ON THE SUM-OF-DIVISORS FUNCTION

John A. Ewell
Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115
(Submitted July 2005-Final Revision October 2005)

ABSTRACT

For each integer $n > 0, \sigma(n)$ denotes the sum of all positive divisors of $n; b(n)$ denotes the exponent (≥ 0) of the largest power of 2 dividing n, and then $0d(n) := n2^{-b(n)}$. For each integer $n \geq 0$, $q(n)$ denotes the number of partitions of n into distinct parts; and $q_0(n)$ denotes the number of partitions of n into distinct odd parts. Conventionally, $q(0) = q_0(0) := 1$. It is here demonstrated that the composite function $\sigma \circ 0d$ can be expressed additively in terms of the functions q, q_0.

1. INTRODUCTION

Recall that $\mathbb{P} := \{1, 2, 3, \ldots\}$, $\mathbb{N} := \mathbb{P} \cup \{0\}$ and $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. Then, for each $n \in \mathbb{P}$, $\sigma(n)$ denotes the sum of all positive divisors of n; $b(n)$ denotes the exponent (≥ 0) of the largest power of 2 dividing n, and then $0d(n) := n2^{-b(n)}$.

For each $n \in \mathbb{N}, q(n)$ denotes the number of partitions of n into distinct parts, and $q_0(n)$ denotes the number of partitions of n into distinct odd parts. Conventionally, $q(0) = q_0(0) := 1$.

In some ways the restricted partition functions $q(\cdot)$ and $q_0(\cdot)$ are better described by their generating functions:

$$\prod_1^\infty (1 + x^n) = \sum_{n=0}^\infty (q(n)x^n \tag{1.1}$$

and

$$\prod_1^\infty (1 + x^{2n-l}) = \sum_{n=0}^\infty q_0(n)x^n, \quad |x| < 1. \tag{1.2}$$

In this note we prove the following theorem, which shows how to evaluate the multiplicative function σ in terms of the restricted partition functions $q(\cdot)$, $q_0(\cdot)$.

Theorem 1.1: *For each $n \in \mathbb{P}$, if n is odd, then*

$$\sigma(n) = \sum_{j=1}^n (-1)^{j+l} jq(j)q_0(n-j) \tag{1.3}$$

and, for each $m \in \mathbb{P}$,

$$\sigma(0d(m)) = \sum_{j=1}^{2m} (-1)^j jq(j)q_0(2m-j). \tag{1.4}$$

2. PROOF OF THEOREM 1.1

Our point of departure is a famous identity due to Euler [2, p. 277], viz.,

$$\prod_1^\infty \frac{1}{1 - x^{2n-1}} = \prod_1^\infty (1 + x^n), \tag{2.1}$$

which is valid for each complex number x such that $|x| < 1$. Clearly, (2.1) and (1.1) imply

$$\prod_{1}^{\infty} \frac{1}{1 - x^{2n-1}} = \sum_{n=0}^{\infty} q(n)x^n.$$

Now, with D_x denoting differentiation with respect to x, operate on both sides of the foregoing identity with xD_x to get

$$\sum_{1}^{\infty} \frac{(2n-1)x^{2n-1}}{1 - x^{2n-1}} \cdot \prod_{1}^{\infty} \frac{1}{1 - x^{2n-1}} = \sum_{n=1}^{\infty} nq(n)x^n. \tag{2.2}$$

It follows easily that

$$\sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-1}}{1 - x^{2n-1}} = \sum_{n=1}^{\infty} \sigma(0d(n))x^n.$$

Hence, in view of the fact that $0d(2n+1) = 2n+1$, for each $n \in \mathbb{N}$, we multiply both sides of (2.2) by the infinite product $\prod(1 - x^{2n-1})$, and subsequently appeal to (1.2) to get

$$\begin{aligned}
\sum_{m=0}^{\infty} \sigma(2m+1)x^{2m+1} &+ \sum_{m=1}^{\infty} \sigma(0d(2m))x^{2m} \\
&= \sum_{n=1}^{\infty} \sigma(0d(n))x^n \\
&= \prod_{n=1}^{\infty} (1 - x^{2n-1}) \sum_{n=1}^{\infty} nq(n)x^n \\
&= \sum_{j=0}^{\infty} (-1)^j q_0(j)x^j \sum_{k=1}^{\infty} kq(k)x^k \\
&= \left\{ \sum_{j=0}^{\infty} q_0(2j)x^{2j} - \sum_{j=0}^{\infty} q_0(2j+1)x^{2j+1} \right\} \\
&\quad \times \left\{ \sum_{k=1}^{\infty} (2k)q(2k)x^{2k} + \sum_{k=0}^{\infty} (2k+1)q(2k+1)x^{2k+1} \right\} \\
&= \sum_{m=0}^{\infty} x^{2m+1} \sum_{k=0}^{m} q_0(2m-2k)(2k+1)q(2k+1) \\
&\quad - \sum_{m=1}^{\infty} x^{2m+1} \sum_{k=1}^{m} q_0(2m-2k+1)(2k)q(2k) \\
&\quad + \sum_{m=1}^{\infty} x^{2m} \sum_{k=1}^{m} q_0(2m-2k)(2k)q(2k) \\
&\quad - \sum_{m=1}^{\infty} x^{2m} \sum_{k=0}^{m-1} q_0(2m-2k-1)(2k+1)q(2k+1).
\end{aligned}$$

Equating coefficients of like powers of x, we thus prove our theorem. (Note that in (1.4) $0d(2m) = 0d(m)$, for each $m \in \mathbb{P}$.)

The formulas of Theorem 1.1 do not provide an efficient way to obtain $\sigma(n)$ for large n because they require computation of too many terms. But, for smaller values of n one can first compute the desired values of the functions $q(\cdot)$ and $q_0(\cdot)$ by recurrences for these [1, pp. 1-2], and subsequently $\sigma(n)$.

3. HISTORICAL COMMENTS

It certainly seems fair to say that Euler would not have been surprised by Theorem 1.1. However, it seems equally fair to say that he was not in possession of tools to establish recurrences for the functions $q(\cdot)$, $q_0(\cdot)$. For, derivation of these depends on a couple of special cases of the celebrated triple-product identity [2, pp. 282-283]:

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+tx^{2n-l})(1+t^{-1}x^{2n-l}) = \sum_{-\infty}^{\infty} x^{n^2}t^n,$$

which is valid for each pair of complex numbers t, x such that $t \neq 0$ and $|x| < 1$. This identity was first stated and proved by Gauss about 25 years after Euler's death (1783). E.g. see [2, p. 296].

Finally, we observe that Theorem 1.1 provides yet another link between multiplicative number theory (since $\sigma(\cdot)$ is known to be multiplicative) and additive number theory (since the functions $q(\cdot)$ and $q_0(\cdot)$ play substantial roles in the theory of partitions of natural numbers).

ACKNOWLEDGMENTS

The author would like to thank the referee for suggestions which led to improved exposition.

REFERENCES

[1] J.A. Ewell. "Recurrences for Two Restricted Partition Functions." *The Fibonacci Quarterly* **18** (1980): 1-2.

[2] G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers*, 4th ed., Clarendon Press, Oxford, 1960.

AMS Classification Numbers: 11E25. 05A20

✠ ✠ ✠