A PRIMER FOR THE FIBONACCI NUMBERS

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EDITED BY

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PREFACE

The original Fibonacci Primer Series I - XIII were conceived to acquaint beginning students with certain information and techniques related to the Fibonacci numbers and related sequences. We decided to enclose these articles between two covers along with certain other articles from the *Fibonacci Quarterly*, selected and edited to round out a comprehensive offering.

It is hoped that this booklet will be useful to individuals wishing to study the properties of the Fibonacci and Lucas sequences but who do not have access to the back issues of the *Fibonacci Quarterly* containing most of the included articles.

Problems which occurred in early Elementary Problem Sections of The *Fibonacci Quarterly* are included as fillers both as a challenge to the beginning Fibonacci enthusiast and as an illustration of the wide diversity of problems related to the Fibonacci sequence.

The Editors are grateful for the help of Brother Alfred Brousseau, who proofed the entire manuscript and produced the booklet. The remaining errors are the responsibility of the Editors.

Marjorie Bicknell
Verner E. Hoggatt, Jr.
Editors

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EXPLORING FIBONACCI NUMBERS

Brother Alfred Brousseau
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What are currently known as Fibonacci numbers came into existence as part of a mathematical puzzle problem proposed by Leonardo Pisano (also known as Fibonacci) in his famous book on arithmetic, the Liber Abaci (1202). He set up the following situation for the breeding of rabbits.

Suppose that there is one pair of rabbits in an enclosure in the month of January; that these rabbits will breed another pair of rabbits in the month of February; that pairs of rabbits always breed in the second month following birth and thereafter produce one pair of rabbits monthly. What is the number of pairs of rabbits at the end of December?

To solve this problem, let us set up a table with columns as follows:

(1) Number of pairs of breeding rabbits at the beginning of the given month;

(2) Number of pairs of non-breeding rabbits at the beginning of the month;

(3) Number of pairs of rabbits bred during the month;

(4) Number of pairs of rabbits at the end of the month.

MONTH	(1)	(2)	(3)	(4)
January	1	0	1	2
February	1	1	1	3
March	2	1	2	5
April	3	2	3	8
May	5	3	5	13
June	8	5	8	21
July	13	8	13	34
August	21	13	21	55
September	34	21	34	89
October	55	34	55	144
November	89	55	89	233
December	144	89	144	377

The answer to the original question is that there are 377 pairs of rabbits at the end of December. But the curious fact that characterizes the series of numbers evolved in this way is: any one number is the sum of the two previous numbers. Furthermore, it will be observed that all four columns in the above table are formed from numbers of the same series which has since come to be known as THE Fibonacci series: 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

1. EXPLORATION

Did anybody ever find out what happened to the "Fibonacci rabbits" when they began to die? Since they have been operating with such mathematical regularity in other respects, let us assume the following as well. A pair of rabbits that is bred in February of one year breeds in April and every month thereafter including February of the following year. Then this pair of rabbits dies at the end of February.

(1) How many pairs of rabbits are there at the end of December of the second year?

(2) How many pairs of rabbits would there be at the end of n months, where n is greater than or equal to 12? (See what follows for notation.)

Assume that the original pair of rabbits dies at the end of December of the first year.

2. NAMES FOR ALL FIBONACCI NUMBERS

The inveterate Fibonacci addict tends to attribute a certain individuality to each Fibonacci number. Mention 13 and he thinks F_7; 55 and F_{10} flashes through his mind. But regardless of this psychological quirk, it is convenient to give the Fibonacci numbers identification tags and since they are infinitely numerous, these tags take the form of subscripts attached to the letter F. Thus 0 is denoted F_0; the first 1 in the series is F_1; the second 1 is F_2; 2 is F_3; 3 is F_4; 5 is F_5; etc. The following table for F_n shows a few of the Fibonacci numbers and then provides additional landmarks so that it will be convenient for each Fibonacci explorer to make up his own table.

n	F_n	n	F_n
0	0	11	89
1	1	12	144
2	1	13	233
3	2	14	377
4	3	20	6765
5	5	30	832040
6	8	40	102334155
7	13	50	12586269025
8	21	60	1548008755920
9	34	70	190392490709135
10	55	80	23416728348467685

3. SUMMATION PROBLEMS

The first question we might ask is: What is the sum of the first n terms of the series? A simple procedure for answering this question is to make up a table in which we list the Fibonacci numbers in one column and their sum up to a given point in another.

n	F_n	Sum
1	1	1
2	1	2
3	2	4
4	3	7
5	5	12
6	8	20
7	13	33
8	21	54

What does the sum look like? It is not a Fibonacci number, but if we add 1 to the sum, it is the Fibonacci number two steps ahead. Thus we could write:

$$1 + 2 + 3 + \ldots + 34 + 55\ (F_{10}) = 143 = 144 - 1 = (F_{12}) - 1,$$

where we have indicated the names of the key Fibonacci numbers in parentheses. It is convenient at this point to introduce the summation notation. The above can be written more concisely:

$$\sum_{k=1}^{10} F_k = F_{12} - 1.$$

The Greek letter Σ (sigma) means: Take the sum of quantities F_k, where k runs from 1 to 10. We shall use this notation in what follows.

It appears that the sum of any number of consecutive Fibonacci numbers starting with F_1 is found by taking the Fibonacci number two steps beyond the last one in the sum and subtracting 1. Thus if we were to add the first hundred Fibonacci numbers together we would expect to obtain for an answer $F_{102} - 1$. Can we be sure of this? Not completely, unless we have provided some form of proof. We shall begin with a numerical proof meaning a proof that uses specific numbers. The line of reasoning employed can then be readily extended to the general case.

Let us go back then to the sum of the first ten Fibonacci numbers. We have seen that this sum is $F_{12} - 1$. Now suppose that we add 89 (or F_{11}) to both sides of the equation. Then on the lefthand side we have the sum of the first eleven Fibonacci numbers and on the right we have

$$144 - 1 + 89 = F_{12} - 1 + F_{11} = 233 - 1 = F_{13} - 1$$

Thus, proceeding from the sum of the first ten Fibonacci numbers to the sum of the first eleven Fibonacci numbers, we have shown that the same type of relation must hold. Is it not evident that we could now go on from eleven to twelve; then from twelve to thirteen; etc., so that the relation must hold in general?

This is the type of reasoning that is used in the general proof by mathematical induction. We suppose that the sum of the first n Fibonacci numbers is $F_{n+2} - 1$. In symbols:

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1$$

We add F_{n+1} to both sides and obtain

$$\sum_{k=1}^{n+1} F_k = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$$

by reason of the fundamental property of Fibonacci series that the sum of any two consecutive Fibonacci numbers is the next Fibonacci number. We have now shown that if the summation holds for n, it holds also for n + 1. All that remains to be done is to go back to the beginning of the series and draw a complete conclusion. Let us suppose, as can readily be done, that the formula for the sum of the first n terms of the Fibonacci sequence holds for $n \leq 7$. Since the formula holds for seven, it holds for eight; since it holds for eight, it holds for nine; etc., etc. Thus the formula is true for all integral positive values of n.

We have seen from this example that there are two parts to our mathematical exploration. In the first we observe and thus arrive at a formula. In the second we prove that the formula is true in general.

Let us take one more example. Suppose we wish to find the sum of all the odd-numbered Fibonacci numbers. Again, we can form our table.

n	F_n	Sum
1	1	1
3	2	3
5	5	8
7	13	21
9	34	55
11	89	144

This is really too easy. We have come up with a Fibonacci number as the sum. Actually it is the very next after the last quantity added. We shall leave the proof to the explorer. However, the question of fitting the above results into notation might cause some trouble. What we need is a type of subscript that will give us just the odd numbers and no others. For the above sum to 11, we would write

$$\sum_{k=1}^{6} F_{2k-1} = F_{12}$$

It will be seen that when k is 1, 2k-1 is 1; when k is 2, 2k-1 is 3; etc., and when k is 6, 2k-1 is 11. In general, the relation for the sum of the first n odd-subscripted Fibonacci numbers would be:

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n}.$$

4. PROBLEMS FOR EXPLORATION

1. Determine the sum of the first n even-subscripted Fibonacci numbers.

2. If we take every fourth Fibonacci number and add, four series are possible:

   (a) Subscripts 1, 5, 9, 13, ...
   (b) Subscripts 2, 6, 10, 14, ...
   (c) Subscripts 3, 7, 11, 15, ...
   (d) Subscripts 4, 8, 12, 16, ...

Determine the sum of the first n terms in each of these series. Hint: Look for products or squares or near-products or near-squares of Fibonacci numbers as the result.

3. If we take every third Fibonacci number and add, three series are possible:

   (a) Subscripts 1, 4, 7, 10, ...
   (b) Subscripts 2, 5, 8, 11, ...
   (c) Subscripts 3, 6, 9, 12, ...

Find the sum of the first n terms in each of those series. Hint: Double the sum and see whether you are near a Fibonacci number.

***** ***** *****

RESEARCH PROJECT: FIBONACCI NIM

Consider a game involving two players in which initially there is a group of 100 or less objects. The first player may reduce the pile by any Fibonacci number. The second does likewise. The player who makes the last move wins the game. Can the first player always win the game?

THE FIBONACCI SEQUENCE AS IT APPEARS IN NATURE

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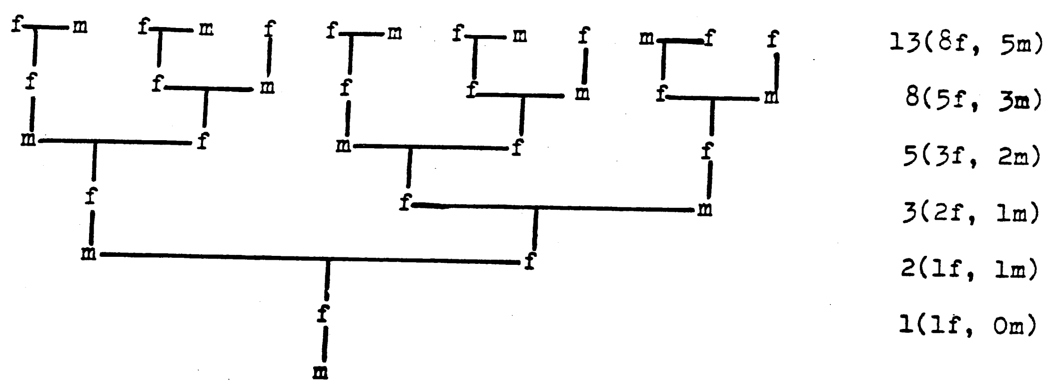
1. INTRODUCTION

The regular spiral arrangement of leaves around plant stalks has enjoyed much attention by botanists and mathematicians in their attempt to unravel the mysteries of this organic symmetry. Because of the abundance of literature on phyllotaxis no more attention will be devoted to it here. However, the Fibonacci numbers have the strange habit of appearing where least expected in other natural phenomena. The following snapshots will demonstrate this fact. (See [1] and [3] for discussions of phyllotaxis.)

2. THE GENEALOGICAL TREE OF THE MALE BEE

We shall trace the ancestral tree of the male bee backwards, keeping in mind that the male bee hatches from an unfertilized egg. The fertilized eggs hatch into females, either workers or queens.

The following diagram clearly shows that the number of ancestors in any one generation is a Fibonacci number. The symbol (m) represents a male and the symbol (f) represents a female.

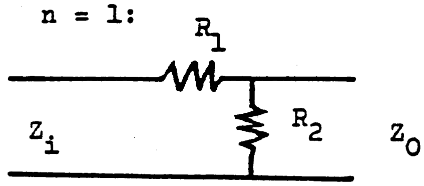


3. SIMPLE ELECTRICAL NETWORKS

Even those people interested in electrical networks cannot escape from our friend Fibonacci. Consider the following simple network of resistors known as a ladder network. This circuit consists of n L-sections in cascade and can be characterized or described by calculating the attenuation which is simply the input voltage divided by the output voltage and denoted by A, the input impedance Z_i and the output impedance Z_o. (See [4].)

Proceeding in a manner similar to mathematical induction, consider the following ladder networks.

n = 1:

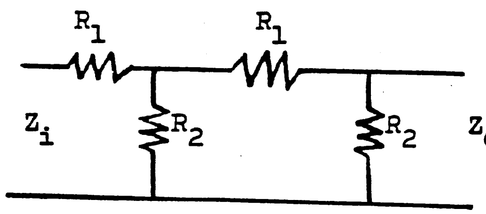


$$Z_O = R_2$$

$$Z_i = R_1 + R_2$$

$$A = R_1/R_2 + 1$$

n = 2:

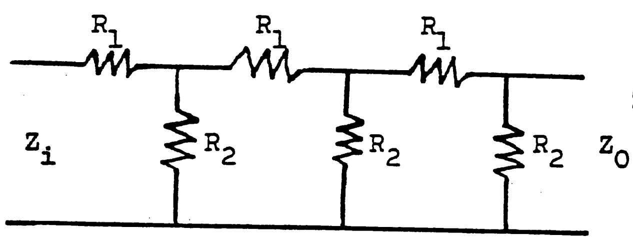


$$Z_O = \frac{R_2(R_1 + R_2)}{R_1 + 2R_2}$$

$$Z_i = \frac{R_1(R_1 + 2R_2) + R_2(R_1 + R_2)}{R_1 + 2R_2}$$

$$A = \frac{(R_1 + R_2)(R_1 + 2R_2) - R_2^2}{R_2^2}$$

n = 3:



$$Z_O = \frac{R_1R_2(R_1 + 2R_2) + R_2^2(R_1 + R_2)}{(R_1 + R_2)(R_1 + 3R_2)}$$

$$Z_i = \frac{R_1^3 + 5R_1^2R_2 + 6R_1R_2^2 + R_2^3}{R_1^2 + 4R_1R_2 + 3R_2^2}$$

$$A = \frac{R_1^3 + 5R_1^2R_2 + 6R_1R_2^2 + R_2^3}{R_2^3}$$

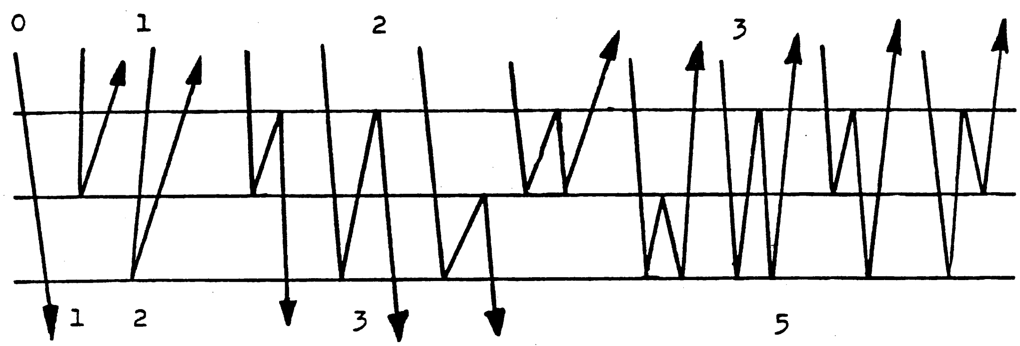
Now suppose all the resistors have the same value, namely, $R_1 = R_2 = 1$ ohm. We have by induction:

$$Z_O = \frac{F_{2n-1}}{F_{2n}}, \qquad Z_i = \frac{F_{2n+1}}{F_{2n}}, \qquad A = (F_{2n-1} + F_{2n}) = F_{2n+1}.$$

In other words, the ladder network can be analyzed by inspection; as n is allowed to increase, n = 1, 2, 3, 4, ..., the value of Z_O for n L-sections coincides with the nth term in the sequence of Fibonacci ratios, i.e., 1/1, 2/3, 5/8, 13/21, ... The value of A is given by the sum of the numerator and denominator of Z_O. The value of Z_i is also clearly related to the expression for A and Z_O.

4. SOME REFLECTIONS (Communicated to us by Leo Moser)

The reflection of light rays within two plates of glass is expressed in terms of the Fibonacci numbers, i. e., if no reflections are allowed, one ray will emerge; if one reflection is allowed, two rays will emerge; if two reflections are allowed, three rays will emerge; ..., and if n reflections are allowed, F_{n+2} rays will emerge. (See [6].)



FOR ADDITIONAL READING

1. H. S. M. Coxeter, Introduction to Geometry, John Wiley and Sons, 1961, pp. 169-172. A complete chapter on Phyllotaxis and Fibonacci numbers appears in easily digestible treatment.

2. N. N. Vorobyov, The Fibonacci Numbers, Blaisdell, New York, 1961. (Translation from the Russian by Halina Moss) This booklet discusses the elementary properties of Fibonacci numbers, their application to geometry, and their connection with the theory of continued fractions.

3. Robert Land, The Language of Mathematics, John Murray, London, 1960. Chapter XIII, pp. 215-225. A very interesting chapter including some phyllotaxis.

4. S. L. Basin, "Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory", Mathematics Magazine, March, 1963.

5. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Mathematics Enrichment Series, Houghton Mifflin Company, Boston, 1969. An introductory study of the Fibonacci numbers and their properties and relationships to algebra and geometry, as well as an entire chapter on phyllotaxis and on the Golden Section.

6. Leo Moser, Elementary Problem B-6. Solution by J. L. Brown, Jr. Fibonacci Quarterly, Vol. 1, No. 4, December, 1963, pp. 75-76.

PHYLLOTAXIS

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When Nehemiah Grew remarked in 1682 that "from the contemplation of plants, men might be invited to Mathematical Enquirys," [5] he might not have been thinking of the amazing relationship between phyllotaxis and Fibonacci numbers, but he could well have been; for the phenomenon of phyllotaxis, literally "leaf arrangement," has long been a subject of special investigation, much speculation, and even heated debate among mathematicians and botanists alike.

By right it is the botanists who deserve the credit for bringing to light the discovery that plants of every type and description seem to have their form elements, that is, their branches, leaves, flowers, or seeds, assembled and arranged according to a certain general pattern; but surely even the old Greek and Egyptian geometers could not have failed to observe the spiral nature of the architecture of plants. Many and varied and even contradictory are the theories on this fascinating phenomenon of phyllotaxis, but it would be beyond the scope of this paper to investigate them here; instead we shall simply try to describe the manifestation of it in the interval-spacing of leaves around a cylindrical stem, in the florets of the sunflower and, finally, in the scales of fir cones and pineapples.

Before we proceed to consider the actual arrangement of the form elements, however, it is interesting to note the relationship between the number of petals of many well-known flowers and the Fibonacci numbers. Two-petalled flowers are not common but enchanter's nightshade is one such example. Several members of the iris and lily families have three petals, while five-petalled flowers, including the common buttercup, some delphiniums, larkspurs and columbines, are the most common of all. Other varieties of delphiniums have eight petals, as does the lesser celandine, and in the daisy family, squalid and field senecio likewise have eight petals in the outer ring of ray florets. Thirteen-petalled flowers are quite common and include the globe flower and some double delphiniums as well as ragwart, corn marigold, mayweed, and several of the chamomiles. Many garden and wild flowers, including some heleniums and asters, chicory, doronicum, and some hawk-bits, have twenty-one petals, while thirty-four is the most common number in the daisy family and is characteristic of the field daisies, ox-eye daisies, some heleniums, gaillardias, plantains, pyrethrums, and a number of hawk-bits and hawkweeds. Some field daisies have fifty-five petals, and Michaelmas daisies often have either fifty-five or eighty-nine petals. It is difficult to trace this relationship much further, but it must be remembered that this number pattern is not necessarily followed by every plant of a species but simply seems to be characteristic of the species as a whole.

Fibonacci numbers occur in other types of patterns too. The milkwort will commonly be found to have two large sepals, three smaller sepals, five petals and eight stamens, and Frank Land [4] reports that he found a clump of alstroemerias in his garden in which one plant had two flowers growing on each of three stalks and that, where the three stalks grew out from the top of the main stem, a whorl of five leaves grew out radially; while another plant had three flowers on each of five stems with a whorl of eight leaves at the base of the flower stalks.

The Fibonacci number pattern, however, which has received the most attention is that associated with the spiral arrangement of the form elements of the plants. In its simplest manifestation it may be observed in plants and trees which have their leaves or buds or branches arranged at intervals around a cylindrical stem. If we should take a twig or branch of a tree, for instance, and choose a certain bud, then by revolving the hand spirally around the branch until we came to a bud directly above the first one counted, we would find that the number of buds per revolution as well as the number of revolutions itself are both Fibonacci numbers, consecutive or alternate ones depending on the direction of revolution, and different for various plants and trees. If the number of revolutions is m and the number of leaves or buds is n, then the leaf or bud arrangement is commonly called an <u>m/n spiral</u> or <u>m/n phyllotaxis</u>. Hence in some trees, such as the elm and basswood, where the leaves along a twig seem to occur directly opposite one another, we speak of 1/2 phyllotaxis, whereas in the beech and the hazel, where the leaves are separated by one-third of a revolution, we speak of 1/3 phyllotaxis. Likewise, the oak, the apricot, and the cherry tree exhibit 2/5 phyllotaxis, the poplar and the pear 3/8, while that of the willow and the almond is 5/13. Much investigation along these lines seems to indicate that, at least as far as leaves and blossoms are concerned, each species is characterized by its own particular phyllotaxis ratio, and that almost always, except where damage or abnormal growth has modified the arrangement, the ratios encountered are ratios of consecutive or alternate terms of the Fibonacci sequence.

When the form elements of certain plants are assembled in the form of a disk rather than along a cylindrical stem, we have a slightly different form of phyllotaxis. It is best exemplified in the head of a sunflower, which consists of a number of tightly packed florets, in reality the seeds of the flower. Very clearly the seeds can be seen to be distributed over the head in two distinct sets of spirals which radiate from the center of the head to the outermost edge in both clockwise and counterclockwise directions. These spirals, logarithmic in character, are of the same nature as those mentioned earlier in plants with cylindrical stems, but in those instances, the adjacent leaves being generally rather far apart along the stem, it is more difficult for the eye to detect the regular spiral arrangement. Here in the close-packed arrangement of the head of the sunflower, we can see the phenomenon in almost two-dimensional form. As was the case with the cylindrical-stemmed plants, the number pattern exhibited by the double sets of spirals is intimately bound up with Fibonacci numbers. The normal sunflower head, which is about five or six inches in diameter, will generally have thirty-four spirals winding in one direction and fifty-five in the other. Smaller sunflower heads will commonly exhibit twenty-one spirals in one direction and thirty-four in the other or a combination of thirteen and twenty-one. Abnormally large heads have been developed with a combination of fifty-five and eighty-nine spirals and even a gigantic one at Oxford with eighty-nine spirals in one direction and a hundred and forty-four in the other. In each instance the combination of clockwise and counterclockwise spirals consists of successive terms of the Fibonacci sequence.

One other interesting manifestation of phyllotaxis and its relation to the Fibonacci numbers is observed in the scales of fir cones and pineapples. These scales are really modified leaves crowded together on relatively short stems, and so, in a sense, we have a combination of the other two forms of the phenomenon; namely, a short conical or cylindrical stem and a close-packed arrangement which easily enables us to observe that the scales are arranged in ascending spirals or helical whorls called parastichies. In the fir cone, as in the sunflower head, two sets of spirals are obvious, and hence in many cones, such as those of the Norway spruce or the American larch, five rows of scales may be seen to be winding steeply up the cone in one direction while three rows wind less steeply the other way; in the common larch we usually find eight rows winding in one direction and five in the other, and frequently the two arrangements cross each other on different parts of the cone. In the pineapple, on the other hand, three distinct groups of parastichies may be observed; five rows winding slowly up the pineapple in one direction, eight rows ascending more steeply in the opposite direction, and, finally, thirteen rows winding upwards very steeply in the first direction. The fact that pineapple scales are of irregular hexagonal shapes accounts for the three sets of whorls, for three distinct sets of scales can consequently be contiguous, and hence, constitute a different formation. Moreover, Fibonacci numbers manifest themselves in still another way in connection with the scales of the pineapple. If the scales should be numbered successively around the fruit from the bottom to the top, the numbering being based on the corresponding lateral distances of the scales along the axis of the pineapple, we would find that each of the three observable groups of parastichies winds through numbers which constitute arithmetic sequences with common differences of 5, 8, and 13, the same three successive Fibonacci numbers observed above. Thus a spiral of the first group would ascend through the numbers 0, 5, 10, ...; one of the second group through the numbers 0, 8, 16, ...; and, finally, a spiral of the third group would wind steeply up the pineapple through the numbers 0, 13, 26, ...

In all these many and varied ways, then, in the number of petals possessed by different species of flowering plants, in the interspacing of leaves or buds around a cylindrical stem, in the double spirals of the close-packed florets of sunflowers, and in the ascending spirals or parastichies of the fir cone and the pineapple, we have encountered number patterns which again and again involve particular terms of the Fibonacci sequence. These Fibonacci number patterns or combinations occur so continually in the varied manifestations of phyllotaxis that we often hear of the "law" of phyllotaxis. However, it must be admitted that not all four-petalled flowers are so rare as the four-leaf clover is reputed to be, and that other combinations also occur, notably in those species exhibiting symmetrical arrangements. Moreover, in the cases of fir cones and some large sunflowers, where the spiral pattern can be verified more carefully, deviations, sometimes even large ones, from the Fibonacci pattern have been found. If this is at all disturbing to the modern botanist, it is not at all so to the Fibonacci devotee, for whom the whole phenomenon, if not a "law", is at least, in the words of H. S. M. Coxeter [1], a fascinatingly prevalent tendency!

REFERENCES

1. H. S. M. Coxeter, Introduction to Geometry, New York: John Wiley and Sons, 1961.
2. J. Hambridge, Practical Applications of Dynamic Symmetry, New Haven: Yale Press, 1932.
3. J. A. H. Hunter and J. S. Madachy, Mathematical Diversions, New Jersey: Van Nostrand Co., 1963.
4. Frank Land, The Language of Mathematics, London: John Murray, 1960.
5. D'Arcy W. Thompson, On Growth and Form, New York: McMillan Co., 1944.

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V. E. Hoggatt, Jr., raised a large sunflower in Santa Clara, California, which exhibited Lucas numbers 76 and 123 in its spiral arrangements. The Lucas numbers 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ..., which have the same rule of formation as the Fibonacci numbers, seem to occur when Fibonacci numbers do not in phyllotaxis, at least frequently enough to be interesting.

A PRIMER ON THE FIBONACCI SEQUENCE: PART I

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SIMPLE PROPERTIES OF THE FIBONACCI SEQUENCE AND MATHEMATICAL INDUCTION

The proofs of Fibonacci identities serve as very suitable examples of certain techniques encountered in a first course in college algebra. With this in mind, it is the intention of this series of articles to introduce the beginner to the Fibonacci sequence and a few techniques in proving some number theoretic identities as well as furnishing examples of well-known methods of proof such as mathematical induction. The collection of proofs that will be given in this series may serve as a source of elementary examples for classroom use.

