COMBINATORIAL IDENTITIES FOR THE PADOVAN NUMBERS

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Abstract. We interpret the Padovan numbers combinatorially by having them count the number of tilings of an n-strip using dominoes and triominoes. Using this interpretation, we develop a collection of identities satisfied by the sequence of Padovan numbers.

1. Introduction

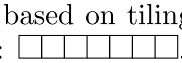
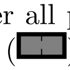
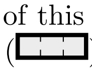
For as long as people have studied recursively defined sequences, they have attempted to find additional identities satisfied by the terms of these sequences. This includes famous sequences such as the Fibonacci sequence, the Lucas sequence, and the Catalan sequence, as well as many lesser known sequences. We follow in their footsteps and develop identities for one such lesser known sequence, the sequence of the Padovan numbers.

People have studied the sequence of Padovan numbers for centuries, but Padovan [6] first introduced it formally and then Stewart [9] gave them the name Padovan numbers. The combinatorial approach we take has been known for decades – for example, DeTemple and Webb [3] reference the connection between the Padovan numbers and tilings with dominoes and triominoes in their Appendix B and Shannon, et al. [7] hint at this interpretation. However, there does not appear to be any literature using this interpretation to derive identities for the terms of the Padovan numbers. It seems to be more common [2, 4, 8] to use matrix methods to derive identities for this sequence.

For our purposes, it will be most convenient to define the Padovan numbers as follows:

$$P_0 = P_1 = P_2 = 1;\ P_n = P_{n-2} + P_{n-3} \text{ for } n \geq 3.$$

Using this recursive definition, we can extend the Padovan numbers to negative indices. Although it is possible to find P_{-n} for any $n \geq 0$, the only additional term we will need to consider is $P_{-1} = 0$. This is known as Sequence A134816 in the Online Encyclopedia of Integer Sequences [5].

To have a combinatorial interpretation of the Padovan numbers, we need to show that they count something, i.e., that they represent the size of a set of objects. The set of objects we consider is based on tilings of an *n-strip* of squares – a row of n squares. For example, here is a 7-strip: . Given an n-strip, we will consider all possible tilings of this strip, where there are two different types of tiles, either a domino () or a triomino (). We let $\mathcal{T}_n$ denote the collection of all such tilings for an n-strip. For example, $\mathcal{T}_7$ is given below:

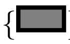
$\mathcal{T}_7 = \{$  $\}$.

Finally, we set $p_n = |\mathcal{T}_n|$ for $n \geq 1$.

We first show that $\{p_n\}_{n\geq 1}$ is the sequence of the Padovan numbers with its index shifted by two.

Theorem 1.1. *For $n \geq 1$, $p_n = P_{n-2}$.*

Proof. We need to show that p_n satisfies the same recursive definition and initial conditions as P_{n-2}. We first consider $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$.

$\mathcal{T}_1 = \emptyset$, $\mathcal{T}_2 = \{$$\}$, $\mathcal{T}_3 = \{$$\}$, $\mathcal{T}_4 = \{$$\}$

Given these sets, it is clear that $p_1 = 0 = P_{-1}$, $p_2 = 1 = P_0$, $p_3 = 1 = P_1$, and $p_4 = 1 = P_2$. Thus, $\{p_n\}_{n\geq 1}$ has the same initial conditions as $\{P_{n-2}\}_{n\geq 1}$.

Suppose $n \geq 3$. Split $\mathcal{T}_n$ into two subsets: Let D be the subset of n-tilings that end in a domino and T be the subset of n-tilings that end in a triomino. Thus, $\mathcal{T}_n = D \cup T$ and $|\mathcal{T}_n| = |D| + |T|$. Since $|D| = p_{n-2}$ and $|T| = p_{n-3}$, this implies that $p_n = |\mathcal{T}_n| = p_{n-2} + p_{n-3}$. Thus, p_n satisfies the same recursive formula with initial conditions as P_{n-2}. Therefore, $p_n = P_{n-2}$ for all $n \geq 1$. □

This interpretation is equivalent to the combinatorial interpretation where the Padovan numbers count the number of ordered compositions of n with 2's and 3's. We prefer this approach because, it will allow us to visualize the Padovan numbers as counting something, and this will assist in proving additional identities for the Padovan numbers.

