THE ORDER OF APPEARANCE OF THE PRODUCT OF CONSECUTIVE LUCAS NUMBERS

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Abstract. Let F_n be the nth Fibonacci number and let L_n be the nth Lucas number. The order of appearance $z(n)$ of a natural number n is defined as the smallest natural number k such that n divides F_k. For instance, $z(L_n) = 2n$, for all $n > 1$. In this paper, among other things, we prove that

$$z(L_nL_{n+1}L_{n+2}L_{n+3}) = \tfrac{n(n+1)(n+2)(n+3)}{3},$$

for all positive integers $n \equiv 0 \pmod 3$.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [4] together with its very extensive annotated bibliography for additional references and history). We cannot go very far in the lore of Fibonacci numbers without encountering its companion Lucas sequence $(L_n)_{n\geq 0}$ which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. Let n be a positive integer number, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n \mid F_k$ (some authors also call it *order of apparition*, or *Fibonacci entry point*). There are several results about $z(n)$ in the literature. For instance, $z(m) < m^2 - 1$, for all $m > 2$ (see [13, Theorem, p. 52]) and in the case of a prime number p, one has the better upper bound $z(p) \leq p + 1$, which is a consequence of the known congruence $F_{p-(\frac{p}{5})} \equiv 0 \pmod p$, for $p \neq 2$, where $(\frac{a}{q})$ denotes the Legendre symbol of a with respect to a prime $q > 2$. Very recently, it was proved that all fixed points of $z(n)$ are of the form 5^k or $12 \cdot 5^k$, for some $k \geq 0$ (see [10]).

In recent papers, the author [6, 7, 8, 9] found explicit formulas for the order of appearance of integers related to Fibonacci numbers, such as $F_m \pm 1, F_nF_{n+1}F_{n+2}$ and F_n^k. We remark that most of those results have a “Lucas-version”. For example, it was proved that $z(L_{4n+1} + 1) = 4n(2n + 1)$, and $z(L_n^{k+1}) = nL_n^k/4$, for all integers $k \geq 4$ and $n \equiv 6 \pmod{12}$. However, for instance, nothing was proved about $z(L_nL_{n+1}L_{n+2}L_{n+3})$.

In this note, in order to bridge this gap, we will study the order of appearance of product of consecutive Lucas numbers. Our main result is the following.

Theorem 1.1. *We have*

(i) *For* $n \geq 1$,

$$z(L_nL_{n+1}) = 2n(n + 1).$$

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(ii) *For* $n \geq 1$,

$$z(L_nL_{n+1}L_{n+2}) = \begin{cases} 2n(n+1)(n+2), & \text{if } n \equiv 1 \pmod 2, \\ n(n+1)(n+2), & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

(iii) *For* $n \geq 1$,

$$z(L_nL_{n+1}L_{n+2}L_{n+3}) = \begin{cases} n(n+1)(n+2)(n+3), & \text{if } n \not\equiv 0 \pmod 3, \\ (n(n+1)(n+2)(n+3))/3, & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

We organize the paper as follows. In Section 2, we will recall some useful properties of Fibonacci and Lucas numbers such as a result concerning the p-adic order of F_n and L_n. The last section will be devoted to the proof of the theorem.

2. Auxiliary Results

Before proceeding further, we recall some facts on Fibonacci numbers for the convenience of the reader.

Lemma 2.1. *We have*

(a) $F_n \mid F_m$ *if and only if* $n \mid m$.
(b) $L_n \mid F_m$ *if and only if* $n \mid m$ *and* m/n *is even.*
(c) $L_n \mid L_m$ *if and only if* $n \mid m$ *and* m/n *is odd.*
(d) $F_{2n} = F_nL_n$.
(e) $\gcd(L_n, L_{n+1}) = \gcd(L_n, L_{n+2}) = 1$.

Proofs of these assertions can be found in [4]. We refer the reader to [1, 3, 4, 11] for more details and additional bibliography.

The second lemma is a consequence of the previous one.

Lemma 2.2. *(Cf. Lemma 2.2 of* [7]*) We have*

(a) *If* $F_n \mid m$, *then* $n \mid z(m)$.
(b) *If* $L_n \mid m$, *then* $2n \mid z(m)$.
(c) *If* $n \mid F_m$, *then* $z(n) \mid m$.

The p-adic order (or valuation) of r, $\nu_p(r)$, is the exponent of the highest power of a prime p which divides r. Throughout the paper, we shall use the known facts that $\nu_p(ab) = \nu_p(a)+\nu_p(b)$ and that $a \mid b$ if and only if $\nu_p(a) \leq \nu_p(b)$, for all primes p.