1. SOME SIMPLE PROPERTIES OF THE FIBONACCI SEQUENCE

By observation of the sequence 1, 1, 2, 3, 5, 8, ..., it is easily seen that each term is the sum of the two preceding terms. In mathematical language, we define this sequence by

(A) $$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n$$

for all integers n. The first few Fibonacci numbers are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

The Lucas numbers L_n satisfy the same recurrence relation but have different starting values, namely,

(B) $$L_1 = 1, \quad L_2 = 3, \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n$$

for all integers n. The first few Lucas numbers are:

1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, ...

The following are some simple formulas which are called Fibonacci Number Identities or Lucas Number Identities for $n \geq 1$.

(1) $$F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1$$

(2) $$L_1 + L_2 + L_3 + \dots + L_n = L_{n+2} - 3$$

(3) $$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

(4) $$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}$$

(5) $$L_n = F_{n+1} + F_{n-1}$$

(6) $$5 \cdot F_n = L_{n+1} + L_{n-1}$$

(7) $$F_{2n+1} = F_{n+1}^2 + F_n^2$$

(8) $$F_{2n} = F_{n+1}^2 - F_{n-1}^2$$

(9) $$F_{2n} = F_n L_n$$

(10) $$F_{n+p+1} = F_{n+1}F_{p+1} + F_n F_p$$

(11) $$F_1^2 + F_2^2 + F_3^2 + \dots + F_n^2 = F_n F_{n+1}$$

(12) $$L_n^2 - 5 \cdot F_n^2 = 4(-1)^n$$

(13) $$F_{-n} = (-1)^{n+1} F_n$$

(14) $$L_{-n} = (-1)^n L_n$$

2. MATHEMATICAL INDUCTION

Any proofs of the foregoing identities ultimately depend upon the postulate of complete mathematical induction.

First one has a formula involving an integer n. For some values of n the formula has been seen to be true. This may be one, two, or, say, twenty times. Now the excitement sets in. Is it true for all positive n? One may prove this by appealing to mathematical induction, whose three phases are:

A. Statement P(1) is true by trial. (If you can't find a first true case...why do you think it's true for any n, let alone all n? Here you need some true cases to start with.) An example of a statement P(n) is $1 + 2 + 3 + \dots + n = n(n + 1)/2$. It is simple to see that P(1) is true; that is, that $1 = 1(1 + 1)/2$.

B. The truth of statement P(k) logically implies the truth of P(k+1). In other words, <u>if</u> P(k) is true, <u>then</u> P(k+1) is true. This step is commonly referred to as the inductive transition. The actual method used to prove this implication may vary from simple algebra to very profound theory.

C. The statement that 'The proof is complete by mathematical induction'.

As an example, let us prove identity (1). Recall from (A) that $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Statement P(n) is

P(n): $$F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1.$$

A. P(1) is true, since $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, so that $F_1 = 2 - 1 = F_3 - 1$.

B. Assume that P(k) is true; that is,

P(k): $$F_1 + F_2 + F_3 + \dots + F_k = F_{k+2} - 1.$$

From this we will show that the truth of P(k) <u>demands</u> the truth of P(k+1), which is

P(k+1): $$(F_1 + F_2 + F_3 + \dots + F_k) + F_{k+1} = F_{k+3} - 1.$$

Since we assume P(k) is true, we may therefore assume that, in P(k+1), we may replace $(F_1 + F_2 + F_3 + \dots + F_k)$ by $(F_{k+2} - 1)$. That is, P(k+1) may be rewritten as

$$(F_{k+2} - 1) + F_{k+1} = (F_{k+2} + F_{k+1}) - 1 = F_{k+3} - 1.$$

This is now clearly true from (A), which for n = k + 1 becomes $F_{k+3} = F_{k+2} + F_{k+1}$.

C. The proof is complete by mathematical induction.

3. A BIT OF THEORY (Cramer's Rule)

Next we need a bit of determinant theory. Given a second order determinant, by definition

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The following theorem can be proved using the definition and simple algebra.

THEOREM: For any real numbers x and y,

$$\begin{vmatrix} ax + by & b \\ cx + dy & d \end{vmatrix} = \begin{vmatrix} ax & b \\ cx & d \end{vmatrix} = xD$$

Suppose a system of two simultaneous equations possesses a unique solution (x_0, y_0); that is,

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

is satisfied if and only if $x = x_0$ and $y = y_0$. This is specified by saying that

$$\text{(C)} \qquad \begin{cases} ax_0 + by_0 = e \\ cx_0 + dy_0 = f \end{cases}$$

are true statements. From our definition of determinant and the theorem, for $x = x_0$ and $y = y_0$, we may write

$$x_0 D = \begin{vmatrix} ax_0 + by_0 & b \\ cx_0 + dy_0 & d \end{vmatrix} = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

where we used (C) to rewrite the determinant. Thus,

$$x_0 = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y_0 = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad D \neq 0,$$

which is Cramer's Rule.

Algebraically, we see that we must take $D \neq 0$. Geometrically, $D = 0$ if the graphs of the two linear equations are distinct parallel lines (inconsistent equations) or if the graphs are the same line (redundant equations). If the graphs (lines) are not parallel or coincident, then the common point of intersection is (x_0, y_0).

4. A CLEVER DEVICE IN ACHIEVING AN INDUCTIVE TRANSITION

Now, to apply our theory in a proof by mathematical induction, suppose we write two examples of definition (A), for $n = k$ and for $n = k - 1$, obtaining $F_{k+2} = F_{k+1} + F_k$ and $F_{k+1} = F_k + F_{k-1}$, and then let us try to solve the pair of simultaneous linear equations,

(D) $$\begin{cases} F_{k+2} = xF_{k+1} + yF_k \\ F_{k+1} = xF_k + yF_{k-1}. \end{cases}$$

This is silly because we know that the answer is $x_0 = 1$ and $y_0 = 1$, but using Cramer's Rule we note:

(E) $$y_0 = 1 = \frac{\begin{vmatrix} F_{k+1} & F_{k+2} \\ F_k & F_{k+1} \end{vmatrix}}{\begin{vmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{vmatrix}} = \frac{F_{k+1}^2 - F_kF_{k+2}}{F_{k+1}F_{k-1} - F_k^2}$$

Let us now use Mathematical Induction to prove Identity (3) which is

P(n): $$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

If we note that $F_0 = 0$ is valid, then P(1): $F_2F_0 - F_1^2 = (-1)^1 = -1$ is true, and part A is done.

Suppose P(k) is true. From (E) and the induction hypothesis,

$$1 = \frac{F_{k+1}^2 - F_kF_{k+2}}{F_{k+1}F_{k-1} - F_k^2} = \frac{F_{k+1}^2 - F_kF_{k+2}}{(-1)^k}$$

so that P(k+1): $F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}$ is indeed true! Thus part B is done. The proof is complete by mathematical induction, and part C is done.

All of the other identities given in this article can be proved by mathematical induction. To test your understanding, you should prove several of them.

* * * * *

Problem B-1 (Proposed by I. D. Ruggles) Show that the sum of twenty consecutive Fibonacci numbers is divisible by F_{10}.

B-2 (Proposed by Verner E. Hoggatt, Jr.) Show that

$$u_{n+1} + u_{n+2} + \cdots + u_{n+10} = 11u_{n+7}$$

holds for generalized Fibonacci numbers such that $u_{n+2} = u_{n+1} + u_n$, where $u_1 = p$ and $u_2 = q$.

B-3 (Proposed by J. E. Householder) Show that F_{n+24} is congruent to F_n (modulo 9), where F_n is the nth Fibonacci number.

A PRIMER ON THE FIBONACCI SEQUENCE: PART II

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A MATRIX WHICH GENERATES FIBONACCI IDENTITIES

The proofs of existing Fibonacci identities and the discovery of new identities can be greatly simplified if matrix algebra and a particular 2 x 2 matrix are introduced. The matrix approach to the study of recurring sequences has been used for some time [1] and the Q matrix appeared in a thesis by C. H. King [2]. We first present the basic tools of matrix algebra.

1. THE ALGEBRA OF (TWO-BY-TWO) MATRICES

The two-by-two matrix A is an array of four elements a, b, c, d:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The zero matrix Z and the identity matrix I are defined as

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix C, which is the <u>matrix sum</u> of two matrices A and B, is

$$C = A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

The matrix P, which is the <u>matrix product</u> of two matrices A and B, is

$$P = AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

The <u>determinant D(A)</u> of matrix A is

$$D(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Two matrices are equal if and only if the corresponding elements are equal. That is, for the matrices A and B above, A = B if and only if a = e, b = f, c = g, and d = h.

The proof of the following simple theorem is left as an exercise in algebra.

THEOREM: The determinant D(P) of the product P = AB of two matrices A and B is the product of the determinants D(A) and D(B). That is, D(P) = D(AB) = D(A)·D(B).

2. THE Q MATRIX

The Q matrix and the determinant of Q are

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D(Q) = -1.$$

If we designate $Q^0 = I$, the identity matrix, then $Q = Q^1 = Q^0Q = IQ = QI = QQ^0$.

<u>Definition</u>: $Q^{n+1} = Q^nQ^1$, an inductive definition where $Q^1 = Q$. This provides the law of exponents for matrices.

It is easily proved by mathematical induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

where F_n is the nth Fibonacci number, and the determinant of Q^n is $(-1)^n$.

3. MORE PROOFS

We may now prove several of the identities very nicely. Let us prove identity (3) from Part I:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

Proof: Evaluate the determinant of Q^n in two ways. $D(Q^n) = D^n(Q) = (-1)^n$, but by definition of determinant, $D(Q^n) = F_{n+1}F_{n-1} - F_n^2$.

Now let us prove identity (7), $F_{2n+1} = F_{n+1}^2 + F_n^2$. Since $Q^{n+1}Q^n = Q^{2n+1}$,

$$Q^nQ^{n+1} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}\begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} = \begin{pmatrix} F_{n+1}F_{n+2} + F_nF_{n+1} & F_{n+1}^2 + F_n^2 \\ F_nF_{n+2} + F_{n-1}F_{n+1} & F_nF_{n+1} + F_{n-1}F_n \end{pmatrix}$$

can also be written as

$$Q^{2n+1} = \begin{pmatrix} F_{2n+2} & F_{2n+1} \\ F_{2n+1} & F_{2n} \end{pmatrix}$$

Since these two matrices are equal, we may equate corresponding elements, so that

$$F_{2n+2} = F_{n+1}F_{n+2} + F_nF_{n+1}$$ (Upper Left)

(7) $$F_{2n+1} = F_{n+1}^2 + F_n^2$$ (Upper Right)

$$F_{2n+1} = F_n F_{n+2} + F_{n-1}F_{n+1}$$ (Lower Left)

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n$$ (Lower Right)

establishing identity (7) as well as two others with some simple algebra. If we accept identity (5), $L_n = F_{n+1} + F_{n-1}$, then

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n = F_n(F_{n+1} + F_{n-1}) = F_nL_n,$$

which gives identity (9). From $F_{k+2} = F_{k+1} + F_k$, for $k = n - 1$, one can write $F_n = F_{n+1} - F_{n-1}$, so that we also have identity (8):

(8) $$F_{2n} = F_n(F_{n+1} + F_{n-1}) = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) = F_{n+1}^2 - F_{n-1}^2$$

It is a simple task to verify that $Q^2 = Q + I$, leading to

$$Q^{n+2} = Q^{n+1} + Q^n \qquad \text{and} \qquad Q^n = Q\cdot F_n + I\cdot F_{n-1},$$

where F_n is the nth Fibonacci number and the multiplication of matrix A, by a number q, is defined by

$$qA = q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aq & bq \\ cq & dq \end{pmatrix}$$

4. GENERATION OF FIBONACCI NUMBERS BY LONG DIVISION

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^2 + \dots + F_nx^{n-1} + \dots$$

In the process of long division below

$$1 - x - x^2 \overline{)\ 1 \qquad\qquad}$$

there is no ending. As far as you care to go the process will yield Fibonacci numbers as the coefficients.

5. F_n AS A FUNCTION OF ITS SUBSCRIPT

It is not difficult to show by mathematical induction that

$$P(n): \qquad F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

This can be derived in many ways. P(1) and P(2) are clearly true. From $F_k = F_{k-1} + F_{k-2}$ and the inductive assumption that P(k-2) and P(k-1) are true,

$$P(k-2): \qquad F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right\}$$

$$P(k-1): \qquad F_{k-1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right\}$$

Adding, after a simple algebra step, we get

$$F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\left(\frac{1+\sqrt{5}}{2} + 1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\left(\frac{1-\sqrt{5}}{2} + 1\right)\right\}$$

Observing that

$$\frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 \quad \text{and} \quad \frac{1-\sqrt{5}}{2} + 1 = \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2$$

it follows simply that if P(k-2) and P(k-1) are true, then for n = k,

$$P(k): \qquad F_k = F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right\}$$

making the proof complete by mathematical induction.

Similarly, it may be shown that

$$L_n = F_{n+1} + F_{n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The formulas given ıor F_n and L_n in terms of the subscripts are called the <u>Binét forms</u> for F_n and L_n.

Now let us use the Binét forms for F_n and L_n to prove identity (9), $F_{2n} = F_nL_n$, a second way:

$$F_{2n} = \frac{1}{\sqrt{5}}\left\{\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\right]^2 - \left[\left(\frac{1-\sqrt{5}}{2}\right)^n\right]^2\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} = F_n L_n.$$

6. MORE IDENTITIES

(15) $$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

(16) $$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

(17) $$F_1^3 + F_2^3 + F_3^3 + \dots + F_n^3 = \frac{F_{3n+2} + (-1)^{n+1}6\cdot F_{n-1} + 5}{10}$$

(18) $$1\cdot F_1 + 2\cdot F_2 + 3\cdot F_3 + \dots + n\cdot F_n = (n+1)F_{n+2} - F_{n+4} + 2$$

(19) $$F_2 + F_4 + \dots + F_{2n} = F_{2n+1} - 1$$

(20) $$F_1F_2 + F_2F_3 + F_3F_4 + \dots + F_{n-1}F_n = \frac{1}{2}(F_{2n-1} + F_nF_{n-1} - 1)$$

(21) $$\sum_{i=0}^{n}\binom{n}{i} F_{n-i} = F_{2n}, \text{ where } \binom{n}{i} = \frac{n!}{(n-i)!i!}, \; m! = 1\cdot 2\cdot 3\cdot \dots \cdot m.$$

(22) $$F_{3n+3} = F_{n+1}^3 + F_{n+2}^3 - F_n^3$$

(23) $$F_nF_m - F_{n-k}F_{m+k} = (-1)^{n-k}F_kF_{m+k-n}$$

(24) $$F_{n+4}^3 = 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3$$

REFERENCES

1. J. S. Frame, "Continued Fractions and Matrices", American Mathematical Monthly, February, 1949, p. 98.
2. Charles H. King, "Some Properties of the Fibonacci Numbers", Unpublished Master's Thesis, San Jose State College, June, 1960.

FIBONACCI MATRICES AND LAMBDA FUNCTIONS

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When we speak of a Fibonacci matrix, we shall have in mind matrices which contain members of the Fibonacci sequence as elements. An example of a Fibonacci matrix is the Q matrix as defined by King in [1], where

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The determinant of Q is -1, written det Q = -1. From a theorem in matrix theory, $\det Q^n = (\det Q)^n = (-1)^n$. By mathematical induction, it can be shown that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

so that we have the familiar Fibonacci identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ by finding $\det Q^n$.

The lambda function of a matrix was studied extensively in [2] by Fenton S. Stancliff, who was a professional musician. Stancliff defined the lambda function $\lambda(M)$ of a matrix M as the change in the value of the determinant of M when the number one is added to each element of M. If we define (M + k) to be that matrix formed from M by adding any given number k to each element of M, we have the identity

$$(1) \qquad \det(M + k) = \det M + k\lambda(M).$$

For an example, the determinant $\lambda(Q^n)$ is given by

$$\lambda(Q^n) = \begin{vmatrix} F_{n+1} + 1 & F_n + 1 \\ F_n + 1 & F_{n-1} + 1 \end{vmatrix} - \det Q^n$$

$$= (F_{n+1}F_{n-1} - F_n^2) + (F_{n-1} + F_{n+1} - 2F_n) - \det Q^n$$

$$= F_{n-3}$$

which follows by use of Fibonacci identities. Now if we add k to each element of Q^n, the resulting determinant is

$$\begin{vmatrix} F_{n+1} + k & F_n + k \\ F_n + k & F_{n-1} + k \end{vmatrix} = \det Q^n + k\, F_{n-3}.$$

However, there are more convenient ways to evaluate the lambda function. For simplicity, we consider only 3 x 3 matrices.

THEOREM. For the given general 3 x 3 matrix M, $\lambda(M)$ is expressed by either of the expressions (2) or (3). For

$$(2)\qquad M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}, \qquad \lambda(M) = \begin{vmatrix} a+e-b-d & b+f-c-e \\ d+h-g-e & e+j-h-f \end{vmatrix},$$

$$(3)\qquad \lambda(M) = \begin{vmatrix} 1 & b & c \\ 1 & e & f \\ 1 & h & j \end{vmatrix} + \begin{vmatrix} a & 1 & c \\ d & 1 & f \\ g & 1 & j \end{vmatrix} + \begin{vmatrix} a & b & 1 \\ d & e & 1 \\ g & h & 1 \end{vmatrix}$$

Proof: This is made by direct evaluation and a simple exercise in algebra.

An application of the lambda function is in the evaluation of determinants. Whenever there is an obvious value of k such that det (M + k) is easy to find, we can use equation (1) advantageously. To illustrate this fact, consider

$$M = \begin{pmatrix} 1000 & 998 & 554 \\ 990 & 988 & 554 \\ 675 & 553 & 554 \end{pmatrix}.$$

We notice that, if we add k = -554 to each element of M, then det (M + k) = 0 since every element in the third column will be zero. From (2) we compute

$$\lambda(M) = \begin{vmatrix} 0 & 10 \\ -120 & 435 \end{vmatrix} = 1200;$$

and from (1) we find that 0 = det M + (-554)(1200), so that det M = (554)(1200).

Readers who enjoy mathematical curiosities can create determinants which are not changed in value when any given number k is added to each element, by writing any matrix D such that $\lambda(D) = 0$.

LEMMA: If two rows (or columns) of a matrix D have a constant difference between corresponding elements, then $\lambda(D) = 0$.

Proof: Evaluate $\lambda(D)$ directly, by (2) or (3).

For example, we write the matrix D, where corresponding elements in the first and second rows differ by 4, such that

$$\det D = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 4 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 1+k & 2+k & 3+k \\ 5+k & 6+k & 7+k \\ 4+k & 9+k & 8+k \end{vmatrix} = 24.$$

Now, we consider other Fibonacci matrices. Suppose that we want to write a Fibonacci matrix U such that det U = F_n. We can write $F_n = F_1F_2F_n$ for any n, and for some n we will also have other Fibonacci factorizations. Hence, for

$$U = \begin{pmatrix} F_1 & F_0 & F_0 \\ F_m & F_2 & F_0 \\ F_k & F_p & F_n \end{pmatrix},$$

det U = F_n where $F_0 = 0$. If we choose m = k = 3 and p = 2, we find that $\lambda(U) = 0$. If we choose m = 1 or 2, k = 1 or 2, and let p be an arbitrary integer, then $\lambda(U) = F_n$.

A more elegant way to write such a matrix was suggested by Ginsburg in [3], who wrote a matrix with the same first two columns as U below but with all elements in the third column equal to n and thus with determinant value n. We can write F_m = det U, where

$$U = \begin{pmatrix} F_{2p} & F_{2p+1} & F_m \\ F_{2p+1} & F_{2p+2} & F_m \\ F_{2p+2} & F_{2p+3} & F_m \end{pmatrix}$$

We have, using equation (3),

$$\lambda(U) = \begin{vmatrix} 1 & F_{2p+1} & F_m \\ 1 & F_{2p+2} & F_m \\ 1 & F_{2p+3} & F_m \end{vmatrix} + \begin{vmatrix} F_{2p} & 1 & F_m \\ F_{2p+1} & 1 & F_m \\ F_{2p+2} & 1 & F_m \end{vmatrix} + \begin{vmatrix} F_{2p} & F_{2p+1} & 1 \\ F_{2p+1} & F_{2p+2} & 1 \\ F_{2p+2} & F_{2p+3} & 1 \end{vmatrix}$$

$$= 0 + 0 + 1 = 1$$

If we let k = F_{m-1}, from (1) we see that det $(U + F_{m-1}) = F_m + (F_{m-1})(1) = F_{m+1}$.

Notice the possibilities for finding Fibonacci identities using the lambda function and evaluation of determinants. As a brief example, we let k = F_n and consider det $(Q^n + F_n)$, which gives us

$$\begin{vmatrix} F_{n+1} + F_n & F_n + F_n \\ F_n + F_n & F_{n-1} + F_n \end{vmatrix} = \det Q^n + F_n\,\lambda(Q^n)$$

or

$$\begin{vmatrix} F_{n+2} & 2\,F_n \\ 2\,F_n & F_{n+1} \end{vmatrix} = (-1)^n + F_nF_{n-3}$$

so that

$$4\,F_n^2 = F_{n+2}F_{n+1} - F_nF_{n-3} + (-1)^{n+1}.$$

As a final example of a Fibonacci matrix, we take the matrix R, given by

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

which has been considered by Brennan [4]. It can be shown that

$$R^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2\,F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & 2\,F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix}$$

by mathematical induction. The reader may verify that by equation (2) and by Fibonacci identities,

$$\lambda(R^n) = (-1)^n(F_{n-1}^2 - F_{n-3}F_{n-2}),$$

the center element of R^{n-2} multiplied by $(-1)^n$.

REFERENCES

1. Charles H. King, "Some Properties of the Fibonacci Numbers", (Master's Thesis), San Jose State College, June, 1960, pp. 11-27.
2. From the unpublished notes of Fenton S. Stancliff.
3. Jekuthiel Ginsburg, "Determinants of a Given Value", Scripta Mathematica, Vol. 18, Issues 3-4, Sept.-Dec., 1952, p. 219.
4. From the unpublished notes of Terry Brennan.

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Problem B-24 (Proposed by Brother Alfred Brousseau): It is evident that the determinant below has a value of zero. Prove that if the same quantity k is added to each element, the value becomes $(-1)^{n-1}k$.

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+2} & F_{n+3} & F_{n+4} \end{vmatrix}$$

A PRIMER FOR THE FIBONACCI SEQUENCE: PART III

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MORE FIBONACCI IDENTITIES FROM MATRICES AND VECTORS

The algebra of vectors and matrices will be further pursued to derive some more Fibonacci identities.

1. THE ALGEBRA OF (TWO-DIMENSIONAL) VECTORS

The two-dimensional vector V is an ordered pair of elements, called scalars, of a field: $V = (a, b)$. (The real numbers, for example, form a field.)

The zero vector, $\emptyset$, is a vector whose elements are each zero; $\emptyset = (0, 0)$.

Two vectors, $U = (a, b)$ and $V = (c, d)$, are equal if and only if their corresponding elements are equal; that is, if and only if $a = c$ and $b = d$.

The vector W, which is the product of a scalar k and a vector $U = (a, b)$, is $W = kU = (ka, kb) = Uk$. We see that if $k = 1$, then $kU = U$. We shall define the additive inverse of U by $-U = (-1)U$.

The vector W, which is the vector sum of two vectors $U = (a, b)$ and $V = (c, d)$ is

$$W = U + V = (a, b) + (c, d) = (a + c, b + d).$$

The vector $W = U - V = U + (-V)$, which defines subtraction.

The only binary multiplicative operation between two vectors, $U = (a, b)$ and $V = (c, d)$, considered here is the scalar or inner product,

$$U \cdot V = (a, b) \cdot (c, d) = ac + bd,$$

which is a scalar.

2. A GEOMETRIC INTERPRETATION OF A TWO-DIMENSIONAL VECTOR

One interpretation of the vector $U = (a, b)$ is a directed line segment from the origin $(0, 0)$ to the point (a, b) in a rectangular coordinate system. Every vector, except the zero vector $\emptyset$, will have the direction from the origin to the point (a, b) and a magnitude or length, $|U| = \sqrt{a^2 + b^2}$. The zero vector $\emptyset$ has a zero magnitude and no defined direction.

The inner or scalar product of two vectors, $U = (a, b)$ and $V = (c, d)$ can be shown to equal

$$U \cdot V = |U||V| \cos \theta,$$

where θ is the angle between the two vectors.

3. TWO-BY-TWO MATRICES AND TWO-DIMENSIONAL VECTORS

If U = (a, b) is written as (a b), then U is a 1 x 2 matrix which we shall call a row-vector. Similarly, if U = (a, b) is written vertically, then U becomes a 2 x 1 matrix which we shall call a column-vector.

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for example, can be considered as two row-vectors $R_1 = (a \quad b)$ and $R_2 = (c \quad d)$ in special position, or, as two column-vectors $C_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $C_2 = \begin{pmatrix} b \\ d \end{pmatrix}$ in special position.

The product W of a matrix A and a column-vector X is a column-vector X',

$$W = AX = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} .$$

Thus matrix A, operating upon the vector X, yields another vector, X'. The zero vector Ø is transformed into the zero vector again. In general, the direction and magnitude of vector X are different from those of vector X'.

4. THE INVERSE OF A TWO-BY-TWO MATRIX

If the determinant D(A) of a two-by-two matrix A is non-zero, then there exists a matrix A^{-1}, the inverse of matrix A, such that $A^{-1}A = AA^{-1} = I$. From the equation AX = X' or pair of equations $ax + by = x'$ and $cx + dy = y'$, one can solve for the variables x and y provided that $D(A) = ad - bc \neq 0$. Suppose this has been done, letting $D = D(A) \neq 0$, so that

$$\frac{d}{D} x' - \frac{b}{D} y' = x$$

$$\frac{-c}{D} x' + \frac{a}{D} y' = y .$$

Thus the matrix B, such that BX' = X, is given by

$$B = \begin{pmatrix} \frac{d}{D} & \frac{-b}{D} \\ \frac{-c}{D} & \frac{a}{D} \end{pmatrix}, \qquad D \neq 0.$$

It is easy to verify that BA = AB = I. Thus B is A^{-1}, the inverse matrix to matrix A. The inverse of the Q matrix is $Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

5. FIBONACCI IDENTITY USING THE Q MATRIX

Suppose we prove, recalling that $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, that $F_1 + F_2 + \dots + F_n = F_{n+2} - 1$.

