ON THE CONSTRUCTION OF A FAMILY OF TRANSCENDENTAL VALUED INFINITE PRODUCTS

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1. INTRODUCTION

In recent time there has been much progress made on the problem of determining sufficiency conditions for a positive rational termed series to converge to either an irrational or transcendental number (see [1], [4], [6] and the references cited therein). Surprisingly, in comparison, very little attention has been paid to finding such sufficiency conditions in the case of infinite products. One such sufficiency condition is attributable to Cantor (see [3]) however some generalisations of this condition have also been obtained in [8]. Cantor, in particular, proved that if $\{a_n\}$ is a sequence of positive integers such that $a_{n+1} > a_n^2$ then the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{a_n}\right) \tag{1}$$

converges to an irrational number. In this paper we shall improve upon Cantor's result by showing that, if for a fixed $\lambda > 2$, the sequence of integers $\{a_n\}$ satisfies the growth condition

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n^{\lambda+1}} > 2\ , \tag{2}$$

then the infinite product, in (1), will, in fact, converge to a transcendental number. The above condition, which is similar in form to the one the author developed in the case of infinite series, (see [5]) will follow as an application of the following Diophantine approximation theorem of Roth (see [7])

Theorem 1.1: *If α is an algebraic number of degree greater than or equal to 2 and ε is any positive number then the inequality*

$$\left|\alpha-\frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

can have only finitely many solution $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(p,q) = 1$.

As an application of the main result we shall exhibit some examples from the class of infinite products defined by (2). These products as we shall see will be formed using $a_n = U_{f(n)}$ and $a_n = V_{f(n)}$, where U_n, V_n are the generalised Fibonacci and Lucas sequences, respectively, and $f(\cdot)$ a predefined integer valued function.

2. MAIN RESULT

To establish transcendence of the infinite products in question we shall need to make use of the following technical lemma.

Lemma 2.1: *If a real number $A > 1$ has an infinite product representation of the form $\prod_{n=1}^{\infty}(1+\frac{1}{a_n})$, with $a_n \in \mathbb{N}$ and such that $a_{n+1} \geq a_n^2 > 1$, then the representation is unique. Moreover each element of the sequence a_n can be calculated in succession via*

$$a_n = \left\lfloor \frac{A_n}{A_n - 1} \right\rfloor, \tag{3}$$

where $A_n = \prod_{r=n}^{\infty}(1+\frac{1}{a_r})$ and $\lfloor x \rfloor$ denotes the largest integer not greater than x.

Proof: Under the assumption of the infinite product representation of A it will suffice to demonstrate (3) in order to prove uniqueness of the representation. From repeated application of the inequality $a_{n+1} \geq a_n^2$, it is clear that $a_{n+s} \geq a_n^{2^s}$ for $s \in \mathbb{N}$. Consequently, by recalling the infinite product identity of Euler's namely, $\prod_{n=0}^{\infty}(1+x^{2^n}) = (1-x)^{-1}$ valid for $|x| < 1$, one obtains after substituting $x = a_n^{-1}$ the following upper bound for A_n

$$A_n = \prod_{r=n}^{\infty}\left(1+\frac{1}{a_r}\right) \leq \prod_{r=1}^{\infty}\left(1+\frac{1}{a_n^{2^{r-1}}}\right) = \frac{a_n}{a_n - 1} = 1 + \frac{1}{a_n - 1}. \tag{4}$$

From (4) it readily follows that $\frac{A_n}{A_n-1} \geq a_n$. Moreover as $A_n > 1 + \frac{1}{a_n}$ one also deduces that $\frac{A_n}{A_n-1} < a_n + 1$. Hence

$$a_n \leq \frac{A_n}{A_n - 1} < a_n + 1,$$

and, as $a_n \in \mathbb{N}$, it is clear that (3) holds from definition of the function $\lfloor x \rfloor$. □

Using this Lemma we can now deduce the following result.

Theorem 2.1: *Suppose $\{a_n\}$ is a sequence of positive integers greater than unity and such that, for a fixed $\lambda > 2$*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n^{\lambda+1}} > 2,$$

then the infinite product $\prod_{n=1}^{\infty}(1+\frac{1}{a_n})$ converges to a transcendental number.

Proof: From the assumption it is clear that the infinite products in question are convergent. Denoting the value of the infinite product by θ, we will demonstrate via Roth's theorem that θ cannot be an algebraic number of degree greater than or equal to two. Transcendence will then follow upon showing that θ in addition cannot be rational. To this end, consider a sequence of rational approximations p_m/q_m to θ generated from the m^{th} partial products, expressed in reduced form. We begin by obtaining an upper bound for $q_m^{\lambda}|\theta - p_m/q_m|$, when m is sufficiently large. From the assumption there must exist a $\delta > 0$ such that $a_{n+1}/2a_n^{\lambda+1} \geq (1+\delta)$, for all $n \geq N(\delta)$ say, moreover, we can take $N(\delta) = \min\{r \in \mathbb{N} : a_{n+1}/2a_n^{\lambda+1} \geq (1+\delta) \text{ for all } n \geq r\}$. Furthermore, choose $m > N(\delta)$ and note from Lemma 2.1, as $a_{m+1} = \lfloor \frac{A_{m+1}}{A_{m+1}-1} \rfloor$, that $a_{m+1}(A_{m+1}-1) \leq A_{m+1}$. Consequently, one obtains the following sequence of inequalities

