ON RECURRENCES OF FAHR AND RINGEL: AN ALTERNATE APPROACH

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Abstract. In a recent article, Hirschhorn found the generating functions of two sequences introduced by Fahr and Ringel. We use a matrix method to obtain the same results in a simpler and more direct manner.

1. Introduction

In a recent paper, Fahr and Ringel [1] introduced two sequences $b_t[r]$ and $c_t[r]$ defined by the initial values $b_0[r] = c_0[r] = \delta_{r,0}$, and the recurrence relations

$$\begin{aligned} b_{t+1}[r] &= c_t[r-1] + 2c_t[r] - b_t[r], \\ c_{t+1}[t] &= b_{t+1}[r] + 2b_{t+1}[r+1] - c_t[r], \end{aligned}$$

for $t, r \geq 0$, with the convention that $c_t[-1] = c_t[0]$. Their first few values are listed below.

$b_t[r]$					
$t\backslash r$	0	1	2	3	4
0	1				
1	2	1			
2	7	4	1		
3	29	18	6	1	
4	130	85	33	8	1

$c_t[r]$					
$t\backslash r$	0	1	2	3	4
0	1				
1	3	1			
2	12	5	1		
3	53	25	7	1	
4	247	126	42	9	1

Define

$$B_r = B_r(q) = \sum_{t\geq 0} b_t[r]q^t, \qquad \text{and} \qquad C_r = C_r(q) = \sum_{t\geq 0} b_t[r]q^t$$

as the "vertical" generating functions for $b_t[r]$ and $c_t[r]$. Hirschhorn [2] observed that

$$B_0 = \frac{1 + 3qC_0}{1 + q},$$

and, for $r \geq 0$,

$$B_{r+1} = \frac{1}{2}\left[(1+q)C_r - B_r\right], \tag{1.1}$$

$$C_{r+1} = \frac{1}{2q}\left[(1+q)B_{r+1} - qC_r\right]. \tag{1.2}$$

After rather lengthy computation that involves solving four auxillary recurrences, he found explicit formulas for B_r and C_r. The purpose of this short note is to use a simpler and more direct approach to derive the same results.

2. A Matrix Approach

Equations (1.1) and (1.2) form a system of recurrences. Such a system can sometimes be solved rather effectively by a transfer matrix, as demonstrated by the author in [3]. For a detail discussion of the technique, see [6].

We first write (1.2) as

$$C_{r+1} = \frac{1+q}{2q}\left(\frac{1+q}{2}\,C_r - \frac{1}{2}\,B_r\right) - \frac{1}{2}\,C_r = -\frac{1+q}{4q}\,B_r + \frac{1+q^2}{4q}\,C_r, \tag{2.1}$$

so that (1.1) and (2.1) can be written in a matrix equation

$$\begin{bmatrix} B_{r+1} \\ C_{r+1} \end{bmatrix} = \frac{1}{4q}\begin{bmatrix} -2q & 2(1+q)q \\ -(1+q) & 1+q^2 \end{bmatrix}\begin{bmatrix} B_r \\ C_r \end{bmatrix}, \qquad r \geq 0.$$

Let A denote the transfer matrix. It is clear that $\begin{bmatrix} B_r \\ C_r \end{bmatrix} = A^r \begin{bmatrix} B_0 \\ C_0 \end{bmatrix}$ for $r \geq 0$. Hence,

$$\sum_{r\geq 0}\begin{bmatrix} B_r \\ C_r \end{bmatrix} x^r = \left(\sum_{r\geq 0}(xA)^r\right)\begin{bmatrix} B_0 \\ C_0 \end{bmatrix} = (I - xA)^{-1}\begin{bmatrix} B_0 \\ C_0 \end{bmatrix}. \tag{2.2}$$

Finding the inverse of $I - xA$ is straightforward:

$$(I - xA)^{-1} = \frac{1}{4q\left(1 - \frac{(1-q)^2}{4q}\,x + \frac{1}{4}\,x^2\right)}\begin{bmatrix} 4q - (1+q^2)x & 2(1+q)qx \\ -(1+q)x & 4q + 2qx \end{bmatrix}.$$