It is easy to establish by mathematical induction that

$$(I + Q + Q^2 + \dots + Q^n)(Q - I) = Q^{n+1} - I .$$

If $(Q - I)$ has an inverse $(Q - I)^{-1}$, then multiplying on each side yields

$$I + Q + Q^2 + \dots + Q^n = (Q^{n+1} - I)(Q - I)^{-1} .$$

It is easy to verify that Q satisfies the matrix equation $Q^2 = Q + I$. Thus $(Q - I)Q = Q^2 - Q = I$ and $(Q - I)^{-1} = Q$. Therefore

$$Q + Q^2 + \dots + Q^n = Q^{n+2} - (Q + I) = Q^{n+2} - Q^2.$$

Equating elements in the upper right in the above matrix equation yields

$$F_1 + F_2 + \dots + F_n = F_{n+2} - F_2 = F_{n+2} - 1.$$

6. THE CHARACTERISTIC POLYNOMIAL OF A MATRIX A

In section 3, we discussed the transformation $AX = X'$. Generally the direction and magnitude of vector X are different from those of vector X'. If we ask which vectors X have their directions unchanged, we are led to the equation $AX = \lambda X$, where λ is a scalar. This can be rewritten as

$$(A - \lambda I)X = \emptyset .$$

Since we want $|X| \neq 0$, the only possible solution occurs when $D(A - \lambda I) = 0$. This last equation is called the <u>characteristic equation</u> of matrix A. The values of λ are called characteristic values or eigenvalues and the associated vectors are the characteristic vectors of matrix A. The characteristic polynomial of A is $D(A - \lambda I)$.

The characteristic equation for the Q matrix is $\lambda^2 - \lambda - 1 = 0$. The Hamilton-Cayley theorem states that a matrix satisfies its own characteristic equation, so that for the Q matrix

$$Q^2 - Q - I = 0.$$

Of course, this can be rewritten as $Q^2 = Q + I$, in which form we will use the matrix equation in the next section.

7. SOME MORE IDENTITIES

Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which satisfies $Q^2 = Q + I$. Thus, since $Q^0 = I$,

(1) $$Q^{2n} = (Q^2)^n = (Q + I)^n = \sum_{i=0}^{n} \binom{n}{i} Q^i$$

Equating elements in the upper right yields

$$F_{2n} = \sum_{i=0}^{n} \binom{n}{i} F_i \; .$$

From (1)

$$Q^p Q^{2n} = \sum_{i=0}^{n} \binom{n}{i} Q^{i+p} \quad ,$$

which gives

$$F_{2n+p} = \sum_{i=0}^{n} \binom{n}{i} F_{i+p}$$

for $n \geq 0$ and integral p.

From part II, $Q^n = F_n Q + F_{n-1} I$, so that

$$Q^{mn+p} = \sum_{i=0}^{m} \binom{m}{i} Q^{i+p} F_n^i F_{n-1}^{m-i} \; .$$

Equating elements in the upper right of the above matrix equation gives

$$F_{mn+p} = \sum_{i=0}^{m} \binom{m}{i} F_{i+p} F_n^i F_{n-1}^{m-i} \; ,$$

with $m \geq 0$, and for any integral p and n.

SOME NEW FIBONACCI IDENTITIES

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In this paper, some new Fibonacci and Lucas identities are generated by matrix methods.

The matrix

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

satisfies the matrix equation $R^3 - 2R^2 - 2R + I = 0$. Multiplying by R^n yields

(1) $$R^{n+3} - 2R^{n+2} - 2R^{n+1} + R^n = 0 \quad .$$

It has been shown by Brennan [1] and appears in an earlier article [2] that

(2) $$R^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_nF_{n-1} & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix} ,$$

where F_n is the nth Fibonacci number.

By the definition of matrix addition, corresponding elements of R^{n+3}, R^{n+2}, R^{n+1}, and R^n must satisfy the recursion formula given in Equation (1). That is, for example,

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0,$$

$$F_{n+3}F_{n+4} - 2F_{n+2}F_{n+3} - 2F_{n+1}F_{n+2} + F_nF_{n+1} = 0.$$

Returning again to $R^3 - 2R^2 - 2R + I = 0$, this equation can be rewritten as

$$(R + I)^3 = R^3 + 3R^2 + 3R + I = 5R(R + I).$$

In general, by induction, it can be shown that

(3) $$R^p(R + I)^{2n+1} = 5^nR^{n+p}(R + I).$$

Equating the elements in the first row and third column of the above matrices, by means of Equation (2), we obtain

$$(4) \qquad \sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i+p}^2 = 5^n F_{2(n+p)+1} .$$

It is not difficult to show that the Lucas numbers and members of the Fibonacci sequence have the relationship

$$L_n^2 - 5F_n^2 = (-1)^n 4 .$$

Since also

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+p} = 0,$$

we can derive the following sum of squares of Lucas numbers,

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L_{i+p}^2 = 5^{n+1} F_{2(n+p)+1} ,$$

by substitution of the preceding two identities in Equation (4).

Upon multiplying Equation (3) on the right by (R + I), we obtain

$$(5) \qquad R^p(R+I)^{2n+2} = 5^n R^{n+p}(R+I)^2 .$$

Then, using the expression for R^n given in Equation (2) and the identity $L_k = F_{k-1} + F_{k+1}$, we find that

$$(R^{n+1} + R^n)(R+I) = \begin{pmatrix} F_{2n-1} & F_{2n} & F_{2n+1} \\ 2F_{2n} & 2F_{2n+1} & 2F_{2n+2} \\ F_{2n+1} & F_{2n+2} & F_{2n+3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} L_{2n} & L_{2n+1} & L_{2n+2} \\ 2L_{2n+1} & 2L_{2n+2} & 2L_{2n+3} \\ L_{2n+2} & L_{2n+3} & L_{2n+4} \end{pmatrix} .$$

Finally, by equating the elements in the first row and third column of the matrices of Equation (5), we derive the two identities

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} F_{i+p}^2 = 5^n L_{2(n+p)+2},$$

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} L_{i+p}^2 = 5^{n+1} L_{2(n+p)+2}.$$

By similar steps, by equating the elements appearing in the first row and second column of the matrices of Equations (3) and (5), we can write the additional identities,

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i-1+p} F_{i+p} = 5^n F_{2(n+p)} \quad ,$$

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} F_{i-1+p} F_{i+p} = 5^n L_{2(n+p)+1} \quad .$$

REFERENCES

1. From the unpublished notes of Terry Brennan.
2. Marjorie Bicknell and Verner E. Hoggatt, Jr., "Fibonacci Matrices and Lambda Functions," The Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 47-52.

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Editorial Comment

Form the (n + 1) X (n + 1) matrix P_n with Pascal's triangle appearing on and below its secondary diagonal, e. g.,

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Surely the reader will see $R = P_2$ and matrix P_1 very like Q in the lower left. The element occurring in the lower left corner of P_n^k is always F_k^n, and the characteristic equation of P_n has the Fibonomial coefficients appearing, leading to identities such as described in the next article.

FIBONACCI NUMBERS AND GENERALIZED BINOMIAL COEFFICIENTS

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1. INTRODUCTION

The first time most students meet the binomial coefficients is in the expansion

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j , \quad n \geq 0,$$

where

$$\binom{n}{m} = 0 \quad \text{for } m > n , \qquad \binom{n}{n} = \binom{n}{0} = 1, \quad \text{and}$$

(1) $$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} , \quad 0 < m < n .$$

Consistent with the above definition is

(2) $$\binom{n}{m} = \frac{n(n-1)\cdots 2\cdot 1}{m(m-1)\cdots 2\cdot 1\ (n-m)(n-m-1)\cdots 2\cdot 1} = \frac{n!}{m!(n-m)!} ,$$

where

$$n! = n(n-1)(n-2)\cdots 2\cdot 1 \quad \text{and} \quad 0! = 1 .$$

Given the first lines of Pascal's arithmetic triangle one can extend the table to the next line using directly definition (2) or the recurrence relation (1).

We now can see just how the ordinary binomial coefficients $\binom{n}{m}$ are related to the sequence of integers 1, 2, 3, ..., k, ... Let us generalize this observation using the Fibonacci sequence.

2. THE FIBONOMIAL COEFFICIENTS

Let the Fibonomial coefficients (which are a special case of the generalized binomial coefficients) be defined as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{F_n F_{n-1} \cdots F_2 F_1}{(F_m F_{m-1} \cdots F_2 F_1)(F_{n-m} F_{n-m-1} \cdots F_2 F_1)} , \quad 0 < m < n ,$$

and $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1$, where F_n is the nth Fibonacci number, defined by

$$F_n = F_{n-1} + F_{n-2}\,, \quad F_1 = F_2 = 1\,.$$

We next seek a convenient recurrence relation, like (1) for the ordinary binomial coefficients, to get the next line from the first few lines of the Fibonomial triangle.

To find two such recurrence relations we recall the Q-matrix,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

for which it is easily established by mathematical induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \; n \geq 0.$$

The laws of exponents hold for the Q-matrix so that $Q^n = Q^m Q^{n-m}$. Thus

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n-m+1} & F_{n-m} \\ F_{n-m} & F_{n-m-1} \end{pmatrix}$$

$$= \begin{pmatrix} F_{m+1}F_{n-m+1} + F_mF_{n-m} & F_{m+1}F_{n-m} + F_mF_{n-m-1} \\ F_mF_{n-m+1} + F_{m-1}F_{n-m} & F_mF_{n-m} + F_{m-1}F_{n-m-1} \end{pmatrix}$$

yielding, upon equating corresponding elements,

(A) $\quad F_n = F_{m+1}F_{n-m} + F_mF_{n-m-1}$ (upper right),

(B) $\quad F_n = F_mF_{n-m+1} + F_{m-1}F_{n-m}$ (lower left).

Define C so that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{F_nF_{n-1}\cdots F_2F_1}{(F_mF_{m-1}\cdots F_2F_1)(F_{n-m}F_{n-m-1}\cdots F_2F_1)} = F_nC\,.$$

With C defined above, then

$$\begin{bmatrix} n-1 \\ m \end{bmatrix} = F_{n-m}C \quad \text{and} \quad \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = F_mC\,.$$

Returning now to identity (A), we may write for $C \neq 0$,

$$F_n C = F_{m+1}(F_{n-m}C) + F_{n-m-1}(F_m C)$$

but by the definition of C, we have derived

$$\text{(D)} \qquad \begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} .$$

Similarly, using identity (B), one can establish

$$\text{(E)} \qquad \begin{bmatrix} n \\ m \end{bmatrix} = F_{m-1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m+1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$

It is thus now easy to establish by mathematical induction that if the Fibonomial coefficients are integers for an integer n ($m = 0, 1, \ldots, n$), then they are integers for an integer $n + 1$ ($m = 0, 1, 2, \ldots, n + 1$).

Recalling

$$L_m = F_{m+1} + F_{m-1}$$

and adding (D) and (E) yields

$$\text{(3)} \qquad 2\begin{bmatrix} n \\ m \end{bmatrix} = L_m\begin{bmatrix} n-1 \\ m \end{bmatrix} + L_{n-m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} ,$$

where L_m is the mth Lucas number. From (3) it is harder to show that the Fibonomial coefficients are integers.

3. THE FIBONOMIAL TRIANGLE

Pascal's arithmetic triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$\binom{n}{0} \quad \binom{n}{1} \quad \cdots \quad \binom{n}{m} \quad \cdots \quad \binom{n}{n-1} \quad \binom{n}{n}$$

has been the subject of many studies and has always generated interest. We note here to get the next line we merely use the recurrence relation (1). Here we point out two interpretations, one of which shows a direction for generalization. The usual first meeting with Pascal's triangle lies in the binomial theorem expansion of $(x + y)^n$. However, of much interest to us is the difference equation interpretation. The difference equation satisfied by n^0 is

$$(n + 1)^0 - n^0 = 0 \quad ,$$

while the difference equation satisfied by n is

$$(n + 2) - 2(n + 1) + n = 0 \quad .$$

For n^2 the difference equation is

$$(n + 3)^2 - 3(n + 2)^2 + 3(n + 1)^2 - n^2 = 0 \; .$$

Certainly one notices the binomial coefficients with alternating signs appearing here. In fact,

$$\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (n + m + 1 - j)^m = 0 \; .$$

It is this connection with the difference equations for the powers of the integers that leads us naturally to the Fibonomial triangle.

Similar to the difference equation coefficients array for the powers of the positive integers which results in Pascal's arithmetic triangle with alternating signs, there is the Fibonomial triangle made up of the Fibonomial coefficients, with doubly alternated signs. We first write down the Fibonomial triangle for the first six levels.

						1						
					1		1					
				1		1		1				
			1		2		2		1			
		1		3		6		3		1		
	1		5		15		15		5		1	
1		8		40		60		40		8		1

The top line is the zeroth row and the coefficients in the difference equation satisfied by F_n^k are the numbers in the $(k + 1)$st row. Of course, we can get the next line of Fibonomial coefficients by using our recurrence relation (D),

$$\begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1} \begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \quad 0 < m < n .$$

We now rewrite the Fibonomial triangle with appropriate signs so that the rows are properly signed to be the coefficients in the difference equation satisfied by F_n^k.

$$1$$

F_n^0 : 1 -1

F_n^1 : 1 -1 -1

F_n^2 : 1 -2 -2 +1

F_n^3 : 1 -3 -6 +3 +1

F_n^4 : 1 -5 -15 +15 +5 -1

F_n^5 : 1 -8 -40 +60 +40 -8 -1

Thus, from the above we may write

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0 \qquad \text{and}$$

$$F_{n+5}^4 - 5F_{n+4}^4 - 15F_{n+3}^4 + 15F_{n+2}^4 + 5F_{n+1}^4 - F_n^4 = 0 .$$

The auxiliary polynomial for the difference equation satisfied by F_n^m is

$$\sum_{h=0}^{m+1} \begin{bmatrix} m+1 \\ h \end{bmatrix} (-1)^{h(h+1)/2} x^{m+1-h}$$

which shows that the sign pattern of doubly alternating signs persists. (See [1], [2].) (Further generalizations given in the original paper are here omitted.)

REFERENCES

1. Dov Jarden, Recurring Sequences, Riveon Lematematika, Jerusalem, Israel, 1958, pp. 42-45.
2. A. P. Hillman and V. E. Hoggatt, "The Characteristic Polynomial of the Generalized Shift Matrix," Fibonacci Quarterly, Vol. 3, No. 2, April, 1965, pp. 91-94.
3. Terrence A. Brennan, "Fibonacci Powers and Pascal's Triangle in a Matrix," Fibonacci Quarterly, Vol. 2, No. 2, April, 1964, pp. 93-103.

A PRIMER FOR THE FIBONACCI NUMBERS: PART IV

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FIBONACCI AND LUCAS VECTORS

1. INTRODUCTION

In the primer, Part III, it was noted that if $V = (x, y)$ is a two-dimensional vector and A is a 2 x 2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $V' = AV$ is a two-dimensional vector, $V' = (x', y') = (ax + by, cx + dy)$. Here, V and consequently V', are expressed as column vectors. The matrix A is said to transform, or map, the vector V onto the vector V'. The matrix A is called the mapping matrix or transformation matrix.

2. SOME MAPPING MATRICES

The zero matrix, $Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, maps every vector V onto the zero vector $\emptyset = (0, 0)$. The identity matrix, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, maps every vector V onto itself; that is, $IV = V$. For any real number k, the matrix $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ maps vectors $V = (k, -k)$ onto the zero vector $\emptyset$. Such a mapping as determined by B is called a many-to-one mapping.

If the only vector mapped onto $\emptyset$ is the vector $\emptyset$ itself, the mapping is a one-to-one mapping. A matrix A determines a one-to-one mapping of two-dimensional vectors onto two-dimensional vectors if, and only if, $\det A \neq 0$. If $\det A \neq 0$, for each vector U, there exists a vector V such that $AV = U$. Note, however, that for matrix B above, $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 2x + 2y \end{pmatrix}$. There is no vector V such that $BV = (0, 1)$.

3. GEOMETRIC INTERPRETATIONS OF 2x2 MATRICES AND 2-DIMENSIONAL VECTORS

As in Primer III, the vector $V = (x, y)$ is interpreted as a point in a rectangular coordinate system. Thus the geometric concepts of length, direction, slope and angle are associated with the vector V.

A non-zero scalar multiple of the identity matrix, kI, maps the vector $U = (a, b)$ onto the vector $V = (ka, kb)$. The length of V, $|V|$, is equal to

$|k||U|$. There is no change in slope but if $k < 0$ the sense or direction is reversed.

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ maps a vector onto the reflection vector with respect to the line through the origin with slope one. Note that different vectors may be rotated through different angles!

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ preserves the first component of a vector while annihilating the second component. Every vector is mapped onto a vector on the X-axis.

The matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates all vectors through the same angle θ (theta), in a counterclockwise direction if theta is a positive angle. There is no change in length. This seems to contradict the notion of a matrix having vectors whose slopes are not changed, but in this case, the characteristic values are complex; thus, there are no real characteristic vectors.

4. THE CHARACTERISTIC VECTORS OF THE Q-MATRIX

The Q matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ does not generally preserve the length of a vector $U = (x, y)$. Also, different vectors are in general rotated through different angles.

The characteristic equation of the Q matrix is

$$\lambda^2 - \lambda - 1 = 0$$

with roots $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$, which are the characteristic roots, or eigenvalues, for Q.

To solve for a pair of corresponding characteristic vectors consider

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \quad , \qquad x^2 + y^2 \neq 0 .$$

Then

$$(1 - \lambda)x + y = 0 .$$

Thus, a pair of characteristic vectors are X_1 and X_2 with slopes m_1 and m_2,

$$X_1 = (\lambda_1 x, x) , \quad |X_1| \neq 0 , \qquad m_1 = (\sqrt{5} - 1)/2 ,$$

$$X_2 = (\lambda_2 x, x), \quad |X_2| \neq 0, \qquad m_2 = -(\sqrt{5} + 1)/2 .$$

What happens when the matrix Q^2 is applied to the characteristic vectors X_1 and X_2 of matrix Q? Since

$$Q^2X_1 = Q(QX_1) = Q(\lambda X_1) = \lambda QX_1 = \lambda^2 X_1 ,$$

clearly X_1 is a characteristic vector of the matrix Q^2 as well as a characteristic vector of matrix Q. The characteristic roots of Q^2 are the squares of the characteristic roots of matrix Q. In general, if λ_1 and λ_2 are the characteristic roots of Q, then λ_1^n and λ_2^n are the characteristic roots of Q^n. But the characteristic equation for Q^n is

$$0 = \lambda^2 - (F_{n+1} + F_{n-1})\lambda + (F_{n+1}F_{n-1} - F_n^2) = \lambda^2 - L_n\lambda + (-1)^n ,$$

recalling that $L_n = F_{n+1} + F_{n-1}$ and that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. Applying the known identity $L_n^2 = 5F_n^2 + 4(-1)^n$, it follows that

$$\lambda_1^n = [(1 + \sqrt{5})/2]^n = (L_n + \sqrt{5}F_n)/2 \text{ and } \lambda_2^n = [(1 - \sqrt{5})/2]^n = (L_n - \sqrt{5}F_n)/2 .$$

5. FIBONACCI AND LUCAS VECTORS AND THE Q MATRIX

Let $U_n = (F_{n+1}, F_n)$ and $V = (L_{n+1}, L_n)$ be denoted as Fibonacci and Lucas vectors, respectively. We note that

$$|U_n|^2 = F_{n+1}^2 + F_n^2 = F_{2n+1} ,$$

$$|V_n|^2 = L_{n+1}^2 + L_n^2 = (5F_{n+1}^2 + (-1)^{n+1}4) + (5F_n^2 + (-1)^n 4) = 5F_{2n+1} .$$

It is well-known that the slopes of the vectors U_n and V_n (the ratios F_n/F_{n+1} and L_n/L_{n+1}) approach the slope $(\sqrt{5} - 1)/2$ of the characteristic vector X_1.

Since $Q^mQ^n = Q^{m+n}$, it is easy to verify that

$$F_{m+1}F_{n+1} + F_mF_n = F_{m+n+1}$$

by equating elements in the upper left in the above matrix equation. In a similar manner it follows that

$$F_{m+1}F_{n+2} + F_mF_{n+1} = F_{m+n+2} ,$$

$$F_{m+1}F_n + F_mF_{n-1} = F_{m+n} .$$

Adding these two equations and using $L_{n+1} = F_{n+2} + F_n$ it follows that

$$F_{m+1}L_{n+1} + F_m L_n = L_{m+n+1} \quad .$$

From the above identities it is easy to verify that

$$Q^{n+1}V_0 = QV_n = V_{n+1} \quad ,$$

$$Q^{n+1}U_0 = QU_n = U_{n+1} \quad ,$$

$$Q^n V_m = V_{m+n+1} \quad ,$$

$$Q^n U_m = U_{m+n+1} \quad .$$

6. A SPECIAL MATRIX

Let $P = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$; then from

$$L_{n+1} = F_{n+1} + 2F_n \quad , \qquad L_n = 2F_{n+1} - F_n \quad ,$$

$$5F_{n+1} = L_{n+1} + 2L_n \quad , \qquad 5F_n = 2L_{n+1} - L_n \quad ,$$

it follows that

$$PU_n = (F_{n+1} + 2F_n, \; 2F_{n+1} - F_n) = V_n$$

$$PV_n = (L_{n+1} + 2L_n, \; 2L_{n+1} - L_n) = 5U_n$$

Also

$$PQ^n = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}$$

$$P^2Q^n = 5Q^n$$

Notice that $\det(PQ^n) = (\det P)(\det Q^n) = 5(-1)^{n+1} = L_{n+1}L_{n-1} - L_n^2$.

We now discuss two geometric properties of matrix P. Let $U = (x, y)$, $|U|^2 = x^2 + y^2 \neq 0$. Now, $PU = (x + 2y, 2x - y)$ and $|PU|^2 = 5(x^2 + y^2) = 5|U|^2$; thus matrix P magnifies each vector length by $\sqrt{5}$.

If $\tan \alpha = y/x$, we say $\alpha = \mathrm{Tan}^{-1}\, y/x$, read " α is an angle whose tangent is y/x." Let $\tan \alpha = y/x$ and $\tan \beta = (2x - y)/(x + 2y)$. From the identity $\tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$ we may now see what effect P has on the slope of vector $U = (x, y)$.

Now, recalling that $x^2 + y^2 \neq 0$,

$$\tan(\alpha + \beta) = \tan\left(\mathrm{Tan}^{-1}\frac{y}{x} + \mathrm{Tan}^{-1}\frac{2x - y}{x + 2y}\right) = \frac{2(x^2 + y^2)}{x^2 + y^2} = 2 .$$

What does this mean? Consider two vectors A and B, the first inclined at an angle α with the positive X-axis and the second inclined at an angle β with the positive X-axis when the angles are measured positively in the counter-clockwise direction. The angle bisector ψ of the angle between vectors A and B is such that $\alpha - \psi = \psi - \beta$ whether or not α is greater than β or the other way around. Solving for ψ yields

$$\psi = (\alpha + \beta)/2 .$$

Thus ψ is the arithmetic average of α and β. Also we note that $\alpha + \beta = 2\psi$. The tangent of double the angle is given by $\tan 2\psi = (2 \tan \psi)/(1 - \tan^2\psi)$. If we let $\tan \psi = (\sqrt{5} - 1)/2$, then it is an easy exercise in algebra to find that $\tan 2\psi = 2$. But, $\tan(\alpha + \beta) = 2$; therefore, we would like to conclude that the angle bisector between vectors U and PU is precisely one whose slope is $(\sqrt{5} - 1)/2$, which is the slope of X_1, the characteristic vector of Q. Can you show that X_1 is also a characteristic vector of P?

We have shown

<u>Theorem 1</u>. The matrix $P = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ maps a vector (x, y) onto a vector PU such that

(1) $$|PU| = \sqrt{5}|U| \quad ;$$

(2) The angle bisector of the angle between the vector U and the vector PU is X_1, a characteristic vector of Q and P. Thus matrix P reflects vector U across vector X_1.

<u>Theorem 2</u>. The vectors U_n and V_n are equally inclined to the vector X_1 whose slope is $(\sqrt{5} - 1)/2$.

<u>Corollary</u>. The vectors V_n are mapped onto vectors $\sqrt{5}\, U_n$ by P and the vectors U_n are mapped onto V_n by P.

7. SOME INTERESTING ANGLES

An interesting theorem is

<u>Theorem 3</u>.

$$\mathrm{Tan}(\mathrm{Tan}^{-1} L_n/L_{n+1} - \mathrm{Tan}^{-1} L_{n+1}/L_{n+2}) = (-1)^n/F_{2n+2}$$

Theorem 4.

$$\mathrm{Tan}(\mathrm{Tan}^{-1} F_n/F_{n+1} - \mathrm{Tan}^{-1} F_{n+1}/F_{n+2}) = (-1)^{n+1}/F_{2n+2}$$

Theorem 5.

$$\mathrm{Tan}^{-1} F_n/F_{n+1} = \sum_{m=1}^{n} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m}$$

We proceed by mathematical induction. For $n = 1$, it is easy to verify that $\mathrm{Tan}^{-1} 1 = \mathrm{Tan}^{-1} (1/F_2)$.

Assume that Theorem 5 is true for $n = k$; that is, that

$$\mathrm{Tan}^{-1} F_k/F_{k+1} = \sum_{m=1}^{k} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m}$$

But, by Theorem 4,

$$\mathrm{Tan}^{-1} F_{k+1}/F_{k+2} = \mathrm{Tan}^{-1} F_k/F_{k+1} + \mathrm{Tan}^{-1} (-1)^k/F_{2k+2}$$

Thus, if the induction hypothesis is true, then

$$\mathrm{Tan}^{-1} F_{k+1}/F_{k+2} = \sum_{m=1}^{k} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m} + \mathrm{Tan}^{-1} (-1)^k/F_{2k+2}$$

$$= \sum_{m=1}^{k+1} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m}$$

because $\mathrm{Tan}^{-1} (-x) = - \mathrm{Tan}^{-1} x$ and $(-1)^k = (-1)^{k+2}$ and the proof is complete.

8. AN EXTENDED RESULT

Theorem 6. The series

$$A = \sum_{m=1}^{\infty} (-1)^{m+1} \mathrm{Tan}^{-1} 1/F_{2m}$$

converges and $A = \mathrm{Tan}^{-1} (\sqrt{5} - 1)/2$.