We remark that the p-adic order of Fibonacci and Lucas numbers was completely characterized, see [2, 5, 12]. For instance, from the main results of Lengyel [5], we extract the following two results.

Lemma 2.3. *For* $n \geq 1$, *we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$, *and if* p *is prime* $\neq 2$ *or* 5, *then*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4. *Let* $k(p)$ *be the period modulo* p *of the Fibonacci sequence. For all primes* $p \neq 5$, *we have*

$$\nu_2(L_n) = \begin{cases} 0, & \textit{if } n \equiv 1,2 \pmod 3; \\ 2, & \textit{if } n \equiv 3 \pmod 6; \\ 1, & \textit{if } n \equiv 0 \pmod 6) \end{cases}$$

$$\textit{and } \nu_p(L_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \textit{if } k(p) \neq 4z(p) \textit{ and } n \equiv \frac{z(p)}{2} \pmod{z(p)}; \\ 0, & \textit{otherwise.} \end{cases}$$

Observe that the relation $L_n^2 = 5F_n^2 + 4(-1)^n$ implies that $\nu_5(L_n) = 0$, for all $n \geq 1$.

In view of strong relations between Fibonacci and Lucas numbers, the similarities between items (i) and (ii) of [8, Theorem 1.1] and our Theorem 1.1 are very natural. However, the "surprise" appears by comparing their item (iii). While there are three possibilities for $n(n+1)(n+2)(n+3)/z(F_nF_{n+1}F_{n+2}F_{n+3})$ (namely, $2, 3$ and 6), the sequence $n(n+1)(n+2)(n+3)/z(L_nL_{n+1}L_{n+2}L_{n+3})$ assumes only the values 1 and 3. The reason is that the number $n(n+1)(n+2)(n+3)$ is always divisible by 24 and so the 2-adic order of $F_{n(n+1)(n+2)(n+3)}$ is at least 5 (Lemma 2.3). On the other hand, $\nu_2(L_nL_{n+1}L_{n+2}L_{n+3})$ is at most 3.

With all of the above tools in hand, we now move to the proof of Theorem 1.1.

3. The Proof of Theorem 1.1

3.1. **Proof of (i).** For $\epsilon \in \{0, 1\}$, one has that $L_{n+\epsilon} | L_nL_{n+1}$ and so Lemma 2.2 (b) yields $2(n+\epsilon) \mid z(L_nL_{n+1})$. But either $\gcd(2n, n+1) = 1$ or $\gcd(n, 2(n+1)) = 1$ according to the parity of n. Thus, $2n(n+1) \mid z(L_nL_{n+1})$. On the other hand, $F_{2n(n+1)} = F_{n(n+1)}L_{n(n+1)}$ (Lemma 2.1 (d)) implies, by Lemma 2.1 (a) and (b), that $L_{n+\epsilon} \mid F_{2n(n+1)}$. Since $\gcd(L_n, L_{n+1}) = 1$, we have $L_nL_{n+1} \mid F_{2n(n+1)}$ and then $z(L_nL_{n+1}) \mid 2n(n+1)$ (Lemma 2.2 (c)). In conclusion, we have $z(L_nL_{n+1}) = 2n(n+1)$. □

3.2. **Proof of (ii).** The proof splits in two cases according to the parity of n.

Case 1: If n is even. Then Lemma 2.1 (b) together with the fact that $n(n+2) \equiv 0 \pmod 8$ yield $L_{n+\epsilon} \mid F_{n(n+1)(n+2)}$, for $\epsilon \in \{0, 1, 2\}$. Since the numbers L_n, L_{n+1}, L_{n+2} are pairwise coprime, we have $L_nL_{n+1}L_{n+2} \mid F_{n(n+1)(n+2)}$ and so

$$z(L_nL_{n+1}L_{n+2}) \mid n(n+1)(n+2). \tag{3.1}$$

Now, we use that $L_{n+\epsilon} \mid L_nL_{n+1}L_{n+2}$, to conclude that $2(n+\epsilon)$ divides $z(L_nL_{n+1}L_{n+2})$ (we used Lemma 2.2 (b)). Also, there are distinct $a, b \in \{-1, 1\}$ such that $2^an, n+1, 2^b(n+2)$ are pairwise coprime (the choice of a and b depends on the class of n modulo 4). Therefore,

$$n(n+1)(n+2) = 2^{a+b}n(n+1)(n+2) \mid z(L_nL_{n+1}L_{n+2})$$

and the result follows from (3.1).