Let

$$\begin{aligned} \mu &= \frac{(1-q)^2 + (1+q)\sqrt{1-6q+q^2}}{8q} &&= \frac{1}{4q}\,(1 - 2q - 3q^2 - 8q^3 - \cdots), \\ \nu &= \frac{(1-q)^2 - (1+q)\sqrt{1-6q+q^2}}{8q} &&= \frac{1}{4}\,(4q + 8q^2 + 28q^3 + 112q^4 + \cdots), \end{aligned}$$

so that

$$\frac{1}{1 - \frac{(1-q)^2}{4q}\,x + \frac{1}{4}\,x^2} = \frac{1}{(1-\mu x)(1-\nu x)} = \sum_{r\geq 0}\frac{\mu^{r+1} - \nu^{r+1}}{\mu - \nu}\,x^r.$$

After expanding the right-hand side of (2.2), and comparing the coefficients of x^r, we find

$$\begin{aligned} 4q(\mu-\nu)B_r &= 4qB_0(\mu^{r+1} - \nu^{r+1}) + [2(1+q)qC_0 - (1+q^2)B_0](\mu^r - \nu^r) \\ &= [4qB_0\mu + 2(1+q)qC_0 - (1+q^2)B_0]\,\mu^r \\ &\quad - [4qB_0\nu + 2(1+q)qC_0 - (1+q^2)B_0]\,\nu^r, \\ 4q(\mu-\nu)C_r &= 4qC_0(\mu^{r+1} - \nu^{r+1}) + [2qC_0 - (1+q)B_0](\mu^r - \nu^r) \\ &= [4qC_0\mu + 2qC_0 - (1+q)B_0]\,\mu^r - [4qC_0\nu + 2qC_0 - (1+q)B_0]\,\nu^r. \end{aligned}$$

Recall that B_r and C_r are analytic infinite series in q, but μ has a pole at $q = 0$. Therefore, we need

$$4qB_0\mu + 2(1+q)qC_0 - (1+q^2)B_0 = 0, \tag{2.3}$$

$$4qC_0\mu + 2qC_0 - (1+q)B_0 = 0. \tag{2.4}$$

Both lead to

$$B_0 = \frac{1 + q + \sqrt{1-6q+q^2}}{2}\,C_0.$$

Together with $B_0 = (1+3qC_0)/(1+q)$, we find

$$C_0 = \frac{(1+q)\sqrt{1-6q+q^2} - (1-4q+q^2)}{2q(1-7q+q^2)},$$

and

$$B_0 = \frac{3\sqrt{1-6q+q^2} - (1+q)}{2(1-7q+q^2)},$$

as found by Hirschhorn [2]. They are also the generating functions for sequences A110122 and A132262, respectively, in *OEIS* [5]. Furthermore, from (2.3), we find

$$-[4qB_0\nu + 2(1+q)qC_0 - (1+q^2)B_0] = -(4qB_0\nu - 4qB_0\mu) = 4q(\mu-\nu)B_0.$$

A similar result for C_r can be derived from (2.4). From these we obtain the surprisingly simple main results of Hirschhorn:

$$B_r = B_0\nu^r = B_0\left(\frac{(1-q)^2 - (1+q)\sqrt{1-6q+q^2}}{8q}\right)^r,$$

$$C_r = C_0\nu^r = C_0\left(\frac{(1-q)^2 - (1+q)\sqrt{1-6q+q^2}}{8q}\right)^r.$$

3. Closing Remarks

There are other methods that one could use to find the generating functions. For instance, Prodinger [4] used bivariate generating functions and the kernel method to derive identical results.

References

[1] P. Fahr and C. M. Ringel, *A partition function for Fibonacci numbers*, J. Integer Sequences, **11** (2008), Article 08.1.4.

[2] M. D. Hirschhorn, *On recurrences of Fahr and Ringel arising in graph theory*, J. Integer Sequences, **12** (2009), Article 09.6.8.

[3] Y.-H. H. Kwong and D. G. Rogers, *A matrix method for counting Hamiltonian cycles on grid graphs*, European J. Combin., **15** (1994), 277–283.

[4] H. Prodinger, *Generating functions related to partition formulae for Fibonacci numbers*, J. Integer Sequences, **11** (2008), Article 08.1.8.

[5] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.

[6] R. P. Stanley, *Enumerative Combinatorics*, Vol. I, Wadsworth & Brooks/Cole, Monterey, CA, 1986.

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