Proof: Since the series is an alternating series, and, since $\mathrm{Tan}^{-1} x$ is a continuous increasing function, then

$$\mathrm{Tan}^{-1} 1/F_{2n} > \mathrm{Tan}^{-1} 1/F_{2n+2} \quad \text{and} \quad \mathrm{Tan}^{-1} 0 = 0 .$$

The angle A must lie between the partial sums S_N and S_{N+1} for every $N > 2$ by the error bound in the alternating series, but by Theorem 5, $S_N = \mathrm{Tan}^{-1} F_N/F_{N+1}$. Thus the angles of U_N and U_{N+1} lie on opposite sides of A. By the continuity of $\mathrm{Tan}^{-1} x$, then,

$$\lim_{n\to\infty} \mathrm{Tan}^{-1}(F_n/F_{n+1}) = A = \mathrm{Tan}^{-1}(\sqrt{5} - 1)/2 .$$

Comment: the same result can be obtained simply from

$$\mathrm{Tan}\,[\mathrm{Tan}^{-1} F_n/F_{n+1} - \mathrm{Tan}^{-1}(\sqrt{5} - 1)/2\,] = (-1)^{n+1}[(\sqrt{5} - 1)/2]^{2n+1}$$

Which slope gives a better numerical approximation to $(\sqrt{5} - 1)/2$, F_n/F_{n+1} or L_n/L_{n+1}? Hmmm?

* * * * * * * * * *

SOME MORE ELEMENTARY PROBLEMS

B-4 (Proposed by S. L. Basin and Vladimir Ivanoff) Show that

$$\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}$$

and generalize.

B-5 (Proposed by L. Moser) Show that, with order taken into account, in getting paid an integral number n dollars, using only one-dollar and two-dollar bills, that the number of different ways is F_{n+1} where F_n is the nth Fibonacci number.

B-9 (Proposed by R. L. Graham) Prove that

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = 1 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = 2 .$$

B-10 (Proposed by Stephen Fisk) Prove the "de Moivre-type" identity

$$\left(\frac{L_n + \sqrt{5}\, F_n}{2}\right)^p = \frac{L_{np} + \sqrt{5}\, F_{np}}{2}$$

where L_n denotes the nth Lucas number and F_n denotes the nth Fibonacci number.

A PRIMER FOR THE FIBONACCI NUMBERS: PART V

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INFINITE SERIES AND FIBONACCI ARCTANGENTS

1. INTRODUCTION

In Section 8 of Part IV, we discussed an alternating series. This time we shall lay down some brief foundations of sequences and infinite series. This leads to some very interesting results and to the broad topics of generating functions and continued fractions. Many Fibonacci numbers shall appear.

2. SEQUENCES

Definition: An ordered set of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called an infinite sequence of numbers. If there are but a finite number of the a's, $a_1, a_2, \ldots, a_n$, then it is a finite sequence of numbers.

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to have a real number a as a limit, written $\lim_{n\to\infty} a_n = a$, if for every positive real number ϵ, $|a_n - a| < \epsilon$ for all but a finite number of the members of the sequence $\{a_n\}$. If the sequence $\{a_n\}$ has a limit, this limit is unique and the sequence is said to converge to this limit. If the sequence $\{a_n\}$ fails to approach a limit, then the sequence is said to diverge. We now give examples of each kind.

If $a_n = 1$, $\{a_n\} = 1, 1, 1, \ldots$ converges since $\lim_{n\to\infty} a_n = 1$.

If $a_n = 1/n$, $\{a_n\} = 1, 1/2, 1/3, \ldots, 1/n, \ldots$ converges to zero.

If $a_n = (-1)^n$, $\{a_n\} = 1, -1, +1, -1, +1, \ldots$ diverges by oscillation. That is, it does not approach any limit.

If $a_n = n$, $\{a_n\} = 1, 2, 3, \ldots$ diverges to positive infinity.

Finally, if $a_n = n/(n+1)$, then $\{a_n\} = 1/2, 2/3, \ldots$ converges to one.

Some limit theorems for sequences are the following:

If $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers with limits a and b respectively, then

$$\lim_{n\to\infty} (a_n + b_n) = a + b$$

$$\lim_{n\to\infty} (a_n - b_n) = a - b$$

$$\lim_{n\to\infty} (ca_n) = ca, \text{ any real } c$$

$$\lim_{n\to\infty} (a_n b_n) = ab$$

$$\lim_{n\to\infty} (a_n/b_n) = a/b , \quad b \neq 0 .$$

3. BOUNDED MONOTONE SEQUENCES

The sequence $\{a_n\}$ is said to be bounded if there exists a positive number K such that $|a_n| < K$ for all $n \geq 1$. If $a_{n+1} \geq a_n$ for $n \geq 1$, the sequence $\{a_n\}$ is said to be a monotone increasing sequence; if $a_n \geq a_{n+1}$ for $n \geq 1$, the sequence is monotone decreasing. If a sequence is such that it is either monotone increasing or monotone decreasing, it will be called a monotone sequence.

The following useful and important theorem is stated without proof:

Theorem 1: A bounded monotone sequence converges.

As an example, consider the sequence $\{(1 + 1/n)^n\}$, which is monotone increasing and bounded above by 3. The limit of this sequence is well known. We will use Theorem 1 in the material to come.

4. ANOTHER IMPORTANT THEOREM

The following sufficient conditions for the convergence of an alternating series are given below.

Theorem 2: If, for the sequence $\{S_n\}$,

1. $S_1 > 0$,

2. $(S_{n-1} - S_n)(-1)^n > (S_n - S_{n+1})(-1)^{n+1} > 0$, for $n \geq 2$,

3. $\lim_{n\to\infty} (S_n - S_{n+1}) = 0$,

then the sequence $\{S_n\}$ converges to a limit S such that $0 < S < S_1$.

5. AN EXAMPLE OF AN APPLICATION OF THEOREM 2

For the following example a limit is known to exist by the application of Theorem 2 of Section 4.

Let $S_n = F_n/F_{n+1}$, where $\{F_n\}$ is the Fibonacci sequence. Then $S_{n-1} - S_n = (-1)^n/(F_nF_{n+1})$. By Theorem 2 above, $\lim_{n\to\infty} S_n$ exists.

To find the limit, consider

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} ,$$

which in terms of $\{S_n\}$ is $1/S_n = 1 + S_{n-1}$. Let the limit of S_n as n tends to infinity be S. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = S > 0$. Applying the limit theorems of Section 2, it follows that S satisfies

$$S = \frac{1}{1 + S} \qquad \text{or} \qquad S^2 + S - 1 = 0 .$$

Thus $S > 0$ is given by $S = (\sqrt{5} - 1)/2$, the positive root of the quadratic equation $S^2 + S - 1 = 0$.

6. INFINITE SERIES

If we add together the members of a sequence $\{a_n\}$, we get the infinite series $a_1 + a_2 + \dots + a_n + \dots$. We now get another sequence from this infinite series.

Define a sequence $\{S_n\}$ in the following way. Let $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, ... , or in general, $S_n = a_1 + a_2 + a_3 + \dots + a_n$. This is called the sequence of partial sums of the infinite series. The infinite series is said to converge to the limit S if the sequence $\{S_n\}$ converges to the limit S; otherwise, the series is said to diverge.

7. SPECIAL RESULTS CONCERNING SERIES

1. If an infinite series $a_1 + a_2 + a_3 + \dots + a_n + \dots$ converges, then $\lim_{n\to\infty} a_n = 0$. This is immediate since $a_n = S_n - S_{n-1}$.

2. From section 3 above, an infinite series of positive terms converges if the partial sums are bounded above since the partial sums form a monotone increasing sequence.

3. For the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ such that $a_n > 0$, $n \geq 1$; $a_{n+1} \leq a_n$, $n \geq 1$; and $\lim_{n\to\infty} a_n = 0$, by Section 4 above, the infinite series converges. In the theorem, $S_n = \sum_{j=1}^{n} (-1)^j a_j$. An example of an alternati series was seen in Part IV, Section 8, of this Primer.

8. FIBONACCI NUMBERS, LUCAS NUMBERS, AND PI

It is well known and easily verified that

$$\frac{\pi}{4} = \mathrm{Tan}^{-1}\frac{1}{1} = \mathrm{Tan}^{-1}\frac{1}{2} + \mathrm{Tan}^{-1}\frac{1}{3} .$$

Also one can verify

$$\frac{\pi}{4} = \mathrm{Tan}^{-1}\frac{1}{1} = \mathrm{Tan}^{-1}\frac{1}{2} + \mathrm{Tan}^{-1}\frac{1}{5} + \mathrm{Tan}^{-1}\frac{1}{8} ,$$

$$\frac{\pi}{4} = \mathrm{Tan}^{-1}\frac{1}{3} + \mathrm{Tan}^{-1}\frac{1}{5} + \mathrm{Tan}^{-1}\frac{1}{7} + \mathrm{Tan}^{-1}\frac{1}{8} .$$

We note Fibonacci and Lucas numbers here, surely. We shall here easily extend these results in several ways.

In this section we shall use several new identities which are left as exercises for the reader and will be marked with an asterisk.

*Lemma 1: $L_{2n}L_{2n+2} - 1 = 5F_{2n+1}^2$

Lemma 2: $L_n^2 = L_{2n} + 2(-1)^n$

Lemma 3: $L_n^2 - 5F_n^2 = 4(-1)^n$

*Lemma 4: $L_nL_{n+1} = L_{2n+1} + (-1)^n$

We now discuss

Theorem 3: If $\tan \theta_n = 1/L_n$, then $\tan(\theta_{2n} + \theta_{2n+2}) = 1/F_{2n+1}$, or,

$$\mathrm{Tan}^{-1}\frac{1}{F_{2n+1}} = \mathrm{Tan}^{-1}\frac{1}{L_{2n}} + \mathrm{Tan}^{-1}\frac{1}{L_{2n+2}} .$$

Proof:

$$\mathrm{Tan}(\theta_{2n} + \theta_{2n+2}) = \frac{L_{2n} + L_{2n+2}}{L_{2n}L_{2n+2} - 1} = \frac{1}{F_{2n+1}}$$

using the trigonometric identity $\tan(x + y) = (\tan x + \tan y)/(1 - \tan x \tan y)$ with Lemma 1 above and the identity $L_{2n+2} + L_{2n} = 5F_{2n+1}$.

Theorem 4: If $\tan \theta_n = 1/F_n$, then $\tan(\theta_{2n} - \theta_{2n+2}) = 1/F_{2n+1}$, or

$$\text{Tan}^{-1} \frac{1}{F_{2n+1}} = \text{Tan}^{-1} \frac{1}{F_{2n}} - \text{Tan}^{-1} \frac{1}{F_{2n+2}} .$$

Proof:

$$\text{Tan}\,(\theta_{2n} - \theta_{2n+2}) = \frac{F_{2n+2} - F_{2n}}{F_{2n}F_{2n+2} + 1} = \frac{1}{F_{2n+1}}$$

since $F_{2n+2} - F_{2n} = F_{2n+1}$ and $F_{2n}F_{2n+2} - F_{2n+1}^2 = (-1)^{2n+1} = -1$.

From Theorem 4,

$$\sum_{n=1}^{M} \text{Tan}^{-1} \frac{1}{F_{2n+1}} = \sum_{n=1}^{M} \left(\text{Tan}^{-1} \frac{1}{F_{2n}} - \text{Tan}^{-1} \frac{1}{F_{2n+2}}\right) = \text{Tan}^{-1} \frac{1}{F_2} - \text{Tan}^{-1} \frac{1}{F_{2M+2}}$$

and since $\lim_{M\to\infty} \text{Tan}^{-1} \frac{1}{F_{2M+2}} = 0$ by continuity of $\text{Tan}^{-1} x$ at $x = 0$, we may write

<u>Theorem 5</u>:
$$\frac{\pi}{4} = \text{Tan}^{-1} 1 = \sum_{n=1}^{\infty} \text{Tan}^{-1} \frac{1}{F_{2n+1}} .$$

This is the celebrated result of D. H. Lehmer, Nov., 1936, <u>American Mathematical Monthly</u> , p. 632, Problem 3801.

We note in passing that the partial sums

$$S_M = \sum_{n=1}^{M} \text{Tan}^{-1} \frac{1}{F_{2n+1}} = \text{Tan}^{-1} \frac{1}{F_2} - \text{Tan}^{-1} \frac{1}{F_{2M+2}}$$

are all bounded above by $\text{Tan}^{-1} 1 = \pi/4$ and S_M is monotone. Thus Theorem 1 can be applied. From Theorem 3,

$$\sum_{n=1}^{M} \text{Tan}^{-1} \frac{1}{F_{2n+1}} = \sum_{n=1}^{M} \left(\text{Tan}^{-1} \frac{1}{L_{2n}} + \text{Tan}^{-1} \frac{1}{L_{2n+2}}\right)$$

so that

$$\sum_{n=1}^{M} \text{Tan}^{-1} \frac{1}{F_{2n+1}} + \text{Tan}^{-1} \frac{1}{3} = 2 \sum_{n=1}^{M} \text{Tan}^{-1} \frac{1}{L_{2n}} + \text{Tan}^{-1} \frac{1}{L_{2M+2}}$$

The limit on the left tends to $\mathrm{Tan}^{-1} 1 + \mathrm{Tan}^{-1} 1/3 = \mathrm{Tan}^{-1} 2$, and the right-hand side tends to this same limit. Since $\lim_{M\to\infty} \mathrm{Tan}^{-1} \frac{1}{L_{2M+2}} = 0$, then

Theorem 6: $$\sum_{n=1}^{\infty} \mathrm{Tan}^{-1} \frac{1}{L_{2n}} = \mathrm{Tan}^{-1} \frac{\sqrt{5}-1}{2} = \frac{1}{2} \mathrm{Tan}^{-1} 2 \quad .$$

Compare with Theorem 6 in Part IV.

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FIBONACCI DETERMINANTS

Below are reprinted a selection of problems which appeared in early issues of the Fibonacci Quarterly.

H-8 (Proposed by Brother Alfred Brousseau) Prove that

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = 2(-1)^{n+1} ,$$

where F_n is the nth Fibonacci number.

B-28 (Proposed by Brother Alfred Brousseau) Using the nine Fibonacci numbers F_2 through F_{10} (1, 2, 3, 5, 8, 13, 21, 34, 55), determine a third-order determinant having each of these numbers as elements so that the value of the determinant is a maximum.

B-13 (Proposed by S. L. Basin) Prove the (n - 1)st order determinant below has value F_n. (This is a special case of B-13)

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n-1}$$

Such determinants are called continuants.

A problem which predates B-28 is to determine the third-order determinant of maximum value which has each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 as elements, and to determine the complete set of determinant values possible. (See Bicknell and Hoggatt, "An Investigation of Nine-Digit Determinants," Mathematics Magazine, May-June, 1963, pp. 147-152.)

A PRIMER FOR THE FIBONACCI NUMBERS: PART VI

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GENERATING FUNCTIONS FOR THE FIBONACCI SEQUENCES

1. INTRODUCTION

We shall devote this part of the primer to the topic of generating functions. These play an important role both in the general theory of recurring sequences and in combinatorial analysis. They provide a tool with which every Fibonacci enthusiast should be familiar.

2. GENERAL THEORY OF GENERATING FUNCTIONS

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The ordinary generating function of the sequence $\{a_n\}$ is the series

$$A(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{n=0}^{\infty} a_nx^n \quad .$$

Another type of generating function of great use in combinatorial problems involving permutations is the exponential generating function of $\{a_n\}$,

$$E(x) = a_0 + a_1x/1! + a_2x^2/2! + \ldots = \sum_{n=0}^{\infty} a_nx^n/n! \quad .$$

For some examples of the two types of generating functions, first let $a_n = a^n$. The ordinary generating function of $\{a_n\}$ is then the geometric series

$$A(x) = \frac{1}{1 - ax} = \sum_{n=0}^{\infty} a^nx^n , \tag{2.1}$$

while the exponential generating function is

$$E(x) = e^{ax} = \sum_{n=0}^{\infty} a^nx^n/n! \quad .$$

Similarly, if $a_n = na^n$, then

$$A(x) = \frac{ax}{(1 - ax)^2} = \sum_{n=0}^{\infty} na^n x^n \quad ,$$

(2.2)

$$E(x) = axe^{ax} = \sum_{n=0}^{\infty} na^n x^n/n! \quad ,$$

each of these being obtained from the preceding one of the same type by differentiation and multiplication by x. A good exercise for the reader to check his understanding is to verify that if $a_n = n^2$, then

$$A(x) = \frac{x(x + 1)}{(1 - x)^3} = \sum_{n=0}^{\infty} n^2 x^n \quad ,$$

$$E(x) = x(x + 1)e^x = \sum_{n=0}^{\infty} n^2 x^n/n! \quad .$$

(Hint: Differeniate the previous results again.)

For the rest of the time, however, we will deal exclusively with ordinary generating functions.

We adopt the point of view here that x is an indeterminate, a means of distinguishing the elements of the sequence through its powers. Used in this context, the generating function becomes a tool in an algebra of these sequences (see [3]). Then formal operations, such as addition, multiplication, differentiation with respect to x, and so forth, and equating equations of like powers of x after these operations merely express relations in this algebra, so that convergence of the series is irrelevant.

The basic rules of manipulation in this algebra are analogous to those for handling polynomials. If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are real sequences with (ordinary) generating functions A(x), B(x), C(x) respectively, then A(x) + B(x) = C(x) if and only if $a_n + b_n = c_n$, and A(x)B(x) = C(x) if and only if

$$c_n = a_n b_0 + a_{n-1} b_1 + \ldots + a_1 b_{n-1} + a_0 b_n \ .$$

Both results are obtained by expanding the indicated sum or product of generating functions and comparing coefficients of like powers of x. The product here is called the Cauchy product of the sequences $\{a_n\}$ and $\{b_n\}$, and the sequence $\{c_n\}$ is called the convolution of the two sequences $\{a_n\}$ and $\{b_n\}$.

To give an example of the usefulness and convenience of generating functions, we shall derive a well-known but nontrivial binomial identity. First note that for a fixed real number k the generating function for the sequence

$$a_n = \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

is

$$A_k(x) = (1+x)^k$$

by the binomial theorem. If k is a nonnegative integer, the generating function is finite since

$$\binom{k}{n} = 0 \quad \text{if } n > k \geq 0 \quad \text{or} \quad n < 0 \tag{2.3}$$

by definition. Then

$$A_k(x) = (1+x)^k = (1+x)^{k-m}(1+x)^m = A_{k-m}(x)A_m(x) \quad .$$

Using the product rule gives

$$\sum_{n=0}^{k}\binom{k}{n}x^n = \sum_{n=0}^{\infty}\binom{k}{n}x^n = \left(\sum_{n=0}^{\infty}\binom{k-m}{n}x^n\right)\left(\sum_{n=0}^{\infty}\binom{m}{n}x^n\right)$$

$$= \sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\binom{k-m}{j}\binom{m}{n-j}\right]x^n \quad ,$$

so that equating coefficients of x^n shows that

$$\binom{k}{n} = \sum_{j=0}^{n}\binom{k-m}{j}\binom{m}{n-j} \quad .$$

This can be found in Chapter 1 of [8].

If the generating function for $\{a_n\}$ is known, it is sometimes desirable to convert it to the generating function $\{a_{n+k}\}$ as follows. If

$$A(x) = \sum_{n=0}^{\infty} a_n x^n ,$$

then

$$\frac{A(x) - a_0}{x} = \sum_{n=0}^{\infty} a_{n+1} x^n .$$

This can be repeated as often as needed to obtain the generating function for $\{a_{n+k}\}$.

Generating functions are a powerful tool in the theory of linear recurring sequences and the solution of linear difference equations. As an example we shall solve completely a second-order difference equation using the technique of generating functions. Let $\{c_n\}$ be a sequence of real numbers which obey

$$c_{n+2} - pc_{n+1} + qc_n = 0 , \quad n \geq 0 ,$$

where c_0 and c_1 are arbitrary. Then by using the Cauchy product we find

$$(1 - px + qx^2) \sum_{n=0}^{\infty} c_n x^n = c_0 + (c_1 - pc_0)x + 0 \cdot x^2 + \ldots = c_0 + (c_1 - pc_0)x$$

so that

$$\text{(2.4)} \qquad \sum_{n=0}^{\infty} c_n x^n = \frac{c_0 + (c_1 - pc_0)x}{1 - px + qx^2} .$$

Suppose a and b are the roots of the auxiliary polynomial $x^2 - px + q$, so the denominator of the generating function factors as $(1 - ax)(1 - bx)$. We divide the treatment into two cases, namely, $a \neq b$ and $a = b$.

If a and b are distinct (i.e., $p^2 - 4q \neq 0$), we may split the generating function into partial functions, giving

$$\text{(2.5)} \qquad \frac{c_0 + (c_1 - pc_0)x}{1 - px + qx^2} = \frac{c_0 + (c_1 - pc_0)x}{(1 - ax)(1 - bx)} = \frac{A}{1 - ax} + \frac{B}{1 - bx}$$

for some constants A and B. Then using (2.1) we find

$$\sum_{n=0}^{\infty} c_n x^n = A \sum_{n=0}^{\infty} a^n x^n + B \sum_{n=0}^{\infty} b^n x^n = \sum_{n=0}^{\infty} (Aa^n + Bb^n) x^n ,$$

so that an explicit formula for c_n is

$$c_n = Aa^n + Bb^n . \tag{2.6}$$

Here A and B can be determined from the initial conditions resulting from assigning values to c_0 and c_1 .

On the other hand, if the roots are equal (i.e., $p^2 - 4q = 0$), the situation is somewhat different because the partial fraction expansion (2.5) is not valid. Letting $c_0 + (c_1 - pc_0)x = r + sx$, we may use (2.2), however, to find

$$\sum_{n=0}^{\infty} c_n x^n = \frac{r + sx}{(1 - ax)^2} = (r + sx) \sum_{n=0}^{\infty} (n + 1) a^n x^n$$

$$= \sum_{n=0}^{\infty} [r(n + 1)a^n + sna^{n-1}] x^n = \sum_{n=0}^{\infty} [(r + s/a)n + r] a^n x^n ,$$

showing that

$$c_n = (An + B) a^n ,$$

where

$$A = r + s/a , \qquad B = r$$

are constants which again can be determined from the initial values c_0 and c_1.

This technique can be easily extended to recurring sequences of higher order. For further developments, the reader is referred to Jeske [6], where a generalized version of the above is derived in another way. For a discussion of the general theory of generating functions, see Chapter 2 of [8] and Chapter 3 of [2].

3. APPLICATIONS TO FIBONACCI NUMBERS

The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} - F_{n+1} - F_n = 0$, $n \geq 0$. Using the general solution of the second-order difference equation given above, where $p = 1$, $q = -1$, $c_0 + (c_1 - pc_0)x = x$, we find that the generating function for the Fibonacci numbers is

$$(3.1) \qquad F(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n \quad .$$

The reader should actually divide out the middle part of (3.1) by long division to see that Fibonacci numbers really do appear as coefficients.

Since the roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ of the auxiliary polynomial $x^2 - x - 1$ are distinct, we see from (2.6) that

$$(3.2) \qquad F_n = A\alpha^n + B\beta^n \; .$$

Putting $n = 0, 1$ and solving the resulting system of equations shows that

$$A = 1/\sqrt{5} = 1/(\alpha - \beta), \qquad B = -1/\sqrt{5} \quad ,$$

establishing the familiar Binet form,

$$(3.3) \qquad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad .$$

We shall now turn around and use this form to derive the original generating function (3.1) by using a technique first exploited by H. W. Gould [5]. Suppose that some sequence $\{a_n\}$ has the generating function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \quad .$$

Then

$$(3.4) \qquad \frac{A(\alpha x) - A(\beta x)}{\alpha - \beta} = \sum_{n=0}^{\infty} a_n \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) x^n = \sum_{n=0}^{\infty} a_n F_n x^n \quad .$$

In particular, if $a_n = 1$, then $A(x) = 1/(1 - x)$, so that

$$F(x) = \frac{1}{\alpha - \beta} \left(\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x} \right) = \frac{x}{1 - x - x^2} \quad .$$

Next we use (3.1) to prove that the Fibonacci numbers are the sums of terms along the rising diagonals of Pascal's Triangle. We write

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2} = \frac{x}{1 - (x + x^2)} = x \sum_{n=0}^{\infty} x^n (1 + x)^n$$

$$= \sum_{n=0}^{\infty} x^{n+1} \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{n+k+1}$$

$$= \sum_{m=1}^{\infty} \sum_{j=0}^{[(m-1)/2]} \binom{m - j - 1}{j} x^m ,$$

where [m] denotes the greatest integer contained in m. The inner sum is the sum of coefficients of x^m in the preceding sum, and the upper limit of summation is determined by the inequality $m - j - 1 < j$, recalling (2.3). The reader is urged to carry through the details of this typical generating function calculation. Equating coefficients of x^n shows that

$$F_n = \sum_{j=0}^{[(n-1)/2]} \binom{n - j - 1}{j} \tag{3.5}$$

linking the Fibonacci numbers to the binomial coefficients.

It follows from (3.1) upon division by x that

$$G(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n . \tag{3.6}$$

Differentiating this yields

$$G'(x) = \frac{2x + 1}{(1 - x - x^2)^2} = \left(\frac{1}{1 - x - x^2}\right)\left(\frac{1 + 2x}{1 - x - x^2}\right) = \sum_{n=0}^{\infty} (n + 1) F_{n+2} x^n .$$

Now

$$\frac{1 + 2x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_{n+1} x^n ,$$

where L_n are the Lucas numbers defined by $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$, $n \geq 0$. Hence

$$G'(x) = \left(\sum_{n=0}^{\infty} F_{n+1}x^n\right)\left(\sum_{n=0}^{\infty} L_{n+1}x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} F_{n-k+1}L_{k+1}\right)x^n ,$$

so that

$$\sum_{k=0}^{n} F_{n-k+1}L_{k+1} = (n+1)F_{n+2} ,$$

a convolution of the Fibonacci and Lucas sequences.

We leave it to the reader to verify that

$$\frac{x}{(1-x)(1-x-x^2)} = \frac{x}{1-2x+x^3} = \sum_{n=0}^{\infty}(F_{n+2}-1)x^n .$$

Also

$$\frac{x}{(1-x)(1-x-x^2)} = \frac{1}{1-x}\cdot\frac{x}{1-x-x^2} = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} F_n x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} F_j\right)x^n .$$

Equating coefficients shows

$$\sum_{j=0}^{n} F_j = F_{n+2} - 1 ,$$

which is really the convolution of the Fibonacci sequence with the constant sequence $\{1, 1, 1, \ldots\}$.