Case 2: If n is odd. Then by Lemma 2.1 (b) we have that $L_{n+\epsilon} \mid F_{2n(n+1)(n+2)}$ (observe that the factor 2 is necessary because in this case only $n+1$ is even) and so $z(L_nL_{n+1}L_{n+2}) \mid 2n(n+1)(n+2)$, where we used that L_n, L_{n+1}, L_{n+2} are pairwise coprime. On the other hand, as in the previous case, $2(n+\epsilon)$ divides $z(L_nL_{n+1}L_{n+2})$. In particular, $n, 2(n+1), n+2$ divides $z(L_nL_{n+1}L_{n+2})$ yielding $2n(n+1)(n+2) \mid z(L_nL_{n+1}L_{n+2})$. The proof is complete. □

3.3. **Proof of (iii).** Since there are two odd numbers among $n, n+1, n+2, n+3$, we conclude that

$$L_{n+\epsilon} \mid L_{n(n+1)(n+2)(n+3)}, \text{ for } \epsilon \in \{0, 1, 2, 3\}. \tag{3.2}$$

Case 1: If $n \not\equiv 0 \pmod 3$. Then $\gcd(L_n, L_{n+3}) = 1$ and so, by Lemma 2.1 (e), the numbers $L_n, L_{n+1}, L_{n+2}, L_{n+3}$ are pairwise coprime. Thus, (3.2) implies that

$$z(L_nL_{n+1}L_{n+2}L_{n+3}) \mid n(n+1)(n+2)(n+3).$$

On the other hand, $L_{n+\epsilon} \mid L_nL_{n+1}L_{n+2}L_{n+3}$ and so $n+\epsilon$ divides $z(L_nL_{n+1}L_{n+2}L_{n+3})$. Note that there exists only one pair among $(n, n+2)$ and $(n+1, n+3)$ whose greatest common divisor is 2 depending on the parity of n. Suppose, without loss of generality, that n is even. Since $\gcd(n, n+3) = 1$, we can deduce that $n/2^a, n+1, (n+2)/2^b, n+3$ are pairwise coprime, for distinct $a, b \in \{0, 1\}$ suitably chosen depending on the class of n modulo 4. Thus,

$$\frac{n(n+1)(n+2)(n+3)}{2} = \frac{n(n+1)(n+2)(n+3)}{2^{a+b}} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}).$$

Therefore, we have

$$z(L_nL_{n+1}L_{n+2}L_{n+3}) \in \left\{\frac{n(n+1)(n+2)(n+3)}{2}, n(n+1)(n+2)(n+3)\right\}$$

and it suffices to prove that

$$L_nL_{n+1}L_{n+2}L_{n+3} \nmid F_{\frac{n(n+1)(n+2)(n+3)}{2}}, \text{ for all } n \geq 1. \tag{3.3}$$

Since we are supposing that n is even, then $4 \mid n+\delta$, for some $\delta \in \{0, 2\}$. Suppose, to derive a contradiction, that (3.3) is false. Then $L_{n+\delta} \mid F_{n(n+1)(n+2)(n+3)/2}$ and Lemma 2.1 (b) implies that

$$\frac{n(n+1)(n+2)(n+3)}{2(n+\delta)} = \frac{(n+1)(n+3)(n+\delta+2(-1)^{\delta/2})}{2}$$

is even. However, this leads to an absurdity, because

$$\nu_2\left(\frac{n(n+1)(n+2)(n+3)}{2(n+\delta)}\right) = \nu_2(n+\delta+2(-1)^{\delta/2}) - 1 = 0,$$

where we used that $n+\delta+2(-1)^{\delta/2} \equiv 2 \pmod 4$, since $n+\delta \equiv 0 \pmod 4$.

