} \tag{2.15}$$

$$\Delta_m^{(3)} = \left(\alpha^3 + 2\alpha\beta + \gamma\right) \Delta_m^{(0)} \tag{2.16}$$

$$\Delta_m^{(4)} = \left(\alpha^4 + 3\alpha^2\beta + 2\alpha\gamma + \beta^2\right) \Delta_m^{(0)} \tag{2.17}$$

$$\Delta_m^{(5)} = \left(\alpha^5 + 4\alpha^3\beta + 3\alpha^2\gamma + 3\alpha\beta^2 + 2\beta\gamma\right) \Delta_m^{(0)} \tag{2.18}$$

$$\Delta_m^{(6)} = \left(\alpha^6 + 5\alpha^4\beta + 4\alpha^3\gamma + 6\alpha^2\beta^2 + 6\alpha\beta\gamma + \beta^3 + \gamma^2\right) \Delta_m^{(0)} \tag{2.19}$$

$$\vdots \tag{2.20}$$

$$\Delta_m^{(k)} = P_k \Delta_m^{(0)} \tag{2.21}$$

where P_k is the polynomial defined by

$$P_k = \alpha P_{k-1} + \beta P_{k-2} + \gamma P_{k-3} \tag{2.22}$$

with initial values $P_{-2} = 0$, $P_{-1} = 0$, and $P_0 = 1$. Collecting all this we obtain the following theorem.

Theorem 2.1. *If $\{X_n\}$, $\{Y_n\}$, and $\{Z_n\}$ are solutions of the recurrence (2.1) and $\Delta_m^{(k)}$ is defined as in (2.2), then*

$$\Delta_m^{(k)} = \gamma^{m-l} P_k \Delta_l^{(0)} \tag{2.23}$$

where $0 \le l \le m$.

We also observe that whenever $\{X_i\}_{i=0}^{\infty}$ is given with initial values $X_0 = 0$, $X_1 = 0$, $X_2 = 1$ and $Y_n = X_{n+1}$ and $Z_n = X_{n+2}$ we see that $P_k = X_{k+2}$ and that

$$\Delta_m^{(k)} = \gamma^{m-2} X_{k+2} \begin{vmatrix} X_2 & X_3 & X_4 \\ X_1 & X_2 & X_3 \\ X_0 & X_1 & X_2 \end{vmatrix} \tag{2.24}$$

$$= \gamma^{m-2} X_{k+2} \begin{vmatrix} 1 & X_3 & X_4 \\ 0 & 1 & X_3 \\ 0 & 0 & 1 \end{vmatrix} \tag{2.25}$$

$$= \gamma^{m-2} X_{k+2} \tag{2.26}$$

by setting $l = 2$. In addition, if we return to Theorem 2.1, we see that if $\{X_i\}_{i=0}^{\infty}$ is given with initial values $X_0 = 0$, $X_1 = 0$, $X_2 = 1$, and $\{Y_i\}_{i=0}^{\infty}$ is given with initial values $Y_0 = z$, $Y_1 = 0$, $Y_2 = 1$ where z is any non-zero complex constant and $Z_n = Y_{n+1}$, we have by a similar argument that

$$\Delta_m^{(k)} = \gamma^{m-2} X_{k+2} \begin{vmatrix} X_2 & Y_3 & Y_4 \\ X_1 & Y_2 & Y_3 \\ X_0 & Y_1 & Y_2 \end{vmatrix} \tag{2.27}$$

$$= \gamma^{m-2} X_{k+2} \begin{vmatrix} 1 & Y_3 & Y_4 \\ 0 & 1 & Y_3 \\ 0 & 0 & 1 \end{vmatrix} \tag{2.28}$$

$$= \gamma^{m-2} X_{k+2}. \tag{2.29}$$

Hence, we have the following corollary.