$$\begin{aligned} q_m^\lambda \left| \theta - \frac{p_m}{q_m} \right| &= q_m^\lambda \left\{ \prod_{n=1}^{\infty} \left(1 + \frac{1}{a_n}\right) - \prod_{n=1}^{m} \left(1 + \frac{1}{a_n}\right) \right\} \\ &= \frac{q_m^{\lambda-1} p_m}{a_{m+1}} a_{m+1} \left\{ \prod_{n=m+1}^{\infty} \left(1 + \frac{1}{a_n}\right) - 1 \right\} \\ &\leq \frac{q_m^{\lambda-1} p_m}{a_{m+1}} \prod_{n=m+1}^{\infty} \left(1 + \frac{1}{a_n}\right) \\ &< \frac{q_m^{\lambda-1} p_m}{a_{m+1}} \theta. \end{aligned} \tag{5}$$

Now, as the m^{th} partial product is equal to $(a_1+1)(a_2+1)\cdots(a_m+1)/a_1\cdots a_m$ and $(p_m, q_m) = 1$, we must have $p_m \leq (a_1+1)(a_2+1)\cdots(a_m+1) \leq 2^m a_1 a_2 \cdots a_m$ and $q_m \leq a_1 a_2 \cdots a_m$. Thus combining with the inequality in (5) yields, for $m > N(\delta)$, the upper bound

$$q_m^\lambda \left| \theta - \frac{p_m}{q_m} \right| < 2^m \frac{(a_1 a_2 \cdots a_m)^\lambda}{a_{m+1}} \theta = b_m \theta. \tag{6}$$

We now demonstrate that $b_m = o(1)$ as $m \to \infty$. As $N(\delta) \in \mathbb{N}$ is fixed, it will suffice to show $b'_m = o(1)$, as $m \to \infty$ where

$$b'_m = \frac{1}{a_{m+1}} \prod_{r=N(\delta)}^{m} (2^{\frac{1}{\lambda}} a_r)^\lambda.$$

To this end consider

$$\begin{aligned} \log(1/b'_m) &= \sum_{r=N(\delta)}^{m} (\log a_{r+1} - \log a_r) + \log a_{N(\delta)} - \lambda \sum_{r=N(\delta)}^{m} \log(2^{\frac{1}{\lambda}} a_r) \\ &= \sum_{r=N(\delta)}^{m} \log\left(\frac{a_{r+1}}{2a_r^{\lambda+1}}\right) + \log a_{N(\delta)} \\ &\geq \sum_{r=N(\delta)}^{m} \log\left(\frac{a_{r+1}}{2a_r^{\lambda+1}}\right). \end{aligned} \tag{7}$$

However, since $r \geq N(\delta)$ one must have

$$\log\left(\frac{a_{r+1}}{2a_r^{\lambda+1}}\right) \geq \log(1+\delta).$$

Consequently, from (7), we have $\log(1/b'_m) \geq (m - N(\delta) + 1)\log(1+\delta) \to \infty$, as $m \to \infty$. So there exists an integer $N_1 > 0$ such that $b_m < \theta^{-1}$ for $m \geq N_1$. Hence, for all $m > \max\{N_1, N(\delta)\}$, the rational approximations p_m/q_m to θ satisfy the inequality

$$\left|\theta - \frac{p_m}{q_m}\right| < \frac{1}{q_m^\lambda},$$

and as $\lambda > 2$, we conclude from Theorem 1.1 that θ is either transcendental or rational. However, as $|q_m\theta - p_m| < q_m^\lambda|\theta - p_m/q_m|$, we deduce from (6) that $|q_m\theta - p_m| = o(1)$ as $m \to \infty$. Thus via standard criterion of irrationality, θ cannot be rational and so the infinite product must have a transcendental value. □

3. APPLICATION

In this section we shall exhibit some transcendental valued infinite products involving specific integer sequences $\{a_n\}$. We begin with the generalised Fibonacci and Lucas sequences, denoted by U_n and V_n respectively. These sequences can be defined as follows: Let (P, Q) be a relatively prime pair of integers, such that the roots α and β of $x^2 - Px + Q = 0$ are distinct, then U_n, V_n are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

It is well known that when the discriminant $\Delta = P^2 - 4Q > 0$ both $\{U_n\}$ and $\{V_n\}$ are increasing sequences of positive integers. In particular, for $(P, Q) = (1, -1)$ one has $U_n = F_n$ and $V_n = L_n$, where F_n and L_n are the Fibonacci and Lucas numbers respectively. We now establish the transcendence of the infinite products $\prod_{n=1}^{\infty}(1 + \frac{1}{U_{f(n)}})$ and $\prod_{n=1}^{\infty}(1 + \frac{1}{V_{f(n)}})$, where the index function $f : \mathbb{N} \to \mathbb{N}$ satisfies for a fixed $\lambda > 2$ the inequality $f(n+1) \geq (\lambda + 2)f(n)$.