Case 2: If $n \equiv 0 \pmod 3$. As in previous items, we obtain that $n+\epsilon \mid z(L_nL_{n+1}L_{n+2}L_{n+3})$. Note that $\gcd(n, n+3) = 3$ and if $9 \mid n$, then $\gcd(n, (n+3)/3) = 1$, while $\gcd(n/3, n+3) = 1$ when $9 \nmid n$. In any case, for a suitable choice of $a, b, c, d, e, f \in \{0, 1\}$, where $a \neq b$ and only one among c, d, e, f is 1, we obtain that

$$\frac{n}{2^c3^a}, \frac{n+1}{2^d}, \frac{n+2}{2^e}, \frac{n+3}{2^f3^b}$$

are pairwise coprime. Here the sets $\{a, b\}$ and $\{c, d, e, f\}$ depend on the class of n modulo 4 and 9, respectively. Hence, we get

$$\frac{n(n+1)(n+2)(n+3)}{6} = \frac{n(n+1)(n+2)(n+3)}{2^{c+d+e+f}3^{a+b}} \mid z(L_nL_{n+1}L_{n+2}L_{n+3}), \tag{3.4}$$

since $a+b = c+d+e+f = 1$. Therefore, we deduce that

$$z(L_nL_{n+1}L_{n+2}L_{n+3}) \in \left\{\frac{n(n+1)(n+2)(n+3)}{6}, \frac{n(n+1)(n+2)(n+3)}{3}, \dots\right\}.$$

However, from (3.3), we obtain

$$z(L_nL_{n+1}L_{n+2}L_{n+3}) \in \left\{\frac{n(n+1)(n+2)(n+3)}{3}, \frac{2n(n+1)(n+2)(n+3)}{3}, \dots\right\}.$$

Thus, it suffices to prove that $L_nL_{n+1}L_{n+2}L_{n+3} \mid F_{n(n+1)(n+2)(n+3)/3}$. We point out that the tools applied before does not work in this case, mainly because $\gcd(L_n, L_{n+3}) = 2$, for all integers $n \equiv 0 \pmod 3$. Hence, we shall prove that

$$\nu_p(L_nL_{n+1}L_{n+2}L_{n+3}) \leq \nu_p\left(F_{\frac{n(n+1)(n+2)(n+3)}{3}}\right), \text{ for all primes } p \text{ and integers } n.$$

Since $5 \nmid L_n$, we may suppose that $p \neq 5$.

• When $p = 2$, Lemma 2.4 yields that $\nu_2(L_nL_{n+1}L_{n+2}L_{n+3}) \leq 3$. On the other hand, $n(n+1)(n+2)(n+3)/3 \equiv 0 \pmod{24}$ (since $3 \mid n$) and thus, by Lemma 2.3,

$$\nu_2\left(F_{\frac{n(n+1)(n+2)(n+3)}{3}}\right) = \nu_2\left(\frac{n(n+1)(n+2)(n+3)}{3}\right) + 2 \geq 5 > \nu_2(L_nL_{n+1}L_{n+2}L_{n+3}).$$

• When $p \neq 2$ and 5. First, note that only one among $L_n, L_{n+1}, L_{n+2}, L_{n+3}$ may be divisible by p. In fact, on the contrary, there exist distinct $\epsilon_1, \epsilon_2 \in \{0, 1, 2, 3\}$ such that

$$n + \epsilon_1 \equiv n + \epsilon_2 \equiv \frac{z(p)}{2} \pmod{z(p)}.$$

But this implies that $z(p) \mid \epsilon_1 - \epsilon_2$ leading to an absurdity, because $|\epsilon_1 - \epsilon_2| \leq 3$ while $z(p) \geq 4$ for all primes $p > 2$. Without loss of generality we can assume that $p \mid L_n$ and thus, (by Lemma 2.4)

$$\nu_p(L_nL_{n+1}L_{n+2}L_{n+3}) = \nu_p(n) + \nu_p(F_{z(p)}). \tag{3.5}$$

Also, $n \equiv z(p)/2 \pmod{z(p)}$ implies that $z(p) \mid 2n$. Thus, $z(p) \mid n(n+1)(n+2)(n+3)/3$, because $(n+1)(n+2)(n+3)/2$ is even and therefore

$$\nu_p\left(F_{\frac{n(n+1)(n+2)(n+3)}{3}}\right) = \nu_p\left(\frac{n(n+1)(n+2)(n+3)}{3}\right) + \nu_p(F_{z(p)}). \tag{3.6}$$

Now we combine (3.5) and (3.6) to obtain

$$\nu_p\left(F_{\frac{n(n+1)(n+2)(n+3)}{3}}\right) - \nu_p(L_nL_{n+1}L_{n+2}L_{n+3}) = \nu_p(n+1) + \nu_p(n+2) + \nu_p(n+3) - \nu_p(3) \geq 0,$$

where we used that in the case of $p = 3$, $\nu_p(n+3) \geq 1$.

In conclusion, $L_nL_{n+1}L_{n+2}L_{n+3} \mid F_{n(n+1)(n+2)(n+3)/3}$ and the proof is then complete. □

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