Consider the sequence $\{F_{kn}\}_{n=0}^{\infty}$, where $k \neq 0$ is an arbitrary but fixed integer. Since

$$F_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta}$$

we have

$$\sum_{n=0}^{\infty} F_{kn}x^n = \frac{1}{\alpha - \beta}\left(\sum_{n=0}^{\infty} \alpha^{kn}x^n - \sum_{n=0}^{\infty} \beta^{kn}x^n\right) = \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha^k x} - \frac{1}{1-\beta^k x}\right)$$

$$(3.7) \qquad = \frac{1}{\alpha-\beta}\cdot\frac{(\alpha^k - \beta^k)x}{1-(\alpha^k+\beta^k)x+(\alpha^k\beta^k)x^2} = \frac{F_k x}{1 - L_k x + (-1)^k x^2} ,$$

where we have used $\alpha\beta = -1$ and the Binet form $L_n = \alpha^n + \beta^n$ for the Lucas numbers. Incidentally, since here the integer in the numerator must divide all coefficients in the expansion, we have a quick proof that F_k divides F_{nk} for all n. A generalization of (3.7) is given in equation (4.18) of Section 4.

We turn now to generating functions for powers of the Fibonacci numbers. First we expand

$$F_n^2 = \left(\frac{\alpha^n - \beta^n}{\alpha-\beta}\right)^2 = \frac{1}{(\alpha-\beta)^2}\,(\alpha^{2n} - 2(\alpha\beta)^n + \alpha^{2n}) .$$

Then

$$\sum_{n=0}^{\infty} F_n^2 x^n = \frac{1}{(\alpha-\beta)^2}\left(\sum_{n=0}^{\infty}\alpha^{2n}x^n - 2\sum_{n=0}^{\infty}(\alpha\beta)^n x^n + \sum_{n=0}^{\infty}\beta^{2n}x^n\right)$$

$$= \frac{1}{(\alpha-\beta)^2}\left(\frac{1}{1-\alpha^2 x} - \frac{2}{1-\alpha\beta x} + \frac{1}{1-\beta^2 x}\right)$$

$$= \frac{x - x^2}{(1-\alpha^2 x)(1-\alpha\beta x)(1-\beta^2 x)} = \frac{x-x^2}{1-2x-2x^2+x^3}$$

This also shows that $\{F_n^2\}$ obeys

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0 .$$

We remark that Gould's technique (3.4) may be applied to F(x), leading to exactly the same result.

In general, to find the generating function for the pth power of the Fibonacci numbers, first expand F_n^p by the binomial theorem. This gives F_n^p as a linear combination of α^{np}, $\alpha^{n(p-1)}\beta^n$, ..., $\alpha^n\beta^{n(p-1)}$, β^{np} so that as above the generating function will have the denominator

$$(1 - \alpha^p x)(1 - \alpha^{p-1}\beta x) \cdots (1 - \alpha\beta^{p-1}x)(1 - \beta^p x) \quad .$$

Fortunately, this product can be expressed in a better way. Define the Fibonomial coefficients $\begin{bmatrix} k \\ r \end{bmatrix}$ by

$$\begin{bmatrix} k \\ r \end{bmatrix} = \frac{F_k F_{k-1} \cdots F_{k-r+1}}{F_1 F_2 \cdots F_r} \quad , \quad r > 0; \qquad \begin{bmatrix} k \\ 0 \end{bmatrix} = 1 \quad .$$

Then it has been shown [7] that

$$Q_p(x) = \prod_{j=0}^{p} (1 - \alpha^{p-j}\beta^j x) = \sum_{j=0}^{p+1} (-1)^{j(j+1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix} x^j \quad .$$

For example,

$$Q_1(x) = 1 - x - x^2$$

$$Q_2(x) = 1 - 2x - 2x^2 + x^3$$

$$Q_3(x) = 1 - 3x - 6x^2 + 3x^3 + x^4$$

$$Q_4(x) = 1 - 5x - 15x^2 + 15x^3 + 5x^4 - x^5$$

Since any sequence obeying the Fibonacci recurrence relation can be written in the form $A\alpha^n + B\beta^n$, $Q_p(x)$ is the denominator of the generating function of the pth power of any such sequence. The numerators of the generating functions can be found by simply multiplying through by $Q_p(x)$. For example, to find the generating function of $\{F^2_{n+2}\}$, we have

$$\sum_{n=0}^{\infty} F^2_{n+2} x^n = \frac{r(x)}{1 - 2x - 2x^2 + x^3} \quad .$$

Then $r(x)$ can be found by multiplying by $Q_2(x)$, giving

$$r(x) = (1 - 2x - 2x^2 + x^3)(1 + 4x + 9x^2 + 25x^4 + \ldots)$$

$$= 1 + 2x - x^2 + 0 \cdot x^3 + \ldots = 1 + 2x - x^2 \quad .$$

This is (4.7) of Section 4. However, for fixed p, once we have obtained the generating functions for $\{F_n^p\}$, $\{F_{n+1}^p\}$, ... , $\{F_{n+p}^p\}$, the one for $\{F_{n+k}^p\}$ follows directly from the identity of Hoggatt and Lind [4]

(3.8) $$F_{n+k}^p = \sum_{j=0}^{p} (-1)^{(p-j)(p-j+3)/2} \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} p \\ j \end{bmatrix} \left(\frac{F_{k-p}}{F_{k-j}}\right) F_{n+j}^p ,$$

where we use the convention $F_0/F_0 = 1$. For example, for p = 1 this gives

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n .$$

Using the generating function for $\{F_{n+1}\}$ in (3.4) and $\{F_n\}$ in (3.1),

$$\sum_{n=0}^{\infty} F_{n+k} x^n = F_k \sum_{n=0}^{\infty} F_{n+1} x^n + F_{k-1} \sum_{n=0}^{\infty} F_n x^n = \frac{F_k + F_{k-1} x}{1 - x - x^2} .$$

In fact, one of the main purposes for deriving (3.5) was to express the generating function of $\{F_{n+k}^p\}$ as a linear combination of those of $\{F_n^p\}$, ..., $\{F_{n+p}^p\}$.

Alternatively, to obtain the generating function of $\{F_{n+k}^p\}$ from that of $\{F_n^p\}$, we could apply k times in succession the technique mentioned in Section 2 for finding the generating function of $\{a_{n+1}\}$ from that of $\{a_n\}$.

The generating function of powers of the Fibonacci numbers have been investigated by several authors (see [3], [5], and [7]).

4. SOME STANDARD GENERATING FUNCTIONS

We list here for reference some of the generating functions we have already derived along with others which can be established in the same way.

(4.1) $$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n$$

(4.2) $$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n$$

(4.3) $$\frac{2 - x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_n x^n$$

(4.4) $$\frac{1 + 2x}{1 - x - x^2} = \sum_{n=0}^{\infty} L_{n+1} x^n$$

(4.5) $$\frac{x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_n^2 x^n$$

(4.6) $$\frac{1 - x}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_{n+1}^2 x^n$$

(4.7) $$\frac{1 + 2x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_{n+2}^2 x^n$$

(4.8) $$\frac{x}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_n F_{n+1} x^n$$

(4.9) $$\frac{4 - 7x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_n^2 x^n$$

(4.10) $$\frac{1 + 7x - 4x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_{n+1}^2 x^n$$

(4.11) $$\frac{9 - 2x - x^2}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} L_{n+2}^2 x^n$$

(4.12) $$\frac{x - 2x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_n^3 x^n$$

(4.13) $$\frac{1 - 2x - x^2}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+1}^3 x^n$$

(4.14) $$\frac{1 + 5x - 3x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+2}^3 x^n$$

(4.15) $$\frac{8 + 3x - 4x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_{n+3}^3 x^n$$

(4.16) $$\frac{2x}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{n=0}^{\infty} F_n F_{n+1} F_{n+2} x^n$$

(4.17) $$\frac{F_k x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn} x^n$$

(4.18) $$\frac{F_r + (-1)^r F_{k-r} x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn+r} x^n$$

Many thanks to Kathleen Weland and Allan Scott.

REFERENCES

1. S. L. Basin and V. E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence, Part I," The Fibonacci Quarterly, Vol. 1, No. 1, Feb., 1963, pp. 65-72; Part II, Vol. 1, No. 2, April, 1963, pp. 61-68.
2. Edwin F. Beckenbach, editor, Applied Combinatorial Mathematics, Wiley, 1964.
3. L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," Duke Math. Journal, Vol. 29, 1962, pp. 521-538.
4. V. E. Hoggatt, Jr., and D. A. Lind, " A Power Identity for Second-Order Recurrent Sequences," The Fibonacci Quarterly, Vol. 4, 1966, pp. 274-282.
5. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 1-16.
6. James A. Jeske, "Linear Recurrence Relations: Part I," The Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 69-74.
7. John Riordan, "Generating Functions for Powers of Fibonacci Numbers", Duke Math. Journal, Vol. 29, 1962, pp. 5-12.
8. John Riordan, An Introduction to Combinatorial Analysis, Wiley, 1960.

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AN EASY PROBLEM

B-14 (Proposed by Maxey Brooke and C. R. Wall) Show that

$$\sum_{n=1}^{\infty} \frac{F_n}{10^n} = \frac{10}{89} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} F_n}{10^n} = \frac{10}{109}$$

SCOTT'S FIBONACCI SCRAPBOOK

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The following generating functions are submitted to continue the list in "A Primer for the Fibonacci Numbers: Part VI". Thanks to Kathleen Weland for verifying these.

$$\sum_{n=0}^{\infty} L_{n+k}^3 x^n = \frac{P_k(x)}{1 - 3x - 6x^2 + 3x^3 + x^4}, \quad k = 0, 1, 2, 3$$

$P_0(x) = 8 - 23x - 24x^2 + x^3$

$P_1(x) = 1 + 24x - 23x^2 - 8x^3$

$P_2(x) = 27 - 17x - 11x^2 - x^3$

$P_3(x) = 64 + 151x - 82x^2 - 27x^3$

$$\sum_{n=0}^{\infty} F_{n+k}^4 x^n = \frac{P_k(x)}{1 - 5x - 15x^2 + 15x^3 + 5x^4 - x^5}, \quad k = 0, 1, 2, 3, 4$$

$P_0(x) = x - 4x^2 - 4x^3 + x^4$

$P_1(x) = 1 - 4x - 4x^2 + x^3$

$P_2(x) = 1 + 11x - 14x^2 - 5x^3 + x^4$

$P_3(x) = 16 + x - 20x^2 - 4x^3 + x^4$

$P_4(x) = 81 - 220x - 244x^2 - 79x^3 + 16x^4$

(Generating functions for $\{F_{n+k}^5\}$, k = 0, 1, 2, 3, 4, 5 ; $\{F_{n+k}^6\}$, k = 0, 1, 2, 3, 4, 5, 6; and $\{F_{n+k}^7\}$, k = 0, 1, 2, 3, 4, 5, 6, 7 are given in this entire article, which appears in The Fibonacci Quarterly, Vol. 6, No. 2, April, 1968, pages 176, 191, and 166.)

A MOTIVATION FOR CONTINUED FRACTIONS

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This Quarterly is devoted to the study of properties of integers, especially to the study of recurrent sequences of integers. We show below how such sequences and continued fractions arise naturally in the problem of approximating an irrational number to any desired closeness by rational numbers.

We begin with the equation

$$x^2 - x - 1 = 0 \quad . \tag{1}$$

One can easily see that there is a negative root between -1 and 0 and a positive root between 1 and 2, for example by graphing $y = x^2 - x - 1$. We call the positive root r. This number has been known since antiquity as the "golden mean." We now look for a sequence of rational approximations to r.

A rational number is of the form p/q with p and q integers (and $q \neq 0$). We therefore wish two sequences

$$p_1, p_2, p_3, \ldots \qquad \text{and} \qquad q_1, q_2, q_3, \ldots \tag{2}$$

of integers such that the quotients p_n/q_n are approximations which get arbitrarily close to r. It would also be helpful if each new approximation were obtainable simply from previous ones.

We go back to (1) and rewrite it as

$$x = 1 + \frac{1}{x} \quad . \tag{3}$$

This states that if we replace x by r in

$$1 + \frac{1}{x} \tag{4}$$

the result is r and suggests that if we replace x in (4) by an approximation to r we will get another approximation. We now change (3) into the form

$$x_2 = 1 + \frac{1}{x_1} \tag{5}$$

and consider x_1 to be an approximation to r. The relative error of $1/x_1$ is the same as that of x_1 and, if x_1 is positive, the relative error of x_2 (i. e., $1 + 1/x_1$) is lower than that of x_1, since adding 1 increases the number but not the error. It can be shown that x_2 in (5) is a better approximation to r than x_1, if $x_1 > 0$.

We now let our first approximation x_1 be a rational number p_1/q_1 and substitute this in (5) obtaining

$$x_2 = 1 + \frac{1}{(p_1/q_1)} = 1 + \frac{q_1}{p_1} = \frac{p_1 + q_1}{p_1} .$$

We therefore choose p_2 to be $p_1 + q_1$ and q_2 to be p_1 . Similarly, our third approximation is p_3/q_3 with $p_3 = p_2 + q_2$ and $q_3 = p_2$. In general, the (n + 1)st approximation p_{n+1}/q_{n+1} has

(6) $$p_{n+1} = p_n + q_n$$

(7) $$q_{n+1} = p_n \quad .$$

It follows from (7) that $q_n = p_{n-1}$; substituting this in (6) gives

(8) $$p_{n+1} = p_n + p_{n-1} \quad .$$

Since r is between 1 and 2 we use 1 as the first approximation, i.e., we let $p_1 = q_1 = 1$. This means that $p_2 = 2$ and it now follows from (8) that p_n is the Fibonacci number F_{n+1} . Then (7) implies that $q_n = F_n$ and we see that the sequence of quotients F_{n+1}/F_n of consecutive Fibonacci numbers furnishes the desired approximations to the root r of (1). It can be shown that this sequence converges to r in the calculus sense.

We next consider the problem of approximating $s = \sqrt{10}$ in this way. The number s is the positive root of

(9) $$x^2 - 10 = 0 \quad .$$

We write (9) in the forms

$$x^2 - 9 = 1$$

(10)
$$(x - 3)(x + 3) = 1$$
$$(x - 3) = 1/(x + 3)$$
$$x = 3 + 1/(x + 3)$$

and change (10) into

(11)
$$x_{n+1} = 3 + \frac{1}{3 + x_n} \quad .$$

Again, if x_n is a positive approximation to s , it can be seen that x_{n+1} is an approximation with smaller relative error. There is a sequence of rational approximations p_n/q_n with

$$p_{n+1} = 3p_n + 10q_n \;, \qquad q_{n+1} = p_n + 3q_n \quad .$$

Letting the first approximation be 3, i.e., letting $p_1 = 3$ and $q_1 = 1$, we obtain the sequence

$$3/1, \;\; 19/6, \;\; 117/37, \;\; \ldots$$

which can be shown to converge to s.

Equation (11) contains the equations

$$x_2 = 3 + \frac{1}{3 + x_1} \;, \qquad x_3 = 3 + \frac{1}{3 + x_2} \quad .$$

Substituting the first of these into the second gives us

$$x_3 = 3 + \cfrac{1}{6 + \cfrac{1}{3 + x_1}} \quad .$$

If this is substituted into $x_4 = 3 + 1/(3 + x_3)$ and if we let x_1 be 3, we obtain

$$x_4 = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \frac{1}{6}}} \quad .$$

In this way we can write continued fraction expressions for any one of the x_n . Then it is natural to let the infinite continued fraction

$$3 + \cfrac{1}{6 + \cfrac{1}{6 + \cdots}}$$

represent the limit s of the sequences x_n defined by (11) and $x_1 = 3$.

The infinite continued fraction for the root r of $x^2 - x - 1 = 0$ is

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} ,$$

whose elegant simplicity is worthy of the title "golden mean."

* * * * * * * * * *

A CURIOUS FORMULA FOR THE GOLDEN SECTION RATIO

A curious formula which relates the Golden Section Ratio $\phi = \frac{1 + \sqrt{5}}{2}$, the imaginary unit $i = \sqrt{-1}$, and e, the base of natural logarithms, is

$$\phi = 2 \cos \left(\frac{\log_e (i^2)}{5i} \right) ;$$

can you prove it? (See J. A. H. Hunter and Joseph S. Madachy, Mathematical Diversions, D. Van Nostrand, Princeton, New Jersey, 1963. Pp. 14-19.)

Formulas also relate the Golden Section Ratio ϕ to trigonometric functions. (See Bicknell and Hoggatt, "Golden Triangles, Rectangles, and Cuboids", pages 75 and 76.) It can be proved that $\sin 18° = 1/2\phi$ and that $\sin 54° = \phi/2$.

Another interesting formula follows which is related to the first problem.

B-18 (Proposed by J. L. Brown, Jr.) Show that

$$F_n = 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1} \frac{\pi}{5} \sin^k \frac{\pi}{10} , \quad \text{for } n \geq 1 .$$

THE GOLDEN RATIO: COMPUTATIONAL CONSIDERATIONS

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1. INTRODUCTION

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel"-- so wrote Kepler (1571-1630)[1].

The famous <u>golden section</u> involves the division of a given line segment into mean and extreme ratio, i.e., into two parts a and b, such that $a/b = b/(a + b)$, $a < b$. Setting $x = b/a$ we have $x^2 - x - 1 = 0$. Let us designate the positive root of this equation by ϕ (the <u>golden ratio</u>). Thus

$$\phi^2 - \phi - 1 = 0 \quad . \tag{1}$$

Since the roots of (1) are $\phi = (1 + \sqrt{5})/2$ and $-1/\phi = (1 - \sqrt{5})/2$ we may write Binet's formula [2] for the nth Fibonacci number in the form

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \tag{2}$$

2. POWERS OF THE GOLDEN RATIO

Returning to (1), let us "solve for ϕ^2" by writing

$$\phi^2 = \phi + 1 \quad . \tag{3}$$

Multiplying both members by ϕ, we get $\phi^3 = \phi^2 + \phi = (\phi + 1) + \phi = 2\phi + 1$. Proceeding in a similar fashion we can write all of

$$\phi^3 = 2\phi + 1 \; ,$$

$$\phi^4 = 3\phi + 2 \; ,$$

$$\phi^5 = 5\phi + 3 \; .$$

This pattern suggests

(4) $$\phi^n = F_n \phi + F_{n-1}, \quad n = 1, 2, 3, \ldots .$$

To prove (4) by mathematical induction, we note that it is true for $n = 1$ and $n = 2$ (since $F_0 = 0$ by definition). Assume that $\phi^k = F_k\phi + F_{k-1}$. Then

$$\phi^{k+1} = F_k\phi^2 + F_{k-1}\phi = F_k(\phi + 1) + F_{k-1}\phi$$
$$= (F_k + F_{k-1})\phi + F_k = F_{k+1}\phi + F_k ,$$

which completes the proof.

The computational advantage of (4) over expansion of $\left(\frac{1+\sqrt{5}}{2}\right)^n$ by the binomial theorem is striking.

Dividing both members of (3) by ϕ, we obtain

(5) $$\frac{1}{\phi} = \phi - 1 .$$

Thus $1/\phi^2 = 1 - 1/\phi = 1 - (\phi - 1) = -(\phi - 2)$. Using this result and (5), $1/\phi^3 = 2/\phi - 1 = 2(\phi - 1) - 1 = 2\phi - 3$. Proceeding in a similar fashion, one may write all of the following:

$$\frac{1}{\phi^2} = -(\phi - 2) ,$$

$$\frac{1}{\phi^3} = 2\phi - 3 ,$$

$$\frac{1}{\phi^4} = -(3\phi - 5) .$$

Via induction, the reader may provide a painless proof of

(6) $$\phi^{-n} = (-1)^{n+1}(F_n\phi - F_{n+1}) , \quad n = 1, 2, 3, \ldots$$

3. A LIMIT OF FIBONACCI RATIOS

If we "solve" $x^2 - x - 1 = 0$ for x by writing $x = 1 + 1/x$ and then consider the related recursion relation

(7) $$x_1 = 1 , \qquad x_{n+1} = 1 + \frac{1}{x_n} ,$$

Fibonacci numbers start popping out! We immediately deduce $x_2 = 1 + 1/x_1 = 1 + 1/1 = 2/1$, $x_3 = 3/2$, $x_4 = 5/3$, $x_5 = 8/5$, etc. This suggests that $x_n = F_{n+1}/F_n$.

Now suppose the sequence $x_1, x_2, x_3, \ldots$ has a limit, say L, as n tends toward infinity. Then

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = L$$

whence (7) yields $L = 1 + 1/L$ or $L = \phi$ since the x_i are positive. Indeed, there are many ways of proving Kepler's observation that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi \quad . \tag{8}$$

For example, from (2)

$$\frac{F_{n+1}}{F_n} = \frac{\phi^{n+1} - (-\phi)^{-n-1}}{\phi^n - (-\phi)^{-n}} = \frac{\phi - \dfrac{1}{(-\phi)^{n+1}\phi^n}}{1 - \dfrac{1}{(-\phi)^n\phi^n}} \longrightarrow \phi$$

as $n \to \infty$ since $\phi = (1 + \sqrt{5})/2 > 1$ implies that the fractions involving ϕ^n approach 0 as $n \to \infty$.

4. AN APPROXIMATE ERROR ANALYSIS

Just how accurate are the above approximations to the golden ratio? Let us denote the exact error at the nth iteration by

$$e_n = x_n - \phi \tag{9}$$

The trick is to express e_{n+1} in terms of e_n using (7) and then to make use of the identity

$$\frac{1}{1 + w} = 1 - w + w^2 - w^3 + w^4 - \ldots , \qquad w < 1 \quad . \tag{10}$$

(The latter may be discovered by dividing 1 by 1 + w).

Thus

$$e_{n+1} = x_{n+1} - \phi$$

$$= 1 + \frac{1}{x_n} - \phi$$

$$= 1 - \phi + \frac{1}{e_n + \phi}$$

$$= 1 - \phi + \frac{1}{\phi} \cdot \frac{1}{1 + (e_n/\phi)}$$

$$= 1 - \phi + \frac{1}{\phi} [1 - (e_n/\phi) + (e_n/\phi)^2 - (e_n/\phi)^3 + \ldots]$$

$$= - \frac{e_n}{\phi^2} + \frac{e_n}{\phi^3} - \frac{e_n}{\phi^4} + \ldots$$

since $1/\phi = \phi - 1$ by (5). However, the terms involving the higher powers of e_n are quite small in comparison with the first term. Thus, following the customary practice of neglecting high order terms, we will approximate the error at the (n + 1)st step by $e_{n+1} = -e_n\phi^{-2}$. Finally, we may note that $e_2 = -e_1\phi^{-2}$, $e_3 = - e_2\phi^{-2} = +e_1\phi^{-4}$, $e_4 = -e_1\phi^{-6}$, and, in general,

(11) $$e_n = (-1)^{n+1}e_1\phi^{-2(n-1)} .$$

If $x_1 = 1$, then $e_1 = 1 - \phi = -1/\phi$ by (9) and (5), making (11) become

(12) $$e_n = (-1)^n \phi^{-2(n-1)-1} .$$

(Sections 5 and 6 of the original paper are omitted here.)

REFERENCES

1. H. S. M. Coxeter, Introduction to Geometry, John Wiley and Sons, Inc., New York, 1961, p. 160.
2. S. L. Basin and Verner E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence--Part II," Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 61-68.

GOLDEN TRIANGLES, RECTANGLES, AND CUBOIDS

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1. INTRODUCTION

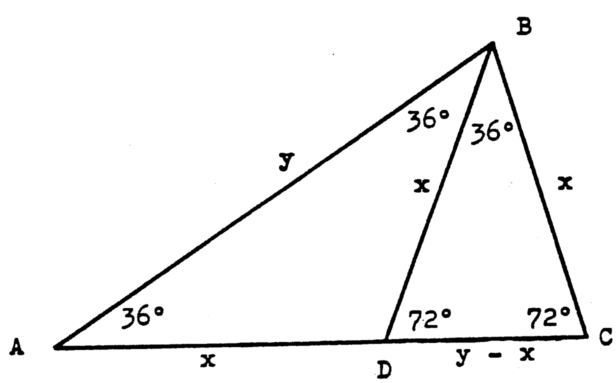
One of the most famous of all geometric figures is the Golden Rectangle, which has the ratio of length to width equal to the Golden Section,

$$\phi = (1 + \sqrt{5})/2 .$$

The proportions of the Golden Rectangle appear consistently throughout classical Greek art and architecture. As the German psychologists Fechner and Wundt have shown in a series of psychological experiments, most people do unconsciously favor "golden dimensions" when selecting pictures, cards, mirrors, wrapped parcels, and other rectangular objects. For some reason not fully known by either artists or psychologists, the Golden Rectangle holds great aesthetic appeal. Surprisingly enough, the best integral lengths to use for sides of an approximation to the Golden Rectangle are adjacent members of the Fibonacci series: 1, 1, 2, 3, 5, 8, 13, ..., and we find 3×5 and 5×8 filing cards, for instance.

Suppose that, instead of a Golden Rectangle, we study a golden section triangle. If the ratio of a side to the base is $\phi = (1 + \sqrt{5})/2$, then we will call the triangle a Golden Triangle. (See [2], [3].)

Now, consider the isosceles triangle with a vertex angle of 36°. On bisecting the base angle of 72°, two isosceles triangles are formed, and ΔBDC is similar to ΔABC as indicated in the figure:



Since ΔABC is similar to ΔBDC,

$$\frac{AB}{BD} = \frac{BC}{DC}, \qquad \text{or} \qquad \frac{y}{x} = \frac{x}{y - x},$$

so that

$$y^2 - yx - x^2 = 0 \ .$$

Dividing through by $x^2 \neq 0$,

$$\frac{y^2}{x^2} - \frac{y}{x} - 1 = 0 \ .$$

The quadratic equation gives

$$\frac{y}{x} = (1 + \sqrt{5})/2 = \phi$$

as the positive root, so that ΔABC is a Golden Triangle. Notice also, that, using the common altitude from B, the ratio of the area of ΔABC to ΔADB is ϕ .

Since the central angle of a regular decagon is 36°, ΔABC above shows that the ratio of the radius y to the side x of an inscribed decagon is ϕ . Also, in a regular pentagon, the angle at a vertex between two adjacent diagonals is 36°. By reference to the figure above, the ratio of a diagonal to a side of a regular pentagon is also ϕ.