In general, one main combinatorial approach to showing an identity is to determine the size of a collection of objects in two different ways. We did this in the previous proof by considering $\mathcal{T}_n$. By conditioning on the last element of each tiling, we were able to prove the recursive identity. For the identities in the next section, we obtain a proof of each identity by conditioning on the location of a given part of each tiling.

2. Identities by Conditioning

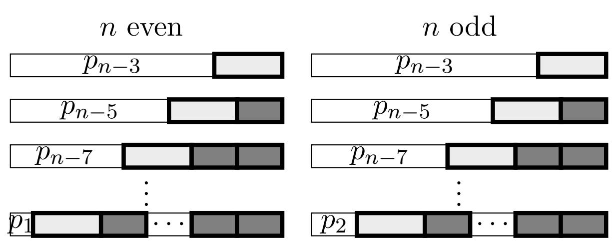
For our first identity, we will consider the collection of n-tilings and condition on the location of the last triomino that occurs in a tiling.

Identity 2.1. *Suppose $n \geq 1$. Let $k = \lfloor \frac{n}{2} \rfloor$.*

(1) *If n is even, then* $\displaystyle\sum_{m=0}^{k-2} p_{2m+1} = p_n - 1.$

(2) *If n is odd, then* $\displaystyle\sum_{m=0}^{k-1} p_{2m} = p_n - 1.$

Proof. For both cases of this identity, we condition on the position of the last triomino that occurs in the n-tiling.



In either of these cases, the diagram above is missing exactly one tiling. For even n, it is missing the tiling consisting of all dominoes, and for odd n, it is missing the tiling that begins with a triomino and has only dominoes after that. Either way, p_n is equal to the sum given plus one. □

If, instead of conditioning on the last triomino, we condition on the location of the last domino, we obtain the following identities.

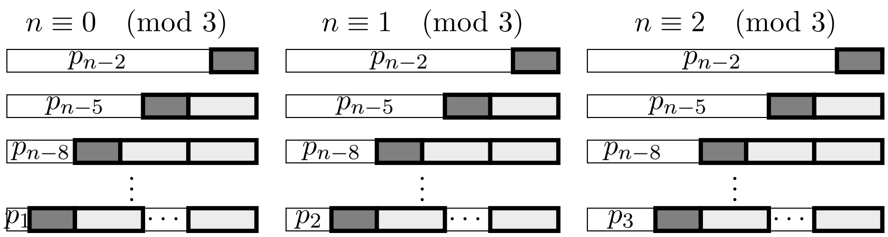
Identity 2.2. *Let $n \geq 1$ and $k = \lfloor \frac{n}{3} \rfloor$.*

(1) *If $n \equiv 0 \pmod 3$, then* $\sum_{m=0}^{k-1} p_{3m+1} = p_n - 1$.

(2) *If $n \equiv 1 \pmod 3$, then* $\sum_{m=0}^{k-1} p_{3m+2} = p_n$.

(3) *If $n \equiv 2 \pmod 3$, then* $\sum_{m=0}^{k-1} p_{3m+3} = p_n - 1$.

Proof. As stated above, we will consider all of these cases by conditioning on the location of the last domino that occurs in the n-tiling.