Corollary 2.2. *Let $\{X_i\}_{i=0}^{\infty}$ be the solution of 2.1 with initial values $X_0 = 0$, $X_1 = 0$, and $X_2 = 1$. Then*

$$\begin{vmatrix} X_{m+k} & X_{m+k+1} & X_{m+k+2} \\ X_{m-1} & X_m & X_{m+1} \\ X_{m-2} & X_{m-1} & X_m \end{vmatrix} = \gamma^{m-2} X_{k+2}. \tag{2.30}$$

If in addition $\{Y_i\}_{i=0}^{\infty}$ is the solution of (2.1) with initial values $Y_0 = z$, $Y_1 = 0$, $Y_2 = 1$ where z is any non-zero complex constant, then

$$\begin{vmatrix} X_{m+k} & Y_{m+k+1} & Y_{m+k+2} \\ X_{m-1} & Y_m & Y_{m+1} \\ X_{m-2} & Y_{m-1} & Y_m \end{vmatrix} = \gamma^{m-2} X_{k+2}. \tag{2.31}$$

3. Applications

We now apply the above results to a few well-known sequences that can be written as four term recurrence relations. Our first sequence is the *Tribonacci* sequence, given by the recurrence in (2.1) with $\alpha = 1$, $\beta = 1$ and $\gamma = 1$. In [4] eight different sets of initial values are given for this sequence although only four of the sequences that arise from these initial values are essentially different when we account for index shift. The first 11 terms of the four sequences are

i	F_i	G_i	H_i	I_i
0	0	0	1	1
1	0	1	1	1
2	1	0	0	1
3	1	1	2	3
4	2	2	3	5
5	4	3	5	9
6	7	6	10	17
7	13	11	18	31
8	24	20	33	57
9	44	37	61	105
10	81	68	112	193

where the values for $i = 0, 1, 2$ are the initial values, i.e we have solutions $\{F_i\}_{i=0}^{\infty}$, $\{G_i\}_{i=0}^{\infty}$, $\{H_i\}_{i=0}^{\infty}$, and $\{I_i\}_{i=0}^{\infty}$ of (2.1). The sequence F_i is the one usually named the Tribonacci sequence in the literature, as for instance done in [2]. By an application of Corollary 2.2, we immediately obtain the identity

$$\begin{vmatrix} F_{m+k} & F_{m+k+1} & F_{m+k+2} \\ F_{m-1} & F_m & F_{m+1} \\ F_{m-2} & F_{m-1} & F_m \end{vmatrix} = \begin{vmatrix} F_2 & F_3 & F_4 \\ F_1 & F_2 & F_3 \\ F_0 & F_1 & F_2 \end{vmatrix} P_k = F_{k+2}. \tag{3.1}$$

Similarly by direct application of Theorem 2.1, we obtain the identity

$$\begin{vmatrix} G_{m+k} & G_{m+k+1} & G_{m+k+2} \\ G_{m-1} & G_m & G_{m+1} \\ G_{m-2} & G_{m-1} & G_m \end{vmatrix} = \begin{vmatrix} G_2 & G_3 & G_4 \\ G_1 & G_2 & G_3 \\ G_0 & G_1 & G_2 \end{vmatrix} P_k = 2F_{k+2} \tag{3.2}$$

and similarly

$$\begin{vmatrix} H_{m+k} & H_{m+k+1} & H_{m+k+2} \\ H_{m-1} & H_m & H_{m+1} \\ H_{m-2} & H_{m-1} & H_m \end{vmatrix} = 7F_{k+2} \tag{3.3}$$

$$\begin{vmatrix} I_{m+k} & I_{m+k+1} & I_{m+k+2} \\ I_{m-1} & I_m & I_{m+1} \\ I_{m-2} & I_{m-1} & I_m \end{vmatrix} = 4F_{k+2}. \tag{3.4}$$