2. A TRIGONOMETRIC PROPERTY OF THE ISOSCELES GOLDEN TRIANGLE

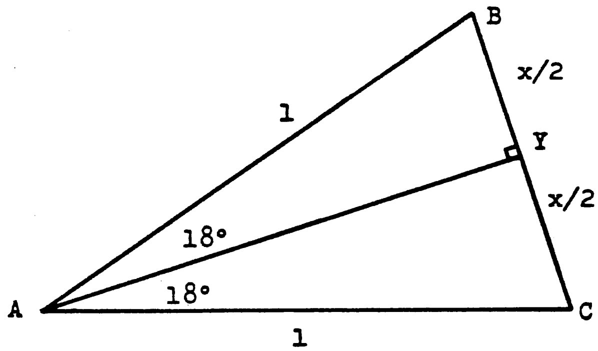
The Golden Triangle with vertex angle 36° can be used for a surprising trigonometric application. Few of the trigonometric functions of an acute angle have values which can be expressed exactly. Usually, a method of approximation is used; most values in trigonometric tables cannot be expressed exactly as terminating decimals, repeating decimals, or even square roots, since they are approximations to transcendental numbers, which are numbers so irrational that they are not the root of any polynomial over the integers.

The smallest positive integral number of degrees for which the trigonometric functions of the angle can be expressed exactly is three degrees. Then, all multiples of 3° can also be expressed exactly by repeatedly using formulas such as sin(A + B). Strangely enough, the Golden Triangle can be used to derive the value of sin 3° .

In our Golden Triangle, the ratio of the side to the base was

$$y/x = (1 + \sqrt{5})/2 \ .$$

Suppose we let AB = y = 1. Then $1/x = (1+\sqrt{5})/2$, or, $x = (\sqrt{5}-1)/2$. Redrawing the figure and bisecting the 36° angle,



we form a right triangle, ΔAYC, with $YC = x/2$. Then,

$$\sin 18° = \frac{YC}{AC} = \frac{x}{2} = \frac{\sqrt{5}-1}{4} = \frac{1}{2\phi} .$$

Since $\sin^2 A + \cos^2 A = 1$,

$$\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{\sqrt{\sqrt{5}\,\phi}}{2} .$$

Since $\sin(A-B) = \sin A \cos B - \sin B \cos A$,

$$\sin 15° = \sin(45° - 30°) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4} .$$

Similarly, using $\cos(A-B) = \cos A \cos B + \sin A \sin B$,

$$\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4} .$$

Again using the formula for $\sin(A-B)$,

$$\sin 3° = \sin(18° - 15°) = \frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}-\sqrt{2}}{4}\cdot\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$= \frac{1}{16}\left[(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - 2(\sqrt{3}-1)\left(\sqrt{5+\sqrt{5}}\right)\right]$$

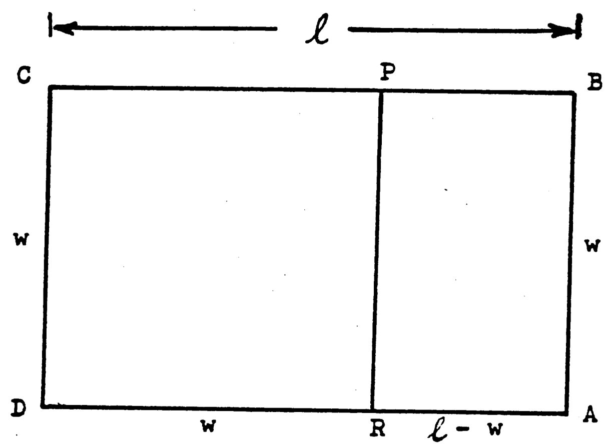
as given by Ransom in [1].

3. GOLDEN RECTANGLE AND GOLDEN TRIANGLE THEOREMS

While a common way to describe the Golden Rectangle is to give the ratio of length to width as $\phi = (1 + \sqrt{5})/2$, this ratio is a consequence of the geometric properties of the Golden Rectangle which are discussed in this section.

__Theorem__. Given that the ratio of length to width of a rectangle is $k > 1$. A square with side equal to the width can be removed to leave a rectangle similar to the original rectangle if and only if $k = (1 + \sqrt{5})/2$.

__Proof__. Let the square PCDR be removed from rectangle ABCD, leaving rectangle BPRA.



If rectangles ABCD and BPRA have the same ratio of length to width, then

$$k = \frac{w}{\ell - w} = \frac{\ell}{w} .$$

Cross-multiplying and dividing by $w^2 \neq 0$ gives a quadratic equation in $\frac{\ell}{w}$ which has $(1 + \sqrt{5})/2$ as its positive root. If

$$\frac{\ell}{w} = (1 + \sqrt{5})/2 = \phi ,$$

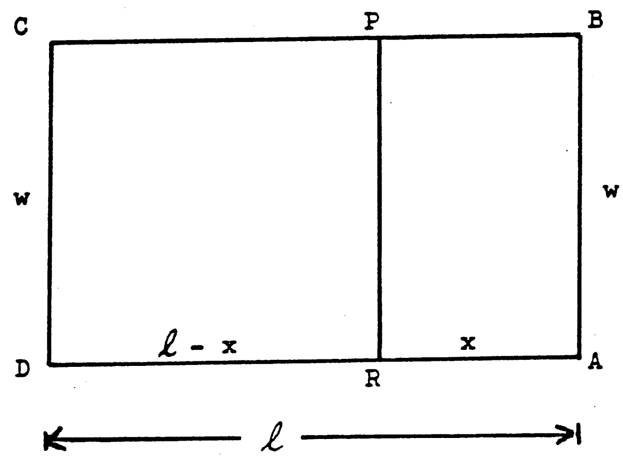
then

$$\frac{w}{\ell - w} = \frac{1}{\frac{\ell}{w} - 1} = \frac{1}{\phi - 1} = \phi$$

so that both rectangles have the same ratio of length to width.

<u>Theorem</u>. Given that the ratio of length to width of a rectangle is $k > 1$. A rectangle similar to the first can be removed to leave a rectangle such that the ratio of the areas of the original rectangle and the rectangle remaining is k, if and only if $k = (1 + \sqrt{5})/2$. Further, the rectangle remaining is a square.

<u>Proof</u>. Remove rectangle BPRA from rectangle ABCD as in the figure:



Then

$$\frac{\text{area ABCD}}{\text{area PCDR}} = \frac{\ell w}{w(\ell - x)} .$$

But,

$$\frac{\ell w}{w(\ell - x)} = \frac{\ell}{w} = k \quad \text{if and only if} \quad \frac{w}{\ell - x} = 1 ,$$

or $w = \ell - x$ or PCDR is a square. Thus, our second theorem is a consequence of the first theorem.

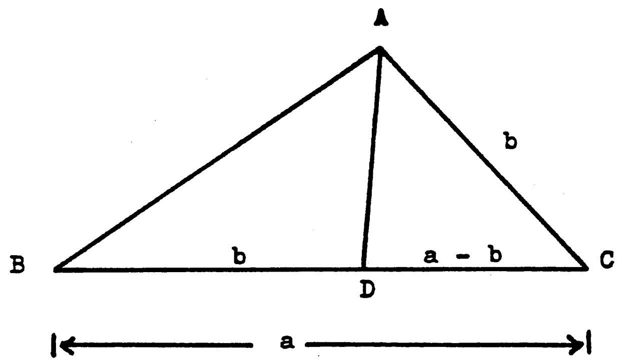
Analogous theorems hold for Golden Triangles.

<u>Theorem</u>. Given that the ratio of two sides a and b of a triangle is $a/b = k > 1$. A triangle with side equal to b can be removed to leave a triangle similar to the first if and only if $k = (1 + \sqrt{5})/2$.

<u>Proof</u>. Remove ΔABD from ΔABC. If ΔADC is similar to ΔBAC, then

$$\frac{AC}{BC} = \frac{DC}{AC} \qquad \text{or} \qquad \frac{b}{a} = \frac{a - b}{b} .$$

Cross-multiply, divide by $b^2 \neq 0$, and solve the quadratic in a/b to give $a/b = (1 + \sqrt{5})/2$ as the only positive root.



If $a/b = (1 + \sqrt{5})/2$, then

$$DC/AC = (a - b)/b = a/b - 1 = (\sqrt{5} - 1)/2$$

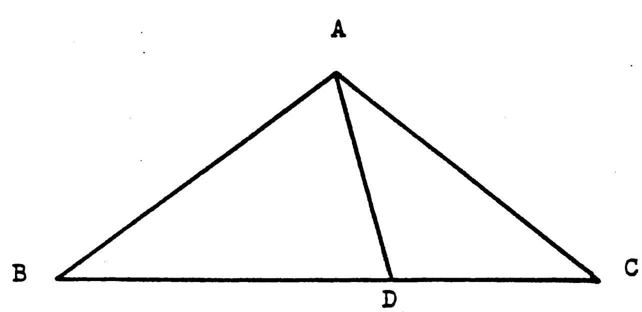
and

$$AC/BC = b/a = 2/(1 + \sqrt{5}) = (\sqrt{5} - 1)/2 = DC/AC .$$

Since $\angle C$ is in both triangles, ΔADC is similar to ΔBAC.

<u>Theorem</u>. Given that the ratio of two sides of a triangle is $k > 1$. A triangle similar to the first can be removed to leave a triangle such that the ratio of the areas of the original triangle and the triangle remaining is k, if and only if $k = (1 + \sqrt{5})/2$.

<u>Proof</u>. Let ΔADC be similar to ΔBAC, such that $BC/AC = AC/DC = k$.



If the ratio of areas of the original triangle and the one remaining is k, since there is a common altitude from A,

$$k = \frac{\text{area } \Delta BAC}{\text{area } \Delta BDA} = \frac{(BC)(h/2)}{(BC - DC)(h/2)} = \frac{BC/AC}{BC/AC - DC/AC} = \frac{k}{k - 1/k} .$$

Again cross-multiplying and solving the quadratic in k gives $k = (1 + \sqrt{5})/2$.

If $k = (1 + \sqrt{5})/2$, then

$$BC/AC = AC/DC = (1 + \sqrt{5})/2,$$

and the ratio of areas $BC/(BC - DC)$ becomes $(1 + \sqrt{5})/2$ upon dividing through by AC and then simply substituting the values of BC/AC and DC/AC.

If $k = (1 + \sqrt{5})/2 = BC/AC$, and the ratio of areas of ΔBAC and ΔBDA is also k, then

$$k = \frac{BC/AC}{BC/AC - DC/AC} = \frac{k}{k - x} ,$$

which leads to

$$x = k - 1 \qquad \text{or} \qquad DC/AC = (1 + \sqrt{5})/2 - 1 = 2/(1 + \sqrt{5})$$

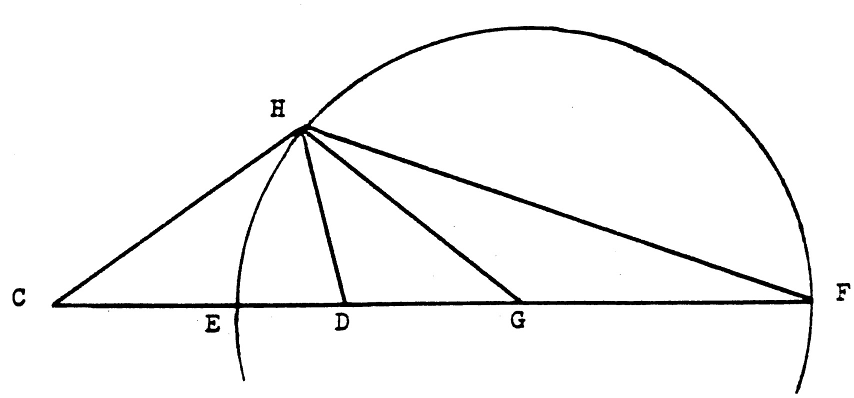
so that

$$AC/DC = (1 + \sqrt{5})/2$$

and ΔBAC is similar to ΔADC.

4. THE GENERAL GOLDEN TRIANGLE

Unlike the Golden Rectangle, the Golden Triangle does not have a unique shape. Consider a line segment $\overline{CD}$ of length $\phi = (1 + \sqrt{5})/2$. Place points E, G, and F on line $\overleftrightarrow{CD}$ such that CE = 1, EG = GF = ϕ as in the diagram.



Then, ED = $\emptyset - 1$, and

$$CE/ED = 1/(\emptyset - 1) = \emptyset ,$$

$$CF/DF = (2\emptyset + 1)/(\emptyset + 1) = \emptyset^3/\emptyset^2 = \emptyset ,$$

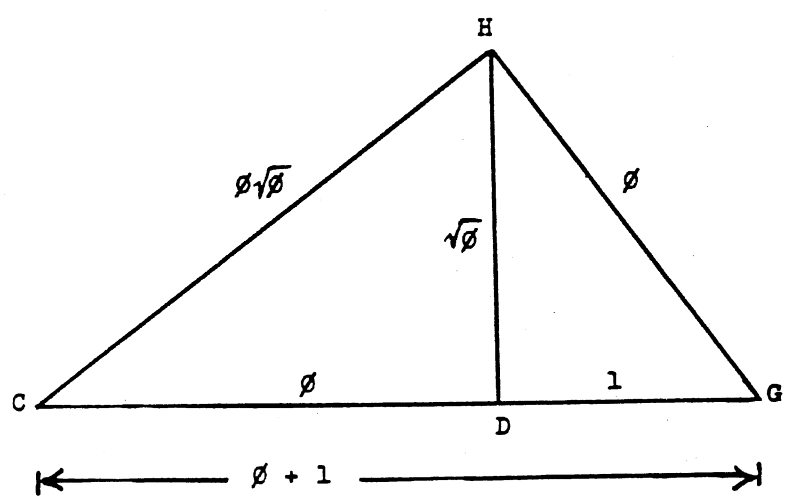
so that E and F divide segment $\overline{CD}$ internally and externally in the ratio $\emptyset$. Then the circle with center G is the circle of Apollonius for $\overline{CD}$ with ratio $\emptyset$. Incidentally, the circle through C, D, and H is orthogonal to the circle with center G and passing through H, and $\overline{HG}$ is tangent to the circle through C, D, and H .

Let H be any point on the circle of Apollonius. Then CH/HD = $\emptyset$, CG/HG = $\emptyset$, and ΔCHG is similar to ΔHDG. The area of ΔCHG is

$$h(1 + \emptyset)/2 = h\emptyset^2/2 ,$$

and when ΔHDG is removed, the area of the remaining ΔCHD is $h\emptyset/2$, so that the areas have the ratio $\emptyset$. Then, ΔCHG is a Golden Triangle, and there are an infinite number of Golden Triangles because H can take an infinite number of positions on circle G.

If we choose H so that CH = $\emptyset + 1$, then we have the isosceles 36-72-72 Golden Triangle of decagon fame. If we erect a perpendicular at D and let H be the intersection with the circle of Apollonius, then we have a right golden triangle by applying the Pythagorean theorem and its converse. In our right golden triangle ΔCHG, CH = $\emptyset\sqrt{\emptyset}$, HG = $\emptyset$, and CG = $\emptyset^2$. The two smaller right triangles formed by the altitude to $\overline{CG}$ are each similar to ΔCHG, so that all three triangles are golden. The areas of ΔHDG, ΔCDH, and ΔCHG form the geometric progression, $\sqrt{\emptyset}/2$, $(\sqrt{\emptyset}/2)\emptyset$, $(\sqrt{\emptyset}/2)\emptyset^2$.



Before going on, notice that the right golden triangle ΔCHG provides an unusual and surprising configuration. While two pairs of sides and all three pairs of angles of ΔCHG and ΔCDH are congruent, yet ΔCHG is not congruent to ΔCDH ! Similarly for ΔCDH and ΔHDG. (See Holt [4].)

5. THE GOLDEN CUBOID

H. E. Huntley [5] has described a Golden Cuboid (rectangular parallelopiped) with lengths of edges a, b, and c, such that

$$a : b : c = \phi : 1 : \phi^{-1} .$$

The ratios of the areas of the faces are

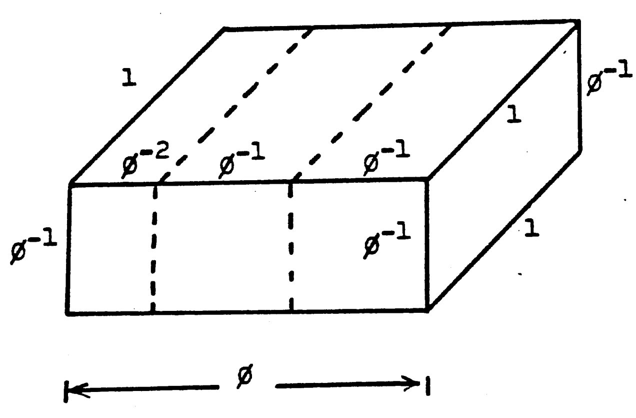
$$\phi : 1 : \phi^{-1} ,$$

and four of the six faces of the cuboid are Golden Rectangles.

If two cuboids of dimension

$$\phi^{-1} \times 1 \times \phi^{-1}$$

are removed from the Golden Cuboid, the remaining cuboid is similar to the original and is also a golden cuboid,



If a cuboid similar to the original is removed and has sides b, c, and d, then $b : c : d = \phi$, so that

$$c = d\phi , \qquad b = d\phi^2 , \qquad a = d\phi^3 .$$

The volume of the original is abc = $\phi^6 d^3$, and the volume removed is bcd = $\phi^3 d^3$. The remaining volume is $(\phi^6 - \phi^3)d^3$. The ratio of the volume of the original to the volume of the remaining cuboid is

$$\frac{\phi^6 d^3}{(\phi^6 - \phi^3)d^3} = \frac{\phi^3}{\phi^3 - 1} = \frac{2 + \sqrt{5}}{1 + \sqrt{5}} = \frac{3 + \sqrt{5}}{4} = \phi^2/2 .$$

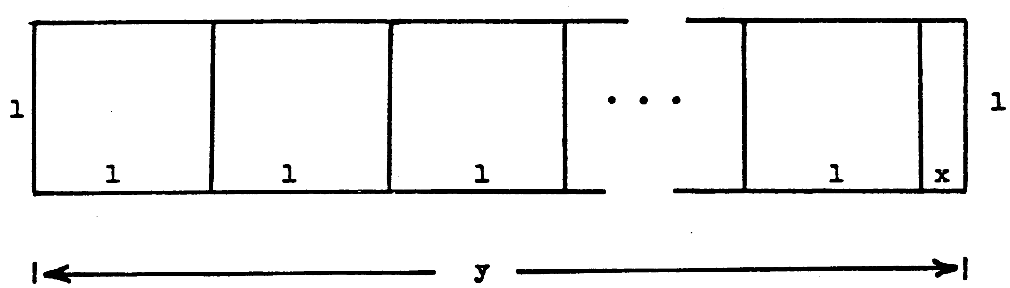
6. LUCAS GOLDEN-TYPE RECTANGLES

Now, in a Golden Rectangle, if one square with side equal to the width is removed, the resulting rectangle is similar to the original. Suppose that we have a rectangle in which when k squares with side equal to the width are removed, a rectangle similar to the original is formed, as discussed by J. A. Raab [6]. In the figure below, the ratio of length to width in the original rectangle and in the similar one formed after removing k squares is $y : 1 = 1 : x$ which gives $x = 1/y$. Since each square has side 1,

$$y - x = y - 1/y = k ,$$

or,

(6.1) $$y^2 - ky - 1 = 0 .$$



Let us consider only Lucas golden-type rectangles. That is, let $k = L_{2m+1}$, where L_{2m+1} is the (2m + 1)st Lucas number defined by

$$L_0 = 2 , \quad L_1 = 1 , \quad L_n = L_{n-1} + L_{n-2} , \quad n \geq 2 .$$

A known identity is

$$L_k = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k = \alpha^k + \beta^k ,$$

where α and β are the roots of $x^2 - x - 1 = 0$. In our problem, when $k = L_{2m+1}$, then (6.1) becomes

$$y^2 - L_{2m+1}y - 1 = 0$$

so that

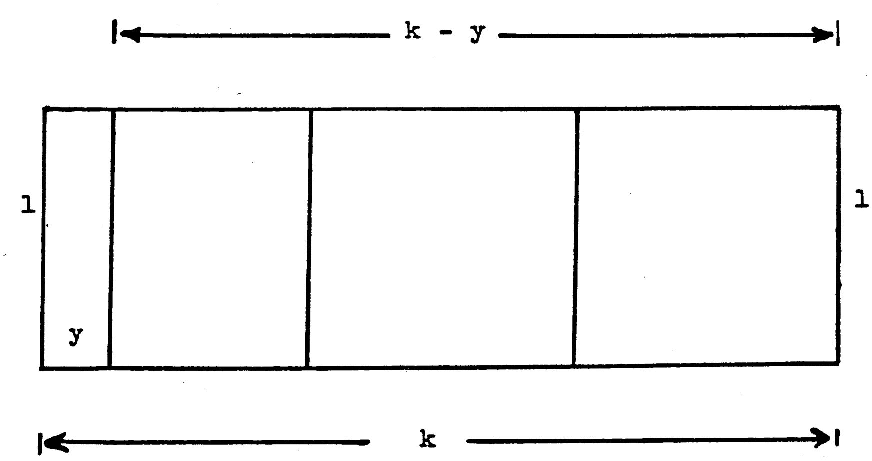
$$y = \alpha^{2m+1} \qquad \text{or} \qquad y = \beta^{2m+1}$$

but $y = \alpha^{2m+1}$ is the only positive root. Then

$$x = 1/\alpha^{2m+1} = -\beta^{2m+1} .$$

On the other hand, suppose we insist that to a given rectangle we add one similar to it such that the result is k squares long. Illustrated for $k = 3$, the equal ratios of length to width in the similar rectangles give

$$\frac{1}{y} = \frac{k-y}{1} \qquad \text{or} \qquad ky - y^2 = 1 \qquad \text{or} \qquad y^2 - ky + 1 = 0 .$$



Now, let $k = L_{2m}$; then $y = \alpha^{2m}$ or $y = \beta^{2m}$. Here, or course, $y = \alpha^{2m}$, so that

$$k - y = L_{2m} - \beta^{2m} = \alpha^{2m} .$$

Both of these cases are, of course, in the plane; the reader is invited to extend these ideas into the third dimension.

(Section 7, entitled "Generalized Golden-Type Cuboids," is omitted here.)

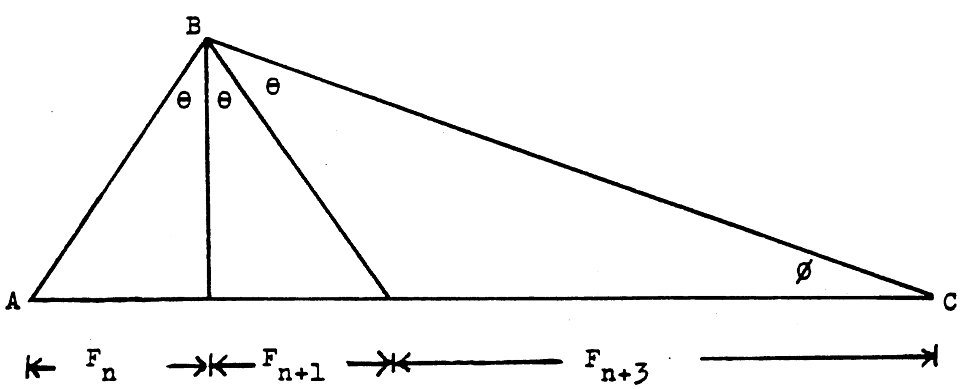
REFERENCES

1. William R. Ransom, Trigonometric Novelties, J. Weston Walch, Portland, Main, 1959, pp. 22-23.
2. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Company, Boston, 1969.
3. Tobias Dantzig, Bequest of the Greeks, Charles Scribner's Sons, New York, 1955, pp. 61-62.
4. Marvin H. Holt, "Mystery Puzzler and Phi," The Fibonacci Quarterly, Vol. 3, No. 2, April, 1965, pp. 135-138.
5. H. E. Huntley, "The Golden Cuboid," The Fibonacci Quarterly, Vol. 2, No. 3, October, 1964, p. 184+.
6. Joseph A. Raab, "A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle," The Fibonacci Quarterly, Vol. 1, No. 3, October, 1963, pp. 21-31.

A SPECIAL GEOMETRY PROBLEM

H-19 (Proposed by Charles R. Wall) In the triangle below [drawn for the case (1, 1, 3)], the trisectors of angle B divide side AC into segments of length F_n, F_{n+1}, and F_{n+3}. Find:

(i) $\lim_{n\to\infty} \theta$ and (ii) $\lim_{n\to\infty} \phi$



A PRIMER FOR THE FIBONACCI NUMBERS: PART VII

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AN INTRODUCTION TO FIBONACCI POLYNOMIALS AND THEIR DIVISIBILITY PROPERTIES

An elementary study of the Fibonacci polynomials yields some general divisibility theorems, not only for the Fibonacci polynomials, but also for Fibonacci numbers and generalized Fibonacci numbers. This paper is intended also to be an introduction to the Fibonacci polynomials.

Fibonacci and Lucas polynomials are special cases of Chebyshev polynomials, and have been studied on a more advanced level by many mathematicians. For our purposes, we define only Fibonacci and Lucas polynomials.

1. THE FIBONACCI POLYNOMIALS

The Fibonacci polynomials $\{F_n(x)\}$ are defined by

(1.1) $$F_1(x) = 1, \quad F_2(x) = x, \quad \text{and} \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x) .$$

Notice that, when $x = 1$, $F_n(1) = F_n$, the nth Fibonacci number. It is easy to verify that the relation

(1.2) $$F_{-n}(x) = (-1)^{n+1}F_n(x)$$

extends the definition of Fibonacci polynomials to all integral subscripts. The first ten Fibonacci polynomials are given below:

$$F_1(x) = 1$$

$$F_2(x) = x$$

$$F_3(x) = x^2 + 1$$

$$F_4(x) = x^3 + 2x$$

$$F_5(x) = x^4 + 3x^2 + 1$$

$$F_6(x) = x^5 + 4x^3 + 3x$$

$$F_7(x) = x^6 + 5x^4 + 6x^2 + 1$$

$$F_8(x) = x^7 + 6x^5 + 10x^3 + 4x$$

$$F_9(x) = x^8 + 7x^6 + 15x^4 + 10x^2 + 1$$

$$F_{10}(x) = x^9 + 8x^7 + 21x^5 + 20x^3 + 5x$$

It is important for Section 4, to notice that the degree of $F_n(x)$ is $|n| - 1$ for $n \neq 0$. Also, $F_0(x) = 0$.

In Table 1, the coefficients of the Fibonacci polynomials are arranged in ascending order. The sum of the nth row is F_n, and the sum of the nth diagonal of slope one, formed by beginning on the nth row, left-most column, and going one up and one right to get the next term, is given by

$$2^{(n-1)/2} = 2 \cdot 2^{(n-3)/2}$$

when n is odd.