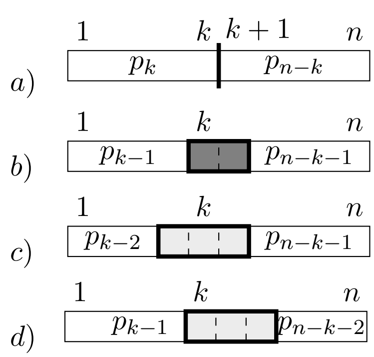
If $n \equiv 0 \pmod 3$, then the tilings listed above miss the n-tiling consisting of all triominos. For $n \equiv 1 \pmod 3$, the tilings listed above consist of all of the n-tilings. Finally, for $n \equiv 2 \pmod 3$, the tiling that begins with a domino and has triominoes as the remainder is missing. For each of these cases, the listing above proves the corresponding part of the identity. □

For the next identity, we consider whether a tiling breaks after the kth square or not ($1 \leq k \leq n$.) Parts *2.* and *3.* of this identity were proven by Sokhuma [8] using a Padovan Q-matrix.

Identity 2.3. *Suppose $n \geq 1$ and $1 \leq k \leq n$. Then,*

(1) $p_n = p_k p_{n-k} + p_{k-1} p_{n-k-1} + p_{k-2} p_{n-k-1} + p_{k-1} p_{n-k-2}$
(2) $p_n = p_k p_{n-k} + p_{k+1} p_{n-k-1} + p_{k-1} p_{n-k-2}$
(3) $p_n = p_k p_{n-k} + p_{k-1} p_{n-k+1} + p_{k-2} p_{n-k-1}$

Proof. We consider the possibilities of whether an n-tiling is breakable at tile k or not.



By considering the four cases separately, we obtain part *1.* of the identity. If we group the tilings from parts *b*) and *c*) together, we obtain part *2.* of the identity. Finally, if we group the tilings from parts *b*) and *d*) together, we obtain part *3.* of the identity.

□

We obtain one additional identity by considering the different possibilities for the first tiles in an n-tiling.

Identity 2.4. *Let $k \geq 1$ and $n \geq 3k$. Then,*

$$p_n = \sum_{i=0}^{k} \binom{k}{i} p_{n-(2k+i)}.$$

Proof. Because $n \geq 3k$, there must be at least k tiles in any n-tiling. Additionally, any combination of dominoes and triominoes are possible as the first k tiles. Therefore, there are p_{n-2k} n-tilings that begin with k dominoes. There are $\binom{k}{1}$ ways to start an n-tiling that begins with $k-1$ dominoes and one triomino. Then, there are p_{n-2k-1} ways to finish the tiling. Thus, there are a total of $\binom{k}{1} \cdot p_{n-(2k+1)}$ n-tilings that begin with $k-1$ dominoes and one triomino. In general, there are $\binom{k}{i} \cdot p_{n-(2k+i)}$ n-tilings that begin with $k-i$ dominoes and i triominoes. Because this covers all the possible n-tilings, this proves the identity. □

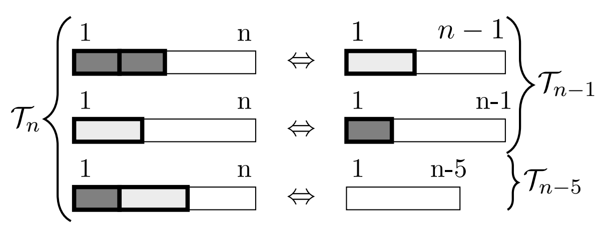
3. Identities Using Bijections

For the identities in this section, we prove them by exhibiting a bijection between two sets of tilings. This bijection then shows that the two sets have the same size, which proves the identity. The benefit of this style of argument is that each proof consists of showing a mapping and then proving that it is a bijection. For the maps we consider, proving each is a bijection is trivial.

We begin with an identity that is sometimes used to define the Padovan numbers.