Only imagination limits the number of identities we can establish in this manner, and a few are given below:

$$\begin{vmatrix} F_{m+k} & F_{m+k+3} & F_{m+k+6} \\ F_{m-1} & F_{m+2} & F_{m+5} \\ F_{m-2} & F_{m+1} & F_{m+4} \end{vmatrix} = F_{k+2} \tag{3.5}$$

$$\begin{vmatrix} G_{m+k} & G_{m+k+3} & G_{m+k+6} \\ G_{m-1} & G_{m+2} & G_{m+5} \\ G_{m-2} & G_{m+1} & G_{m+4} \end{vmatrix} = 2F_{k+2} \tag{3.6}$$

$$\begin{vmatrix} H_{m+k} & H_{m+k+3} & H_{m+k+6} \\ H_{m-1} & H_{m+2} & H_{m+5} \\ H_{m-2} & H_{m+1} & H_{m+4} \end{vmatrix} = 7F_{k+2} \tag{3.7}$$

$$\begin{vmatrix} I_{m+k} & I_{m+k+3} & I_{m+k+6} \\ I_{m-1} & I_{m+2} & I_{m+5} \\ I_{m-2} & I_{m+1} & I_{m+4} \end{vmatrix} = 4F_{k+2} \tag{3.8}$$

$$\begin{vmatrix} F_{m+k} & G_{m+k} & H_{m+k} \\ F_{m-1} & G_{m-1} & H_{m-1} \\ F_{m-2} & G_{m-2} & H_{m-2} \end{vmatrix} = F_{k+2} \tag{3.9}$$

$$\begin{vmatrix} H_{m+k} & G_{m+k} & I_{m+k} \\ H_{m-1} & G_{m-1} & I_{m-1} \\ H_{m-2} & G_{m-2} & I_{m-2} \end{vmatrix} = -F_{k+2}. \tag{3.10}$$

The four term Lucas (Trucas) sequence [5], L_n, is the solution $\{L_i\}_{i=0}^{\infty}$ given by the same recurrence as the Tribonacci sequence but with initial values $L_0 = 1$, $L_1 = 3$, and $L_2 = 4$. We easily obtain the identities.

$$\begin{vmatrix} L_{m+k} & L_{m+k+1} & L_{m+k+2} \\ L_{m-1} & L_{m} & L_{m+1} \\ L_{m-2} & L_{m-1} & L_{m} \end{vmatrix} = 11F_{k+2} \tag{3.11}$$

$$\begin{vmatrix} F_{m+k} & L_{m+k+1} & L_{m+k+2} \\ F_{m-1} & L_{m} & L_{m+1} \\ F_{m-2} & L_{m-1} & L_{m} \end{vmatrix} = -8F_{k+2} \tag{3.12}$$

$$\begin{vmatrix} F_{m+k} & L_{m+k} & L_{m+k+1} \\ F_{m-1} & L_{m-1} & L_{m} \\ F_{m-2} & L_{m-2} & L_{m-1} \end{vmatrix} = 5F_{k+2} \tag{3.13}$$

by application of Theorem 2.1 and Corollary 2.2. The Tribonacci polynomials $T_n(x)$ are defined the solution $\{T_n(x)_i\}_{i=0}^{\infty}$ of the recurrence in (2.1) with coefficients $\alpha = x^2$, $\beta = x$, and $\gamma = 1$, and initial values $T_0(x) = 0$, $T_1(x) = 0$ and $T_2(x) = 1$. The first few polynomials are

$$T_2(x) = 1 \tag{3.14}$$
$$T_3(x) = x^2 \tag{3.15}$$
$$T_4(x) = x^4 + x \tag{3.16}$$
$$T_5(x) = x^6 + 2x^3 + 1 \tag{3.17}$$
$$T_6(x) = x^8 + 3x^5 + 3x^2 \tag{3.18}$$
$$T_7(x) = x^{10} + 4x^7 + 6x^4 + 2x \tag{3.19}$$
$$T_8(x) = x^{12} + 5x^8 + 10x^6 + 7x^3 + 1. \tag{3.20}$$

We observe that $P_k = T_{k+2}(x)$ and by application of Corollary 2.2, we obtain the identity

$$\begin{vmatrix} T_{m+k}(x) & T_{m+k+1}(x) & T_{m+k+2}(x) \\ T_{m-1}(x) & T_m(x) & T_{m+1}(x) \\ T_{m-2}(x) & T_{m-1}(x) & T_m(x) \end{vmatrix} = T_{k+2}(x) \tag{3.21}$$

which is a generalization of an identity given in [1]. We observe however that our identity has the opposite sign compared to the one in [1] due to two interchanged columns.

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