Table 1

Fibonacci Polynomial Coefficients Arranged in Ascending Order

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
1	1								
2	0	1							
3	1	0	1						
4	0	2	0	1					
5	1	0	3	0	1				
6	0	3	0	4	0	1			
7	1	0	6	0	5	0	1		
8	0	4	0	10	0	6	0	1	
9	1	0	10	0	15	0	7	0	1
. . .									

To compare with Pascal's triangle, the sum of the nth row there is 2^n, and the sum of the nth diagonal of slope one is F_n. In fact, the (alternate) diagonals of slope one in Table 1 produce Pascal's triangle.

If the successive binomial expansions of $(x + 1)^n$ are written in descending order,

n = 0:	1								
n = 1:	x	+	1						
n = 2:	x^2	+	2x	+	1				
n = 3:	x^3	+	$3x^2$	+	3x	+	1		
n = 4:	x^4	+	$4x^3$	+	$6x^2$	+	4x	+	1
. . .			. . .						

the sum of the 4th diagonal of slope one is $F_4(x) = x^4 + 3x^2 + 1$, and the sum of the nth diagonal of slope one is $F_n(x)$, or,

$$(1.3) \qquad F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1}$$

for [x] the greatest integer contained in x, and binomial coefficient $\binom{n}{j}$, as given by Swamy [1] and others.

2. LUCAS POLYNOMIALS AND GENERAL FIBONACCI POLYNOMIALS

The Lucas polynomials $\{L_n(x)\}$ are defined by

$$(2.1) \qquad L_0(x) = 2, \quad L_1(x) = x, \qquad L_{n+1}(x) = xL_n(x) + L_{n-1}(x) .$$

Again, when x = 1, $L_n(1) = L_n$, the nth Lucas number. Lucas polynomials have the properties

$$(2.2) \qquad \begin{aligned} L_n(x) &= F_{n+1}(x) + F_{n-1}(x) = xF_n(x) + 2F_{n-1}(x) \\ xL_n(x) &= F_{n+2}(x) - F_{n-2}(x) \end{aligned}$$

and can be extended to negative subscripts by

$$(2.3) \qquad L_{-n}(x) = (-1)^n L_n(x) .$$

If the Lucas polynomial coefficients are arranged in ascending order in a left-justified triangle similar to that of Table 1, the sum of the nth row is L_n, and the sum of the nth diagonal of slope one is given by $3 \cdot 2^{(n-2)/2}$ for even n, $n \geq 2$. The degree of $L_n(x)$ is $|n|$, as can be observed in the following list of the first ten Lucas polynomials:

$$L_1(x) = x$$
$$L_2(x) = x^2 + 2$$
$$L_3(x) = x^3 + 3x$$
$$L_4(x) = x^4 + 4x^2 + 2$$
$$L_5(x) = x^5 + 5x^3 + 5x$$
$$L_6(x) = x^6 + 6x^4 + 9x^2 + 2$$
$$L_7(x) = x^7 + 7x^5 + 14x^3 + 7x$$
$$L_8(x) = x^8 + 8x^6 + 20x^4 + 16x^2 + 2$$
$$L_9(x) = x^9 + 9x^7 + 27x^5 + 30x^3 + 9x$$
$$L_{10}(x) = x^{10} + 10x^8 + 35x^6 + 50x^4 + 25x^2 + 2 .$$

When general Fibonacci polynomials are defined by

(2.4) $$H_1(x) = a, \quad H_2(x) = bx, \quad H_n(x) = xH_{n-1}(x) + H_{n-2}(x) ,$$

then

(2.5) $$H_n(x) = bxF_{n-1}(x) + aF_{n-2}(x) .$$

If the coefficients of the $\{H_n(x)\}$, written in ascending order, are placed in a left-justified triangle such as Table 1, then the sum of the nth diagonal of slope one is

$$(a + b)\cdot 2^{(n-3)/2} = (a + b)\cdot 2^{[(n-2)/2]}$$

for odd n, $n \geq 3$. (Notice that, if $a = 2$, $b = 1$, then $H_{n+1}(x) = L_n(x)$, and if $a = b = 1$, $H_n(x) = F_n(x)$.)

3. A MATRIX GENERATOR FOR FIBONACCI POLYNOMIALS

Since Fibonacci polynomials appear as the elements of the matrix defined below, many identities can be derived for Fibonacci polynomials using matrix theory, as done by Hayes [2] and others, and as done for Fibonacci numbers by Basin and Hoggatt [3].

It is easily established by mathematical induction that the matrix

$$Q = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},$$

when raised to the kth power, is given by

$$Q^k = \begin{pmatrix} F_{k+1}(x) & F_k(x) \\ F_k(x) & F_{k-1}(x) \end{pmatrix}$$

for any integer k, where Q^0 is the identity matrix and Q^{-k} is the matrix inverse of Q^k. Since det Q = -1, det Q^k = (det Q)^k = $(-1)^k$ gives us

$$F_{k+1}(x)F_{k-1}(x) - F_k^2(x) = (-1)^k . \tag{3.2}$$

Since $Q^mQ^n = Q^{m+n}$ for all integers m and n, matrix multiplication of Q^m and Q^n gives

$$Q^mQ^n = \begin{pmatrix} F_{m+1}(x)F_{n+1}(x) + F_m(x)F_n(x) & F_{m+1}(x)F_n(x) + F_m(x)F_{n-1}(x) \\ F_m(x)F_{n+1}(x) + F_{m-1}(x)F_n(x) & F_m(x)F_n(x) + F_{m-1}(x)F_{n-1}(x) \end{pmatrix}$$

while

$$Q^{m+n} = \begin{pmatrix} F_{m+n+1}(x) & F_{m+n}(x) \\ F_{m+n}(x) & F_{m+n-1}(x) \end{pmatrix} .$$

Equating elements in the upper right corner gives

$$F_{m+n}(x) = F_{m+1}(x)F_n(x) + F_m(x)F_{n-1}(x) . \tag{3.3}$$

Replacing n by $(-n)$ and using the identity (1.2) gives

$$F_{m-n}(x) = (-1)^n[-F_{m+1}(x)F_n(x) + F_m(x)F_{n+1}(x)] .$$

Then,

$$F_{m+n}(x) + (-1)^nF_{m-n}(x) = F_m(x)F_{n-1}(x) + F_m(x)F_{n+1}(x) = F_m(x)L_n(x) .$$

If we replace n by k and m by $m - k$ above, we can obtain finally

$$(3.4) \qquad F_m(x) = L_k(x)F_{m-k}(x) + (-1)^{k+1}F_{m-2k}(x)$$

which results in the divisibility theorems of the next section.

4. DIVISIBILITY PROPERTIES OF FIBONACCI AND LUCAS POLYNOMIALS

Lemma. The Fibonacci polynomials $F_m(x)$ satisfy

$$F_m(x) = F_{m-k}(x)\left(\sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{(2i+1)k-2im}(x)\right) + (-1)^{p(m-k)+m+1}F_{(2p-1)m-2pk}(x)$$

for all integers m and k, and for $p \geq 1$.

Proof: If $p = 1$, the Lemma is just Equation (3.4). For convenience, call $Q_p(x)$ the sum of Lucas polynomials in the Lemma. Then, assume that the Lemma holds when $p = j$, or that

$$(A) \qquad F_m(x) = F_{m-k}(x)Q_j(x) + (-1)^{j(m-k)+m+1}F_{(2j-1)m-2jk}(x) .$$

Substitute $[2jk - (2j - 1)m]$ for m in Equation (3.4), giving

$$F_{2jk-(2j-1)m}(x) = L_{k'}(x)F_{2jk-(2j-1)m-k'}(x) + (-1)^{k'+1}F_{2jk-(2j-1)m-2k'}(x) .$$

Since we want to express $F_{2jk-(2j-1)m}(x)$ in terms of $F_{m-k}(x)$, set

$$2jk - (2j - 1)m - k' = m - k$$

and solve for k', yielding $k' = (2j + 1)k - 2jm$, so that

$$F_{2jk-(2j-1)m}(x) = L_{(2j+1)k-2jm}(x)F_{m-k}(x) + (-1)^{k+1}F_{(2j+1)m-(2j+2)k}(x) .$$

Substituting into (A) and using Equation (1.2) to simplify gives

$$F_m(x) = [Q_j(x) + (-1)^{j(m-k)}L_{(2j+1)k-2jm}(x)]F_{m-k}(x) + (-1)^{(j+1)(m-k)+m+1}F_{(2j+1)m-(2j+2)k}(x) ,$$

which is the Lemma when $p = j + 1$, completing a proof by mathematical induction.

Notice that the Lemma yields an interesting identity for Fibonacci numbers, given below:

$$(4.1)\quad F_m = F_{m-k}\left(\sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{m-(2i+1)(m-k)}\right) + (-1)^{p(m-k)}F_{m-2p(m-k)},$$

where $p \geq 1$. To establish (4.1), use algebra on the subscripts of the Lemma and then take $x = 1$.

<u>Theorem 1:</u> Whenever a Fibonacci polynomial $F_m(x)$ is divided by a Fibonacci polynomial $F_{m-k}(x)$, $m \neq k$, of lesser or equal degree, the remainder is always a Fibonacci polynomial or the negative of a Fibonacci polynomial, and the quotient is a sum of Lucas polynomials whenever the division is not exact. Explicitly, for $p \geq 1$,

(i) the remainder is

$$\pm F_{(2p-1)m-2kp}(x) \quad \text{when} \quad \frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1},$$

or, equivalently, the remainder is

$$\pm F_{m-2p(m-k)}(x) \quad \text{for} \quad \frac{|m|}{2p+1} < |m-k| < \frac{|m|}{2p-1};$$

(ii) the quotient is $\pm L_k(x)$ when $|k| < 2|m|/3$;

(iii) the quotient is given by

$$Q_p(x) = \sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{(2i+1)k-2im}(x)$$

$$= \sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{m-(2i+1)(m-k)}(x)$$

for m, k, and p related as in (i), and by $Q_p(x) + (-1)^{p(m-k)}$ if $k = 2pm/(2p+1)$;

(iv) the division is exact when $k = 2pm/(2p+1)$ or $k = (2p-1)m/2p$.

Proof: When

$$\frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1}$$

and the degree of $F_m(x)$ is greater than that of $F_{m-k}(x)$, we can show that

$$|m| > |m-k| > |(2p-1)m - 2pk| .$$

Since the degree of $F_n(x)$ is $|n| - 1$, we can interpret the Lemma in terms of quotients and remainders for the restrictions on m, k, and p above, establishing (i), (ii), and (iii). As for (iv), the division is exact if we have $k = \frac{(2p-1)m}{2p}$, for then

$$F_{(2p-1)m-2pk}(x) = F_0(x) = 0 .$$

When $k = \frac{2pm}{2p+1}$,

$$F_{(2p-1)m-2pk}(x) = F_{k-m}(x) = (-1)^{m-k+1}F_{m-k}(x)$$

$$= (-1)^{m+1}F_{m-k}(x)$$

because k is an even integer. Referring to the Lemma, increasing the quotient by $(-1)^{p(m-k)+m+1+m+1} = (-1)^{p(m-k)}$ will make the division exact.

Corollary 1.1: $F_q(x)$ divides $F_m(x)$ if and only if q divides m.

Proof: If q divides m, then either $m/2p = q$ or $m/(2p+1) = q$. Let $q = m - k$ and apply Theorem 1.

If $F_q(x)$ divides $F_m(x)$, then let $q = m - k$ and consider the remainder of Theorem 1. Either

$$F_{(2p-1)m-2pk}(x) = F_0(x) \quad \text{or} \quad F_{(2p-1)m-2pk}(x) = \pm F_{m-k}(x) ,$$

giving

$$k = \frac{(2p-1)m}{2p} , \quad k = \frac{(2p-2)m}{2p-1} , \quad \text{or} \quad k = \frac{2pm}{2p+1}$$

by equating subscripts. The possibilities give $q = m - k = m/2p$, $q = m/(2p-1)$, or $q = m/(2p+1)$, so that q divides m.

<u>Corollary 1.2</u>: If the Fibonacci number F_m is divided by F_{m-k}, $m \neq k$, then the remainder of least absolute value is always a Fibonacci number or its negative. Further,

(i) the remainder is

$$\pm F_{m-2p(m-k)} \quad \text{when} \quad \frac{|m|}{2p+1} < |m-k| < \frac{|m|}{2p-1},$$

$m - k \neq 2$, and the quotient is the sum of Lucas numbers;

(ii) the quotient is $\pm L_k$ when $|k| < 2|m|/3$, for Lucas number L_k.

<u>Proof</u>: Let $x = 1$ throughout Theorem 1. Since the magnitudes of Fibonacci numbers are ordered by their subscripts, $\pm F_{m-2p(m-k)}$ represents a remainder (unless $m - k = 2$ since $F_2 = F_1 = 1$).

To illustrate Corollary 1.2, divide F_{13} by F_7:

$$233 = 17 \cdot 13 + 12 = 18 \cdot 13 + (-1) .$$

Now, 12 is the remainder in usual division, but we consider the positive and negative remainders with absolute value less than that of the divisor, so that $(-1) = -F_1$ is the remainder of least absolute value. Here, $m = 13$, $k = 6 < 2m/3$, $p = 1$, and the quotient is $L_6 = 18$. The remainders found upon dividing one Fibonacci number by another have been discussed by Taylor [4] and Halton [5].

<u>Corollary 1.3</u>: The Fibonacci number F_q divides F_m if and only if q divides m, $|q| \neq 2$.

<u>Proof</u>: If q divides m, let $x = 1$ in Corollary 1.1. If F_q divides F_m, let $q = m - k$. The remainder of Corollary 1.2 becomes $F_{m-2p(m-k)} = F_0 = 0$ or $F_{m-2p(m-k)} = \pm F_{m-k}$. The algebra on the subscripts follows the proof of Corollary 1.1, which will prove that q divides m, provided that there are no cases of mistaken identity, such as $F_s = F_q$, $|s| \neq |q|$, and such that s does not divide m. Thus, the restriction $|q| \neq 2$ since $F_2 = F_1 = 1$.

Unfortunately, as pointed out by E. A. Parberry, Corollary 1.3 cannot be proved immediately from Corollary 1.1 by simply taking $x = 1$. That F_q

divides F_m does not imply that $F_q(x)$ divides $F_m(x)$, just as that f(1) divides g(1) does not imply that f(x) divides g(x) for arbitrary polynomials f(x) and g(x). Also, Webb and Parberry [8] have proved that a Fibonacci polynomial $F_m(x)$ is irreducible over the integers if and only if m is prime. But, if m is prime, while F_m is not divisible by any other Fibonacci number F_q, $q \geq 3$, F_m is not necessarily a prime. How to determine all values of m for which F_m is prime when m is prime, is an unsolved problem.

Corollary 1.4: There exist an infinite number of sequences $\{S_n\}$ having the division property that, when S_m is divided by S_{m-k}, $m \neq k$, the remainder of least absolute value is always a member of the sequence or the negative of a member of the sequence.

Proof: We can let x be any integer in the Lemma and throughout Theorem 1. If x = 2, one such sequence is ..., 0, 1, 2, 5, 12, 29, 70, 169, ...

Theorem 2: Whenever a Lucas polynomial $L_m(x)$ is divided by a Lucas polynomial $L_{m-k}(x)$, $m \neq k$, of lesser degree, a non-zero remainder is always a Lucas polynomial or the negative of a Lucas polynomial. Explicitly,

(i) non-zero remainders have the form

$$\pm L_{(2p-1)m-2pk}(x) \qquad \text{when} \qquad \frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1},$$

or, equivalently,

$$\pm L_{2p(m-k)-m}(x) \qquad \text{for} \qquad \frac{|m|}{2p+1} < |m-k| < \frac{|m|}{2p-1} \;;$$

(ii) if $|k| < 2|m|/3$, the quotient is $\pm L_k(x)$;

(iii) the division is exact when $k = 2pm/(2p+1)$, $p \neq 0$.

Proof: Since the proof parallels that of the Lemma and Theorem 1, details are omitted. Identity (3.4) is used to establish

$$(4.2) \qquad L_m(x) = L_k(x)L_{m-k}(x) + (-1)^{k+1}L_{m-2k}(x) .$$

Since $L_{-n}(x) = (-1)^n L_n(x)$, it can be proved that

$$L_m(x) = Q_p(x)L_{m-k}(x) \pm L_{(2p-1)m-2pk}(x) ,$$

for $|m| \geq |m - k| \geq |(2p - 1)m - 2pk|$. Since the degree of $L_n(x)$ is $|n|$, the rest of the proof is similar to that of Theorem 1. However, notice that it is necessary to both proofs that $F_{-n}(x) = \pm F_n(x)$ and $L_{-n}(x) = \pm L_n(x)$.

Corollary 2.1: The Lucas polynomial $L_q(x)$ divides $L_m(x)$ if and only if m is an odd multiple of q .

Proof: If $m = (2p + 1)q$, let $q = m - k$ and Theorem 2 guarantees that $L_q(x)$ divides $L_m(x)$.

If $L_q(x)$ divides $L_m(x)$, then let $q = m - k$. For the division to be exact, the term $\pm L_{(2p-1)m-2pk}(x)$ must equal $L_{m-k}(x)$ since it cannot be the zero polynomial. Then, either $k = 2pm/(2p + 1)$ or $k = 2pm/(2p - 1)$, so $q = m - k = m/(2p + 1)$ or $q = m/(2p - 1)$. In either case, m is an odd multiple of q .

Corollary 2.2: If a Lucas number L_m is divided by L_{m-k}, then the non-zero remainder of least absolute value is always a Lucas number or its negative with the form

$$\pm L_{2p(m-k)-m} \quad \text{for} \quad \frac{|m|}{2p + 1} < |m - k| < \frac{|m|}{2p - 1} ,$$

and the quotient is $\pm L_k$ when $|k| < 2|m|/3$.

Proof: Let $x = 1$ throughout the development of Theorem 2.

Corollary 2.3: The Lucas number L_q divides L_m if and only if $m = (2s + 1)q$ for some integer s. (This result is due to Carlitz [6]).

Proof: If $m = (2s + 1)q$, let $x = 1$ in Corollary 2.1. If L_q divides L_m, take $q = m - k$ and examine the remainder $L_{2p(m-k)-m}$ of Corollary 2.2 which must equal L_{m-k} or L_{k-m} since it cannot be zero. The algebra follows that given in Corollary 2.1. Since there are no Lucas numbers such that $L_q = L_s$ where $|q| \neq |s|$, and since $L_q \neq 0$ for any q , there are no restrictions.

Since the generalized Fibonacci polynomials $H_m(x)$ satisfy Equation (2.5), $H_m(x) = bxF_{m-1}(x) + aF_{m-2}(x)$, we can show that

$$(4.3) \qquad H_m(x) = L_k(x)H_{m-k}(x) + (-1)^{k+1}H_{m-2k}(x) ,$$

but since $H_m(x) \neq \pm H_{-m}(x)$, we have a more limited theorem.

Theorem 3: Whenever a generalized Fibonacci polynomial $H_m(x)$ is divided by $H_{m-k}(x)$, $2m/3 > k > 0$, any non-zero remainder is always another generalized Fibonacci polynomial or its negative, and the quotient is $L_k(x)$.

As a consequence of Theorem 3, when a generalized Fibonacci number H_m is divided by H_q, a non-zero remainder of least absolute value is guaranteed to be another generalized Fibonacci number only when $|m - q| < 2m/3$. Taylor [4] has proved that, of all generalized Fibonacci sequences $\{H_m\}$ satisfying the recurrence $H_m = H_{m-1} + H_{m-2}$, the only sequences with the division property that the non-zero remainders of least absolute value are always a member of the sequence or the negative of a member of the sequence, are the Fibonacci and Lucas sequences. For your further reading, Hoggatt [7] gives a lucid description of divisibility properties of Fibonacci and Lucas numbers.

REFERENCES

1. M. N. S. Swamy, "Further Properties of Morgan-Voyce Polynomials," The Fibonacci Quarterly, Vol. 6, No. 2, April, 1968, pp. 167-175.
2. Richard A. Hayes, Fibonacci and Lucas Polynomials, Unpublished Master's Thesis, San Jose State College, January, 1965, pp. 36-39.
3. S. L. Basin and Verner E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence: Part II," Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 61-68.
4. Laurence Taylor, "Residues of Fibonacci-Like Sequences," Fibonacci Quarterly, Vol. 5, No. 3, October, 1967, pp. 298-304.
5. J. H. Halton, "On Fibonacci Residues," Fibonacci Quarterly, Vol. 2, No. 3, October, 1964, pp. 217-218.
6. Leonard Carlitz, "A Note on Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 1, February, 1964, pp. 15-28.
7. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton-Mifflin Mathematics Enrichment Series, Houghton-Mifflin, Boston, 1969, Chapters 7 and 8.
8. W. A. Webb and E. A. Parberry, "Divisibility Properties of Fibonacci Polynomials," Fibonacci Quarterly, Vol. 7, No. 5, December, 1969, pp. 457-463.

A PRIMER FOR THE FIBONACCI NUMBERS: PART VIII

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SEQUENCES OF SUMS FROM PASCAL'S TRIANGLE

There are many ways to generalize Fibonacci numbers, one way being to consider them as a sequence of sums found from diagonals in Pascal's triangle [1], [2]. Since Pascal's triangle and computations with generating functions are so interrelated with the Fibonacci sequence, we introduce a way to find such sums in this section of the Primer.

1. INTRODUCTION

Some elementary but elegant mathematics solves the problem of finding the sums of integers appearing on diagonals of Pascal's triangle. Writing Pascal's triangle in a left-justified manner, the problem is to find the infinite sequence of sums p/q of binomial coefficients appearing on diagonals p/q for integers p and q, $p + q \geq 1$, $q > 0$, where we find entries on a diagonal p/q by counting up p and right q, starting in the left-most column. (Notice that, while the intuitive idea of "slope" is useful in locating the diagonals, the diagonal 1/2, for example, is not the same as 2/4 or 3/6.) As an example, the sums 2/1 on diagonals formed by going up 2 and right 1 are 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, ... , as illustrated below:

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Some sequences of sums are simple to find. For example, the sums 0/1 formed by going up 0 and right 1 are the sums of integers appearing in each row, the powers of 2. The sums 0/2 are formed by alternate integers in a row, also powers of 2. The sums 1/1 give the famous Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ..., defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. The sums -1/2, found by counting down 1 and right 2, give the Fibonacci numbers with odd subscripts, 1, 2, 5, 13, 34, 89, ..., F_{2n+1}, ... While the problem is not defined for negative "slope" less than or equal to -1 nor for summing columns, the diagonals -1/1 are the same as the columns of the array, and the sum of the first j integers in the nth column is the same as the jth entry in the (n + 1)st column.

To solve the problem in general, we develop some generating functions.

2. GENERATING FUNCTIONS FOR THE COLUMNS OF PASCAL'S TRIANGLE

Here, a generating function is an algebraic expression which lists terms in a sequence as coefficients in an infinite series. For example, by the formula for summing an infinite geometric progression,

$$(1) \qquad \frac{a}{1-r} = a + ar + ar^2 + ar^3 + \dots, \qquad |r| < 1 ,$$

we can write the generating function for the powers of 2 as

$$(2) \qquad \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \dots + 2^n x^n + \dots , \qquad |x| < 1/2 .$$

Long division gives a second verification that $1/(1 - 2x)$ generates powers of 2, and long division can be used to compute successive coefficients of powers of x for any generating function which follows.

We need some other generating functions to proceed. By summing the geometric progression,

$$(3) \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} \binom{k}{0} x^k , \qquad |x| < 1 .$$

By multiplying series or by taking successive derivatives of (3), one finds

$$(4) \qquad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + kx^{k-1} + \dots = \sum_{k=0}^{\infty} \binom{k}{1} x^k , \qquad |x| < 1,$$

$$(5) \qquad \frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \dots = \sum_{k=0}^{\infty} \binom{k}{2} x^k , \qquad |x| < 1.$$

Computation of the nth derivative of (3) shows that

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \binom{k}{n} x^k , \quad n = 0, 1, 2, 3, \ldots, \quad |x| < 1 ,$$

is a generating function for the integers appearing in the nth column of Pascal's triangle, or equivalently, the column generator for the nth column, where we call the left-most column the zero-th column. As a restatement, the columns of Pascal's triangle give the coefficients of the binomial expansion of $(1 - x)^{-n-1}$, $n = 0, 1, 2, \ldots, |x| < 1$, or of $(1 + x)^{-n-1}$ if taken with alternating signs.

3. SOME PARTICULAR SUMS DERIVED USING COLUMN GENERATORS

It is easy to prove that the rows in Pascal's triangle have powers of 2 as their sums: merely let $x = 1$ in $(x + 1)^n$, $n = 0, 1, 2, \ldots$. But, to demonstrate the methods, we work out the sums 0/1 of successive rows using column generators.

First write Pascal's triangle to show the terms in the expansions of $(x + 1)^n$. Because we want the exponents of x to be identical in each row so that we will add the coefficients in each row by adding the column generators, multiply the columns successively by $1, x, x^2, x^3, \ldots$, making

	1					
	$1x$	$1x$				
	$1x^2$	$2x^2$	$1x^2$			
	$1x^3$	$3x^3$	$3x^3$	$1x^3$		
	$1x^4$	$4x^4$	$6x^4$	$4x^4$	$1x^4$	
	...	...	...	...	...	...
generators:	$\frac{1}{1-x}$	$\frac{x}{(1-x)^2}$	$\frac{x^2}{(1-x)^3}$	$\frac{x^3}{(1-x)^4}$	$\frac{x^4}{(1-x)^5}$	...

Then the sum S of column generators will have the sums 0/1 of the rows appearing as coefficients of successive powers of x. But, S is a geometric progression with ratio $x/(1 - x)$, so by (1),

$$S = \frac{\frac{1}{1 - x}}{1 - \frac{x}{1 - x}} = \frac{1}{1 - 2x}, \qquad \text{for} \qquad \left|\frac{x}{1 - x}\right| < 1 \quad \text{or} \quad |x| < 1/2,$$

the generating function for powers of 2 given earlier in (2).

If we want the sums 0/2, we sum every other generating function, forming

$$S^* = \frac{1}{1 - x} + \frac{x^2}{(1 - x)^3} + \frac{x^4}{(1 - x)^5} + \cdots,$$

and again sum the geometric progression to find

$$S^* = \frac{1 - x}{1 - 2x} = \frac{1}{1 - 2x} - \frac{x}{1 - 2x}$$

$$= (1 + 2x + 4x^2 + 8x^3 + \ldots + 2^n x^n + \ldots)$$

$$- (x + 2x^2 + 4x^3 + \ldots + 2^{n-1} x^n + \ldots)$$

$$= 1 + (x + 2x^2 + 4x^3 + \ldots + 2^{n-1} x^n + \ldots),$$

which again generates powers of 2 as verified above.