Identity 3.1. *For $n \geq 5$, $p_n = p_{n-1} + p_{n-5}$.*

Proof. We build a bijection between $\mathcal{T}_n$ and $\mathcal{T}_{n-1} \cup \mathcal{T}_{n-5}$. We describe the bijection graphically as follows:

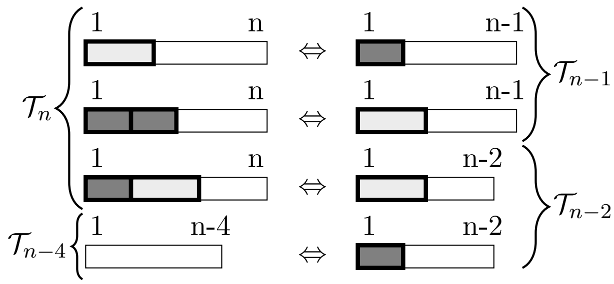


In words, any n-tiling that begins with two dominoes, is mapped to the $n-1$-tiling where the two dominoes have been replaced with a triomino. If an n-tiling begins with a triomino, then it is mapped to an $n-1$ tiling where the triomino has been replaced with a domino. Finally, any n-tiling that begins with a domino then a triomino are mapped to an $n-5$-tiling by dropping the domino and triomino. It is easy to see that this is a bijection between $\mathcal{T}_n$ and $\mathcal{T}_{n-1} \cup \mathcal{T}_{n-5}$. Thus, $p_n = p_{n-1} + p_{n-5}$. □

For the next two identities, we relate the value of p_n to the value of the previous term p_{n-1} and some other terms.

Identity 3.2. *For $n \geq 5$, $p_n = p_{n-1} + p_{n-2} - p_{n-4}$.*

Proof. To prove this identity, we will exhibit a bijection between $\mathcal{T}_n \cup \mathcal{T}_{n-4}$ and $\mathcal{T}_{n-1} \cup \mathcal{T}_{n-2}$. The bijection is shown below:

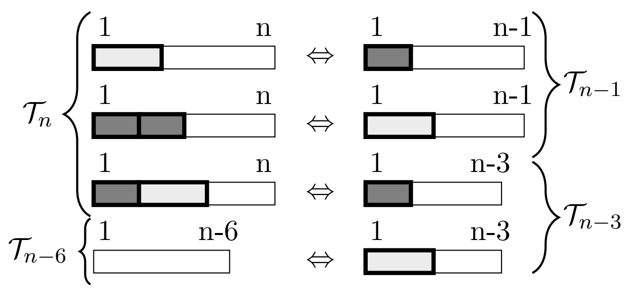


If an n-tiling begins with a triomino, the tiling is mapped to an $n-1$-tiling by replacing the triomino with a domino. If an n-tiling begins with a pair of dominoes, then it is mapped to an $n-1$-tiling by replacing them with a single triomino. Any n-tiling that begins with a domino then a triomino is mapped to an $n-2$-tiling by dropping the domino. Finally, each $n-4$-tiling is mapped to an $n-2$-tiling by adding a domino to the beginning of the tiling.

It is easy to see that this map is a bijection between $\mathcal{T}_n \cup \mathcal{T}_{n-4}$ and $\mathcal{T}_{n-1} \cup \mathcal{T}_{n-1}$, and therefore, $p_n + p_{n-4} = p_{n-1} + p_{n-2}$. □

Identity 3.3. *For $n \geq 7$, $p_n = p_{n-1} + p_{n-3} - p_{n-6}$.*

Proof. As in the previous proof, we will prove this identity by showing $p_n + p_{n-6} = p_{n-1} + p_{n-3}$. Additionally, we will give a bijection between $\mathcal{T}_n \cup \mathcal{T}_{n-6}$ and $\mathcal{T}_{n-1} \cup \mathcal{T}_{n-3}$. The bijection is shown below:



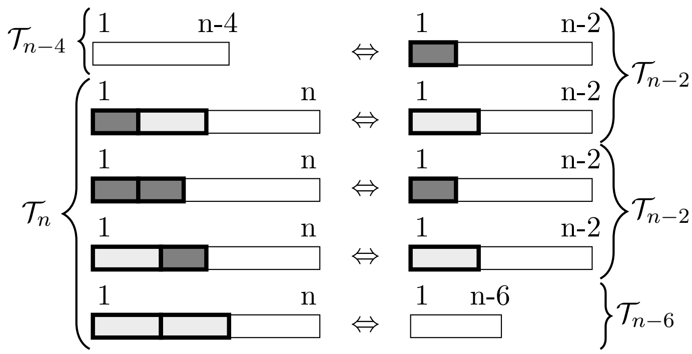
If an n-tiling begins with a triomino, then it is mapped to an $n-1$-tiling by replacing the triomino with a domino. If an n-tiling begins with a pair of dominoes, then it is mapped to an $n-1$-tiling by replacing the pair with a triomino. If an n-tiling begins with a domino then a triomino, then it is mapped to an $n-3$-tiling by removing the triomino. Finally, each $n-6$-tiling is mapped to an $n-3$-tiling by adding a triomino to the beginning of the tiling.

This map is clearly a bijection between $\mathcal{T}_n \cup \mathcal{T}_{n-6}$ and $\mathcal{T}_{n-1} \cup \mathcal{T}_{n-3}$. This proves $p_n + p_{n-6} = p_{n-1} + p_{n-3}$. □

Finally, we consider some members of the sequence of inequalities for the Padovan numbers proved by Cerda-Morales [2]. Each term of this sequence of inequalities is of the form $p_n = \sigma_i p_{n-i} + \mu_i p_{n-2i} + p_{n-3i}$ for $i = 1, 2, 3, \ldots$. After showing a few beginning patterns, Cerda-Morales determines a general formula for μ_i and σ_i by using Padovan numbers with positive and negative indices. Although we do not have a combinatorial argument for the general formula, we do have arguments for the first few cases. In each case, we build a bijection between appropriate sets of tilings.

Identity 3.4. *For $n \geq 7$, $p_n = 2p_{n-2} - p_{n-4} + p_{n-6}$.*

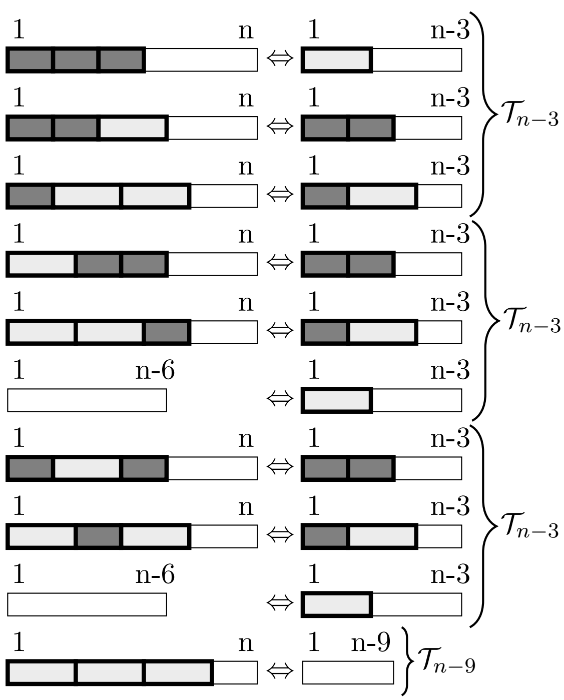
Proof. We build a bijection between $\mathcal{T}_n \cup \mathcal{T}_{n-4}$ and $\mathcal{T}_{n-2} \cup \mathcal{T}_{n-2} \cup \mathcal{T}_{n-6}$, where $\mathcal{T}_{n-2} \cup \mathcal{T}_{n-2}$ should be interpreted as every element of $\mathcal{T}_{n-2}$ appearing twice. This will show that $p_n + p_{n-4} = 2p_{n-2} + p_{n-6}$. The bijection is given in the following diagram:



Using the bijection shown above, it is clear that $|\mathcal{T}_n \cup \mathcal{T}_{n-4}| = |\mathcal{T}_{n-2} \cup \mathcal{T}_{n-2} \cup \mathcal{T}_{n-6}|$, thus proving the identity. $\square$