Corollary 3.1: *Let (P, Q) be a relatively prime pair of integers with $P > |Q + 1|$ and $Q \neq 1$ and $\{U_m\}$, $\{V_m\}$ the associated generalised Fibonacci and Lucas sequences. If, for a fixed $\lambda > 2$, the function $f : \mathbb{N} \to \mathbb{N}$ has the property that $f(n+1) \geq (\lambda+2)f(n)$, then the infinite product $\prod_{n=1}^{\infty}(1 + \frac{1}{a_n})$ converges to a transcendental number, where $a_n = U_{f(n)}$ or $a_n = V_{f(n)}$.*

Remark 3.1: *We note that the restriction on Q is required as the sequence $\{U_m\}$ will contain infinitely many zero elements when $(P, Q) = (1, 1)$.*

Proof: In view of Theorem 2.1, it will suffice to demonstrate in either case that $a_{m+1}/a_m^{\lambda+1} \to \infty$ as $m \to \infty$. Clearly, from definition $\alpha = (P + \sqrt{\Delta})/2$ and $\beta = (P - \sqrt{\Delta})/2$, where $\Delta = P^2 - 4Q$. Now from assumption $\sqrt{\Delta} > \sqrt{(Q+1)^2 - 4Q} = |Q - 1| > 1$ and so

$$|\beta| = \left|\frac{P - \sqrt{\Delta}}{2}\right| = \frac{|2Q|}{P + \sqrt{\Delta}} < \frac{|2Q|}{|Q+1| + |Q-1|} = 1\ ,$$

noting here that the right hand equality holds for all $Q \in R$ with $|Q| \geq 1$. Consequently, $|\alpha| = |Q|/|\beta| > |Q| \geq 1$ and $|\beta/\alpha| < 1$. Now, in the case when $a_m = U_{f(m)}$, observe

$$\frac{a_{m+1}}{a_m^{\lambda+1}} = \alpha^{f(m+1)-(\lambda+1)f(m)}(\sqrt{\Delta})^{\lambda}\frac{(1-(\beta/\alpha)^{f(m+1)})}{(1-(\beta/\alpha)^{f(m)})^{\lambda+1}} \sim \alpha^{f(m+1)-(\lambda+1)f(m)}(\sqrt{\Delta})^{\lambda}\ ,$$

as $m \to \infty$. While in the latter case

$$\frac{a_{m+1}}{a_m^{\lambda+1}} = \alpha^{f(m+1)-(\lambda+1)f(m)}\frac{(1+(\beta/\alpha)^{f(m+1)})}{(1+(\beta/\alpha)^{f(m)})^{\lambda+1}} \sim \alpha^{f(m+1)-(\lambda+1)f(m)}\ ,$$

as $m \to \infty$. However as $f(m+1) - (\lambda+1)f(m) \geq f(m)$ and $\alpha > 1$ one has $\alpha^{f(m)} \to \infty$ as $m \to \infty$, hence the condition of Theorem 2.1 is satisfied. □

As an example, in the case $(P, Q) = (1, -1)$, we have for $\lambda = 3$ and $f(n+1) = 5f(n)$, with $f(1) = 5$, the infinite products $\prod_{n=1}^{\infty}(1 + \frac{1}{F_{5^n}})$ and $\prod_{n=1}^{\infty}(1 + \frac{1}{L_{5^n}})$ converge to transcendental numbers. In [2, p. 86], it was noted that the transcendence of the series $\sum_{n=1}^{\infty} a^{b^n}$, for integers $a \geq 2$ and $b \geq 3$, could be deduced as an application of a result of Schmidt. We now demonstrate, as a special case of the following corollary, the transcendence of those infinite products involving sequence terms of the form $a_n = a^{b^n}$ for integers $a \geq 2$ and $b \geq 5$.

Corollary 3.2: *If, for a fixed $\lambda > 2$, the function $f : \mathbb{N} \to \mathbb{N}$ has the property that $f(n+1) \geq (\lambda+2)f(n)$, then the infinite product $\prod_{n=1}^{\infty}(1 + \frac{1}{a_n})$ converges to a transcendental number, where $a_n = a^{f(n)}$ for any integer $a \geq 2$.*

Proof: Again it will suffice to demonstrate that $a_{n+1}/a_n^{\lambda+1} \to \infty$ as $n \to \infty$. Now by assumption

$$\frac{a_{n+1}}{a_n^{\lambda+1}} = a^{f(n+1)-(\lambda+1)f(n)} \geq a^{f(n)}.$$

But as $a > 1$ one has $a^{f(n)} \to \infty$ as $n \to \infty$. □

Taking $\lambda = 3$ and $f(n) = b^n$, for integer $b \geq 5$, one finds that $f(n+1) \geq 5f(n)$ and so the infinite product $\prod_{n=1}^{\infty}(1 + \frac{1}{a^{b^n}})$ converges to a transcendental number,

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