THE GOLDEN RATIO: COMPUTATIONAL CONSIDERATIONS

Dmitri Thoro
San Jose State College, San Jose, California

1. INTRODUCTION

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel"-- so wrote Kepler (1571-1630)[1].

The famous <u>golden section</u> involves the division of a given line segment into mean and extreme ratio, i.e., into two parts a and b, such that $a/b = b/(a + b)$, $a < b$. Setting $x = b/a$ we have $x^2 - x - 1 = 0$. Let us designate the positive root of this equation by ϕ (the <u>golden ratio</u>). Thus

$$\phi^2 - \phi - 1 = 0 \quad . \tag{1}$$

Since the roots of (1) are $\phi = (1 + \sqrt{5})/2$ and $-1/\phi = (1 - \sqrt{5})/2$ we may write Binet's formula [2] for the nth Fibonacci number in the form

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \tag{2}$$

2. POWERS OF THE GOLDEN RATIO

Returning to (1), let us "solve for ϕ^2" by writing

$$\phi^2 = \phi + 1 \quad . \tag{3}$$

Multiplying both members by ϕ, we get $\phi^3 = \phi^2 + \phi = (\phi + 1) + \phi = 2\phi + 1$. Proceeding in a similar fashion we can write all of

$$\phi^3 = 2\phi + 1 \ ,$$
$$\phi^4 = 3\phi + 2 \ ,$$
$$\phi^5 = 5\phi + 3 \ .$$

This pattern suggests

(4) $$\phi^n = F_n \phi + F_{n-1}, \quad n = 1, 2, 3, \ldots .$$

To prove (4) by mathematical induction, we note that it is true for n = 1 and n = 2 (since $F_0 = 0$ by definition). Assume that $\phi^k = F_k\phi + F_{k-1}$. Then

$$\phi^{k+1} = F_k\phi^2 + F_{k-1}\phi = F_k(\phi + 1) + F_{k-1}\phi$$
$$= (F_k + F_{k-1})\phi + F_k = F_{k+1}\phi + F_k ,$$

which completes the proof.

The computational advantage of (4) over expansion of $\left(\frac{1 + \sqrt{5}}{2}\right)^n$ by the binomial theorem is striking.

Dividing both members of (3) by ϕ, we obtain

(5) $$\frac{1}{\phi} = \phi - 1 .$$

Thus $1/\phi^2 = 1 - 1/\phi = 1 - (\phi - 1) = -(\phi - 2)$. Using this result and (5), $1/\phi^3 = 2/\phi - 1 = 2(\phi - 1) - 1 = 2\phi - 3$. Proceeding in a similar fashion, one may write all of the following:

$$\frac{1}{\phi^2} = -(\phi - 2) ,$$

$$\frac{1}{\phi^3} = 2\phi - 3 ,$$

$$\frac{1}{\phi^4} = -(3\phi - 5) .$$

Via induction, the reader may provide a painless proof of

(6) $$\phi^{-n} = (-1)^{n+1}(F_n\phi - F_{n+1}) , \quad n = 1, 2, 3, \ldots$$

3. A LIMIT OF FIBONACCI RATIOS

If we "solve" $x^2 - x - 1 = 0$ for x by writing $x = 1 + 1/x$ and then consider the related recursion relation

(7) $$x_1 = 1 , \qquad x_{n+1} = 1 + \frac{1}{x_n} ,$$

Fibonacci numbers start popping out! We immediately deduce $x_2 = 1 + 1/x_1 = 1 + 1/1 = 2/1$, $x_3 = 3/2$, $x_4 = 5/3$, $x_5 = 8/5$, etc. This suggests that $x_n = F_{n+1}/F_n$.

Now suppose the sequence $x_1, x_2, x_3, \ldots$ has a limit, say L, as n tends toward infinity. Then

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = L$$

whence (7) yields $L = 1 + 1/L$ or $L = \emptyset$ since the x_i are positive. Indeed, there are many ways of proving Kepler's observation that

(8) $$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \emptyset \quad .$$

For example, from (2)

$$\frac{F_{n+1}}{F_n} = \frac{\emptyset^{n+1} - (-\emptyset)^{-n-1}}{\emptyset^n - (-\emptyset)^{-n}} = \frac{\emptyset - \dfrac{1}{(-\emptyset)^{n+1}\emptyset^n}}{1 - \dfrac{1}{(-\emptyset)^n\emptyset^n}} \longrightarrow \emptyset$$

as $n \to \infty$ since $\emptyset = (1 + \sqrt{5})/2 > 1$ implies that the fractions involving $\emptyset^n$ approach 0 as $n \to \infty$.

4. AN APPROXIMATE ERROR ANALYSIS

Just how accurate are the above approximations to the golden ratio? Let us denote the exact error at the nth iteration by

(9) $$e_n = x_n - \emptyset$$

The trick is to express e_{n+1} in terms of e_n using (7) and then to make use of the identity

(10) $$\frac{1}{1 + w} = 1 - w + w^2 - w^3 + w^4 - \ldots, \qquad w < 1 \quad .$$

(The latter may be discovered by dividing 1 by 1 + w).

Thus

$$e_{n+1} = x_{n+1} - \emptyset$$

$$= 1 + \frac{1}{x_n} - \emptyset$$

$$= 1 - \emptyset + \frac{1}{e_n + \emptyset}$$

$$= 1 - \emptyset + \frac{1}{\emptyset} \cdot \frac{1}{1 + (e_n/\emptyset)}$$

$$= 1 - \emptyset + \frac{1}{\emptyset} [1 - (e_n/\emptyset) + (e_n/\emptyset)^2 - (e_n/\emptyset)^3 + \dots]$$

$$= - \frac{e_n}{\emptyset^2} + \frac{e_n}{\emptyset^3} - \frac{e_n}{\emptyset^4} + \dots$$

since $1/\emptyset = \emptyset - 1$ by (5). However, the terms involving the higher powers of e_n are quite small in comparison with the first term. Thus, following the customary practice of neglecting high order terms, we will approximate the error at the (n + 1)st step by $e_{n+1} = -e_n\emptyset^{-2}$. Finally, we may note that

$e_2 = -e_1\emptyset^{-2}$, $e_3 = - e_2\emptyset^{-2} = +e_1\emptyset^{-4}$, $e_4 = -e_1\emptyset^{-6}$, and, in general,

(11) $$e_n = (-1)^{n+1}e_1\emptyset^{-2(n-1)} .$$

If $x_1 = 1$, then $e_1 = 1 - \emptyset = -1/\emptyset$ by (9) and (5), making (11) become

(12) $$e_n = (-1)^n \emptyset^{-2(n-1)-1} .$$

(Sections 5 and 6 of the original paper are omitted here.)

REFERENCES

1. H. S. M. Coxeter, Introduction to Geometry, John Wiley and Sons, Inc., New York, 1961, p. 160.
2. S. L. Basin and Verner E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence--Part II," Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 61-68.