Identity 3.5. *For $n \geq 10$, $p_n = 3p_{n-3} - 2p_{n-6} + p_{n-9}$.*

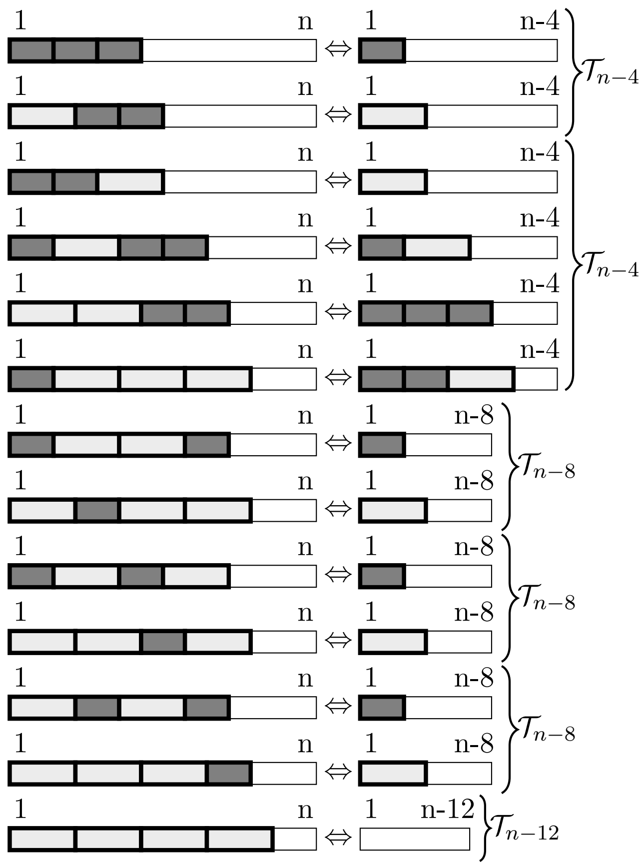
Proof. Once again we build a bijection, this time between $\mathcal{T}_n \cup \mathcal{T}_{n-6} \cup \mathcal{T}_{n-6}$ and $\mathcal{T}_{n-3} \cup \mathcal{T}_{n-3} \cup \mathcal{T}_{n-3} \cup \mathcal{T}_{n-9}$. We exhibit the bijection below.



Using this bijection, it is clear that $|\mathcal{T}_n \cup \mathcal{T}_{n-6} \cup \mathcal{T}_{n-6}| = |\mathcal{T}_{n-3} \cup \mathcal{T}_{n-3} \cup \mathcal{T}_{n-3} \cup \mathcal{T}_{n-9}|$, thus proving the identity. $\square$

Identity 3.6. *For $n \geq 13$, $p_n = 2p_{n-4} + 3p_{n-8} + p_{n-12}$.*

Proof. We build a bijection between $\mathcal{T}_n$ and $\mathcal{T}_{n-4} \cup \mathcal{T}_{n-4} \cup \mathcal{T}_{n-8} \cup \mathcal{T}_{n-8} \cup \mathcal{T}_{n-8} \cup \mathcal{T}_{n-12}$, as shown below:



Using this bijection, it is clear that $|\mathcal{T}_n| = |\mathcal{T}_{n-4} \cup \mathcal{T}_{n-4} \cup \mathcal{T}_{n-8} \cup \mathcal{T}_{n-8} \cup \mathcal{T}_{n-8} \cup \mathcal{T}_{n-12}|$, thus proving the identity. □

Notice that the description for the required bijection is growing in complexity as the value of i increases in the sequence. We hope to find an underlying pattern that can determine combinatorially the coefficients needed for each term in the sequence of identities.

Additionally, Benjamin and Quinn [1] introduced other techniques for considering more complicated identities. We hope that these techniques can be applied to the sequence of Padovan numbers as well to generate combinatorial proofs for additional identities.

References

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