We have already noted that the sums 1/1 give the Fibonacci numbers. To use column generators, we must multiply the columns successively by $1, x^2, x^4, x^6, \ldots$, so that the exponents of x will be the same along each diagonal 1/1. The sum S^{**} of column generators becomes

$$S^{**} = \frac{1}{1 - x} + \frac{x^2}{(1 - x)^2} + \frac{x^4}{(1 - x)^3} + \frac{x^6}{(1 - x)^4} + \cdots,$$

again a geometric progression, so that

$$S^{**} = \frac{\frac{1}{1 - x}}{1 - \frac{x^2}{1 - x}} = \frac{1}{1 - x - x^2}$$

for

$$\left|\frac{x^2}{1 - x}\right| < 1 \qquad \text{or} \qquad |x| < \left(\frac{1 + \sqrt{5}}{2}\right)^{-1}.$$

This means that, for $|x|$ less than the positive root of $x^2 + x - 1 = 0$,

(6) $$\frac{1}{1-x-x^2} = 1 + 1x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \ldots + F_n x^{n-1} + \ldots$$

For generating functions, we are concerned primarily with the coefficients of x rather than values of x, but a particular example is interesting at this point. Both series (4) and (6) converge when $x = 1/2$; let $x = 1/2$ in those series to form

$$4 = 1 + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + 5\cdot\frac{1}{16} + 6\cdot\frac{1}{32} + \ldots + n\cdot\frac{1}{2^{n-1}} + \ldots,$$

$$4 = 1 + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 5\cdot\frac{1}{16} + 8\cdot\frac{1}{32} + \ldots + F_n\cdot\frac{1}{2^{n-1}} + \ldots,$$

the same result whether we use the natural numbers or the Fibonacci numbers as coefficients of the powers of 1/2 !

Also, $x = 0.1$ in (6) gives, upon division by 100,

$$\frac{1}{89} = 0.0112358$$
```
                 13
                  21
                   34
                    55
                     89
                      ...
```

the reciprocal of a Fibonacci number with successive Fibonacci numbers making up its decimal expansion.

We are now in a position to solve the general problem of finding the sums p/q.

4. SEQUENCES OF SUMS p/q APPEARING ALONG ANY DIAGONAL

To find the sequence of sums appearing along the diagonals p/1, multiply the columns of Pascal's triangle successively by $1, x^{p+1}, x^{2(p+1)}, x^{3(p+1)}, \ldots$, so that the exponents of x appearing on each diagonal p/1 will be the same, giving

1					
$1x$	$1x^{p+1}$				
$1x^2$	$2x^{p+2}$	$1x^{2p+2}$			
$1x^3$	$3x^{p+3}$	$3x^{2p+3}$	$1x^{3p+3}$		
$1x^4$	$4x^{p+4}$	$6x^{2p+4}$	$4x^{3p+4}$	$1x^{4p+4}$	
...	...	...	...	...	...

generators: $\frac{1}{1-x}$ $\frac{x^{p+1}}{(1-x)^2}$ $\frac{x^{2p+2}}{(1-x)^3}$ $\frac{x^{3p+3}}{(1-x)^4}$ $\frac{x^{4p+4}}{(1-x)^5}$ ... .

The sum S of column generators is a geometric progression, so that

$$(7)\quad S = \frac{\frac{1}{1-x}}{1 - \frac{x^{p+1}}{1-x}} = \frac{1}{1 - x - x^{p+1}}, \quad p \geq 1, \quad \left|\frac{x^{p+1}}{1-x}\right| < 1,$$

with S convergent for $|x|$ less than the positive root of $x^{p+1} + x - 1 = 0$. Then, the generating function (7) gives the sums p/1 as coefficients of successive powers of x. [Reader: Show $|x| < 1/2$ is sufficient. Editor.]

In conclusion, the sequence of sums p/q are found by multiplying successive qth columns by 1, x^{p+q}, $x^{2(p+q)}$, $x^{3(p+q)}$, ..., making the sum of column generators be

$$S^* = \frac{1}{1-x} + \frac{x^{p+q}}{(1-x)^{q+1}} + \frac{x^{2p+2q}}{(1-x)^{2q+1}} + \frac{x^{3p+3q}}{(1-x)^{3q+1}} + \cdots .$$

Summing that geometric progression yields the generating function

$$S^* = \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}}, \quad p + q \geq 1, \quad q > 0,$$

which converges for $|x|$ less than the absolute value of the root of smallest absolute value of $x^{p+q} - (1-x)^q = 0$ and which gives the sums of the binomial coefficients found along the diagonals p/q as coefficients of successive powers of x. [Reader: Show $|x| < 1/2$ is sufficient. Editor.]

Some references for readings related to the problem of this paper follow but the list is by no means exhaustive. We leave the reader with the problem of determining the properties of particular sequences of sums arising in this paper.

REFERENCES

1. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers", Fibonacci Quarterly, Vol. 2, No. 4, Dec., 1964, pp. 277-289.
2. V. C. Harris and Carolyn C. Styles, "Generalized Fibonacci Sequences Associated with a Generalized Pascal Triangle," Fibonacci Quarterly, Vol. 4, No. 3, October, 1966, pp. 241-248.
3. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Diagonal Sums of Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 7, No. 4, Nov., 1969, pp. 341-358.
4. Mark Feinberg, "New Slants," Fibonacci Quarterly, Oct., 1964, pp. 223-227.
5. V. E. Hoggatt, Jr., "A New Angle on Pascal's Triangle," Fibonacci Quarterly, Vol. 6, No. 4, October, 1968, pp. 221-234.

A PRIMER FOR THE FIBONACCI NUMBERS: PART XIII

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THE FIBONACCI CONVOLUTION TRIANGLE, PASCAL'S TRIANGLE, AND SOME INTERESTING DETERMINANTS

The simplest and most well-known convolution triangle is Pascal's triangle, which is formed by convolving the sequence $\{1, 1, 1, \ldots\}$ with itself repeatedly. The Fibonacci convolution triangle [1] is formed by repeated convolutions of the sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ with itself. We now show three different ways to obtain the Fibonacci convolution triangle, as well as some interesting sequences of determinant values found in Pascal's triangle, the Fibonacci convolution triangle, and the trinomial coefficient triangle.

1. CONVOLUTION OF SEQUENCES

If $\{a_n\}$ and $\{b_n\}$ are two sequences, then the convolution of the two sequences is another sequence $\{c_n\}$ which is calculated as shown:

$$c_1 = a_1b_1$$

$$c_2 = a_1b_2 + a_2b_1$$

$$c_3 = a_1b_3 + a_2b_2 + a_3b_3$$

$$\cdots\cdots\cdots\cdots$$

$$c_n = a_1b_n + a_2b_{n-1} + a_3b_{\ 1} + \cdots + a_nb_1 = \sum_{k=1}^{n} a_kb_{n-k+1} \,.$$

If we convolve the Fibonacci sequence with itself, we obtain the <u>First Fibonacci Convolution Sequence</u> $\{1, 2, 5, 10, 20, 38, 71, \ldots\}$, as follows:

$$F_1^{(1)} = F_1F_1 \qquad = 1\cdot 1 \qquad = 1$$

$$F_2^{(1)} = F_1F_2 + F_2F_1 \qquad = 1\cdot 1 + 1\cdot 1 \qquad = 2$$

$$F_3^{(1)} = F_1F_3 + F_2F_2 + F_3F_1 \qquad = 1\cdot 2 + 1\cdot 1 + 2\cdot 1 \qquad = 5$$

$$F_4^{(1)} = F_1F_4 + F_2F_3 + F_3F_2 + F_4F_1 = 1\cdot 3 + 1\cdot 2 + 2\cdot 1 + 3\cdot 1 = 10$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Next we can obtain the <u>Second Fibonacci Convolution Sequence</u> {1, 3, 9, 22, 51, 111, ...} as indicated below ,

$$F_1^{(2)} = F_1F_1^{(1)} \qquad = 1\cdot1 \qquad = 1$$

$$F_2^{(2)} = F_2F_1^{(1)} + F_1F_2^{(1)} \qquad = 1\cdot1 + 1\cdot2 \qquad = 3$$

$$F_3^{(2)} = F_3F_1^{(1)} + F_2F_2^{(1)} + F_1F_3^{(1)} \qquad = 2\cdot1 + 1\cdot2 + 1\cdot5 \qquad = 9$$

$$F_4^{(2)} = F_4F_1^{(1)} + F_3F_2^{(1)} + F_2F_3^{(1)} + F_1F_4^{(1)} = 3\cdot1 + 2\cdot2 + 1\cdot5 + 1\cdot10 = 22$$

. . . .

by writing the convolution of the first Fibonacci convolution sequence with the Fibonacci sequence. To obtain the succeeding Fibonacci convolution sequences, we continue writing the convolution of a Fibonacci convolution sequence with the Fibonacci sequence. A second method follows.

The Fibonacci sequence is obtained from the generating function

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^2 + \dots + F_{n+1}x^n + \dots$$

which provides Fibonacci numbers as coefficients of successive powers of x as far as one pleases to carry out a long division. The kth convolution of of the Fibonacci numbers appears as the coefficients of successive powers of x in the generating function

$$\frac{1}{(1 - x - x^2)^{k+1}} = F_1^{(k)} + F_2^{(k)}x + F_3^{(k)}x^2 + \dots + F_{n+1}^{(k)}\,x^n + \dots ,$$

$k = 0, 1, 2, \dots$. For $k = 0$, we get just the Fibonacci numbers. In the next section we shall see yet another way to find the convolved Fibonacci sequences.

3. THE FIBONACCI CONVOLUTION TRIANGLE

Suppose someone writes a column of zeroes. To the right and one space down place a one. To generate the elements below the one we add the one element directly above and the one element diagonally left of the element to be written. Such a rule generates a convolution triangle. This rule, of course, generates Pascal's triangle in left-justified form:

0						
0	1					
0	1	1				
0	1	2	1			
0	1	3	3	1		
0	1	4	6	4	1	
0	1	5	10	10	5	1
. . .						

The columns of Pascal's triangle give convolution sequences for the sequence $\{1, 1, 1, \dots\}$. Notice that the row sums give powers of two, and the sums of rising diagonals formed by beginning in the column of ones and going up one and to the right one throughout the array give the Fibonacci numbers 1, 1, 2, 3, 5, ..., F_n, ..., where $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, 5, \dots$

Next suppose we change the rule of formation. Begin as before, but to generate elements below the one, add the two elements directly above and the element diagonally left of the element to be generated. Now we have the Fibonacci convolution triangle in left-justified form,

0						
0	1					
0	1	1				
0	2	2	1			
0	3	5	3	1		
0	5	10	9	4	1	
0	8	20	22	14	5	1
. . .						

The columns give the convolution sequences for the Fibonacci sequence. The row sums are the Pell numbers 1, 2, 5, 12, 29, 70, ..., p_n, ..., where $p_n = 2p_{n-1} + p_{n-2}$. The rising diagonal sums are 1, 1, 3, 5, 11, 21, ..., r_n, ..., where $r_n = r_{n-1} + 2r_{n-2}$. The diagonal sums found by beginning in the column of Fibonacci numbers and going up two and right one throughout the array are 1, 1, 2, 4, 7, 13, 24, ..., T_n, ..., $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, the Tribonacci numbers.

If one changes the rule of formation yet again, so that elements below the initial one are found by adding the one element directly above and the two elements diagonally left of the element to be generated, the array obtained is the trinomial coefficient triangle. The coefficients in successive rows are the same as those found in expansions of the trinomial $(1 + x + x^2)^n$, n = 0, 1, 2, ... The columns do not form convolution sequences as before, but the row sums are now the powers of three, and the sums of elements appearing on the rising diagonals are 1, 1, 2, 4, 7, 13, ..., the Tribonacci numbers just defined. To illustrate, the trinomial triangle is formed as follows:

0									
0	1								
0	1	1	1						
0	1	2	3	2	1				
0	1	3	6	7	6	3	1		
0	1	4	10	16	19	16	10	4	1
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

3. SOME SPECIAL MATRICES

If one looks again at how colvoluted sequences are formed, the arithmetic is much like matrix multiplication. Suppose that we define three matrices. Let P be the n x n matrix formed by using as elements the first n rows of Pascal's triangle in rectangular form. Let F be the n x n matrix formed by writing the first n rows of Pascal's triangle in vertical position on and below the main diagonal, which makes the row sums of F be Fibonacci numbers. Let C be the n x n matrix whose elements are the first n rows of the Fibonacci convolution triangle written in rectangular form. Then it can be proved that FP = C (See [1], [2].) To illustrate, for n = 6,

$$FP = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 9 & 14 & 20 & 27 \\ 3 & 10 & 22 & 40 & 65 & 98 \\ 5 & 20 & 51 & 105 & 190 & 315 \\ 8 & 38 & 111 & 256 & 511 & 924 \end{bmatrix} = C \tag{3.1}$$

Suppose that, instead of multiplying matrix F by the rectangular Pascal array P, we use an n x n matrix A whose elements are given by the first n rows of Pascal's triangle in left-justified form on and below its main diagonal, and zero elsewhere. Let F^t be the transpose of F. Then the matrix product $AF^t = T$, where T is the n x n matrix whose elements are found in the left-justified trinomial coefficient triangle given in Section 2. We illustrate for n = 6:

$$AF^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

$$(3.2) \qquad = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 1 & 3 & 6 & 7 & 6 & 3 \\ 1 & 4 & 10 & 16 & 19 & 16 \\ 1 & 5 & 15 & 30 & 45 & 51 \end{bmatrix} = T$$

4. SPECIAL DETERMINANTS IN PASCAL'S TRIANGLE

A multitude of unit determinants can be found in Pascal's triangle. The following theorems are proved in [2].

<u>Theorem 4.1</u>: The determinant of any k x k array taken with its first column along the column of ones and its first row the ith row of Pascal's triangle written in left-justified form, has value one.

<u>Theorem 4.2</u>: The determinant of any k x k array taken with its first row along the row of ones or with its first column along the column of ones in Pascal's triangle written in rectangular form, is one.

For example,

$$1 = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 10 & 15 & 21 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 35 \\ 1 & 6 & 21 & 56 \end{vmatrix} .$$

Pascal's triangle also has sequences of determinants which have binomial coefficients for their values. Here we have to number the rows and columns of Pascal's triangle; the row of ones is the zeroeth row; the column of ones, the zeroeth column. To illustrate some of the sequences of determinants considered here, we look back at the matrix P of (3.1) which contains the first n rows and columns of Pascal's triangle written in rectangular form. When 2 x 2 determinants are taken across the first and second rows of Pascal's rectangular array,

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \quad \begin{vmatrix} 3 & 4 \\ 6 & 10 \end{vmatrix} = 6, \quad \begin{vmatrix} 4 & 5 \\ 10 & 15 \end{vmatrix} = 10, \quad \ldots,$$

giving values found in the second column of Pascal's triangle. Of course, the 1 x 1 determinants along the first row give the values found in the first column of Pascal's triangle. Taking 3 x 3 determinants yields

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4, \quad \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} = 10, \quad \ldots,$$

successive entries in the third column of Pascal's triangle. In fact, taking successive k x k determinants along the first, second, ..., and kth rows yields the successive entries of the kth column of Pascal's triangle.

The following theorems are proved in [3].

<u>Theorem 4.3</u>: If Pascal's triangle is written in left-justified form, any k x k matrix selected within the array with its first column the first column of Pascal's triangle and its first row the ith row has determinant value given by the binomial coefficient $\binom{i+k-1}{k}$.

<u>Theorem 4.4</u>: The determinant of the k x k matrix taken with its first column the jth column of Pascal's triangle written in rectangular form, and its first row the first row of the rectangular Pascal array, has values given by the binomial coefficient $\binom{j+k-1}{k}$.

5. SPECIAL DETERMINANTS IN THE FIBONACCI CONVOLUTION TRIANGLE AND IN THE TRINOMIAL TRIANGLE ARRAYS

Now we are ready to prove that the unit determinants and binomial coefficient determinants of Section 4 are also found in the Fibonacci convolution triangle and in the trinomial coefficient triangle. Returning to (3.1), the first n entries of the first n rows of the Fibonacci convolution triangle are given by the matrix product $FP = C$. But, notice that $k \times k$ submatrices of C taken along either the first or second matrix row are the product of a $k \times k$ submatrix of F with a unit determinant and a similarly placed $k \times k$ submatrix of P which has been evaluated in Theorem 4.2 or Theorem 4.4. Let us also number the Fibonacci convolution triangle as Pascal's triangle, with the top row the zeroeth row. Thus, we have

Theorem 5.1: Let a $k \times k$ matrix M be selected from the Fibonacci convolution triangle in rectangular form. If M includes the row of ones, then $\det M = 1$. If M has its first column the jth column and its first row along the first row of the Fibonacci array, then $\det M = \binom{j+k-1}{k}$.

Reasoning in a similar fashion from (3.2), the matrix product AF^t and Theorems 4.1 and 4.3 yield the following, where the trinomial coefficient triangle is numbered as Pascal's triangle, with the left-most column the zeroeth column.

Theorem 5.2: Let a $k \times k$ matrix N be selected from the trinomial triangle written in left-justified form. If N includes the column of ones, then $\det N = 1$. If N has its first row the ith row and its first column along the first column of the trinomial triangle, then $\det N = \binom{i+k-1}{k}$.

These results are generalized in [2] and [3]. Other classes of determinants are also developed there. The reader should verify the results given here numerically.

REFERENCES

1. Verner E. Hoggatt, Jr., "Convolution Triangles For Generalized Fibonacci Numbers," Fibonacci Quarterly, Vol. 8, No. 2, March, 1970, pp. 158-171.
2. Marjorie Bicknell and V. E. Hoggatt, Jr., "Unit Determinants in Generalized Pascal Triangles," Fibonacci Quarterly, February, 1973.
3. Marjorie Bicknell and V. E. Hoggatt, Jr., "Special Determinants Found Within Generalized Pascal Triangles," Fibonacci Quarterly, April, 1973.

A PRIMER FOR THE FIBONACCI NUMBERS: PART IX

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TO PROVE: F_n Divides F_{nk}

For many years, it has been known that the n^{th} Fibonacci number F_n divides F_m if and only if n divides m, $n > 2$. Many different proofs have been given; it will be instructive and entertaining to examine some of them.

Some special cases are very easy. It is obvious that F_k divides F_{2k}, for $F_{2k} = F_k L_k$. If we wish only to prove that F_n divides F_{nk} when k is a power of 2, the identity

$$F_{2^j n} = F_n L_n L_{2n} L_{4n} \cdots L_{2^{j-1} n}$$

suffices.

1. PROOFS USING THE BINET FORM

Perhaps the simplest proof to understand is one which depends upon simple algebra and the Binet form (see [1]),

$$(1) \qquad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \,,$$

where

$$\alpha = (1 + \sqrt{5})/2, \qquad \beta = (1 - \sqrt{5})/2$$

are the roots of $x^2 - x - 1 = 0$. Then

$$F_n = \frac{\alpha^{nk} - \beta^{nk}}{\alpha - \beta} = \frac{\alpha^k - \beta^k}{\alpha - \beta} \cdot (M) = F_k M,$$

where

$$M = \alpha^{(n-1)k} + \alpha^{(n-2)k}\beta^k + \alpha^{(n-3)k}\beta^{2k} + \cdots + \alpha^k \beta^{(n-2)k} + \beta^{(n-1)k} .$$

If M is an integer, then F_k divides F_{nk}, $k \neq 0$.

Since $\alpha\beta = -1$, if $(n-1)k$ is odd, pairing the first and last terms, second and next to last terms, and so on,

$$M = (\alpha^{(n-1)k} + \beta^{(n-1)k}) + (-1)^k(\alpha^{(n-3)k} + \beta^{(n-3)k}) + (-1)^{2k}(\alpha^{(n-5)k} + \beta^{(n-5)k}) + \ldots$$

$$= L_{(n-1)k} + (-1)^k L_{(n-3)k} + (-1)^{2k} L_{(n-5)k} + \ldots ,$$

where the n^{th} Lucas number is given by

$$(2) \qquad L_n = \alpha^n + \beta^n .$$

Thus, M is the sum of integers, and hence an integer. If $(n-1)k$ is even, the symmetric pairs can again be formed except for the middle term which is

$$(\alpha\beta)^{(n-1)k/2} = (-1)^{(n-1)k/2} ,$$

again making M an integer. Thus, F_k divides F_{nk}, or, F_n divides F_m if n divides m.

2. PROOFS BY MATHEMATICAL INDUCTION

Other proofs can be derived, starting with a known identity and using mathematical induction. For example, use the known identity (see [2])

$$(3) \qquad F_{m+n} = F_m F_{n+1} + F_{m-1} F_n .$$

Let $m = nk$:

$$(4) \qquad F_{nk+n} = F_{n(k+1)} = F_{nk} F_{n+1} + F_{nk-1} F_n .$$

Obviously, F_n divides F_n and F_n divides F_{2n}, for $F_{2n} = F_n L_n$, so that F_n divides F_{kn} for $k = 1, 2$. Assume that F_n divides F_{in} for $i = 1, 2, \ldots, k$. Then, since F_n divides F_n and F_n divides F_{kn}, identity (4) forces F_n also to divide $F_{n(k+1)}$, so that F_n divides F_{kn} for all positive integers k.

Another identity, easily proved using (2) and (3), which leads to an easy proof by mathematical induction is

(5) $$L_nF_{m-n} + F_nL_{m-n} = 2F_m .$$

Let m = nk, yielding

(6) $$L_nF_{n(k-1)} + F_nL_{n(k-1)} = 2F_{nk} .$$

If F_n divides F_n and F_n divides $F_{n(k-1)}$, then F_n must divide F_{nk}, for $|F_n| > 2$.

A less obvious identity given by Siler [3] also yields a proof by mathematical induction:

(7) $$((-1)^n + 1 - L_n)\left(\sum_{i=1}^{k} F_{in}\right) = (-1)^n F_{kn} - F_{n(k+1)} + F_n .$$

If F_n divides F_{in} for i = 1, 2, 3, ..., k, then F_n is a factor of the left-hand member of (7). Since F_n divides F_n and F_n divides F_{kn}, F_n must also divide $F_{n(k+1)}$, so that F_n divides F_{kn} for all positive integers k.

3. PROOFS FROM GENERATING FUNCTIONS AND POLYNOMIALS

Now let us look for elegance. Suppose that we have proved the generating function identity given in [4],

$$\frac{F_n x}{1 - L_n x + (-1)^n x^2} = \sum_{k=0}^{\infty} F_{nk} x^k .$$

Then, since the leading coefficient of the divisor is one and the resulting operations of division are multiplying, adding, and subtracting integers, the quotient coefficients F_{nk}/F_n of powers of x are integers, and F_n divides F_{nk} for all integers $k \geq 0$.

Let us develop a generating function for a related proof that L_n divides L_{kn} whenever k is odd. Applying (2) and the formula for summing an infinite geometric progression,

$$\sum_{i=0}^{\infty} L_{(2i+1)n} x^i = \sum_{i=0}^{\infty} \alpha^{n(2i+1)} x^i + \sum_{i=0}^{\infty} \beta^{n(2i+1)} x^i$$

$$= \frac{\alpha^n}{1 - \alpha^{2n} x} + \frac{\beta^n}{1 - \beta^{2n} x}$$

$$= \frac{(\alpha^n + \beta^n)(1 - (-1)^n x)}{1 - (\alpha^{2n} + \beta^{2n})x + (\alpha\beta)^{2n} x^2}$$

$$= \frac{L_n(1 - (-1)^n x)}{1 - L_{2n} x + x^2} \quad .$$

Then

$$\sum_{i=0}^{\infty} \frac{L_{(2i+1)n}}{L_n} x^i = \frac{1 - (-1)^n x}{1 - L_{2n} x + x^2} ,$$

so that by the same reasoning given for the Fibonacci generating function above, $L_{(2i+1)n}/L_n$ is an integer.

Next, we prove that $L_{(2k+1)n}/L_n$ is an integer another way. Now it is true that

$$L_{(2k+1)n} = L_n L_{2kn} - (-1)^{n+1} L_{(2k-1)n}$$

so that

$$\frac{L_{(2k+1)n}}{L_n} = L_{2n} - (-1)^{n+1} \frac{L_{(2k-1)n}}{L_n} \quad .$$

Thus, we are set up to use mathematical induction since when $k = 1$, it is clear that L_n divides L_n. Thus, if $L_{(2k-1)n}/L_n$ is an integer, then $L_{(2k+1)n}/L_n$ is also an integer. The proof is complete by mathematical induction.

We can carry this one step further, and prove that L_m is not divisible by L_n if $m \neq (2k+1)n$, $n \geq 2$, for

$$L_{(2k+1)n+j} = L_n L_{2kn+j} + (-1)^n L_{(2k-1)n+j}, \qquad j = 1, 2, 3, \ldots, 2n-1 .$$

Thus, given that some $j = 1, 2, 3, \ldots,$ or $2n - 1$ exists so that $L_{(2k+1)n+j}$ is divisible by L_n, then by the method of infinite descent, $L_{(2k-1)n+j}$ is divisible by L_n for this same $j = 1, 2, 3, \ldots,$ or $2n - 1$. This will ultimately yield the inequality

$$-|L_n| < L_{-n+j} < L_n \qquad ,$$

which is clearly a contradiction since the L_s in that range are all smaller than L_n, $n \geq 2$. The same technique can be used on F_{nk} and F_k to prove that F_n divides F_m only if n divides m, $n > 2$. (Since $F_2 = 1$ divides all F_n, we must make the qualification $n > 2$.)

If the theory of Fibonacci polynomials is at our disposal, the theorem that F_n divides F_m if and only if n divides m, $n > 2$, becomes a special case. (See [5]).

If the following identity is accepted (proved in [5]),

$$F_m = F_n\left(\sum_{i=0}^{p-1} (-1)^{in} L_{m-(2i+1)n}\right) + (-1)^{pn} F_{m-2pn}, \qquad p \geq 1,$$

when $|n| < |m|$, $n \neq 0$, the identity can be interpreted in terms of quotients and remainders; the quotient being a sum of Lucas numbers and the remainder of least absolute value being a Fibonacci number or its negative. The remainder is zero if and only if either $F_{m-2pn} = 0$ or $F_{m-2pn} = \pm F_n$, in which case the quotient is changed by ± 1. In the first case, $m - 2pn = 0$, so that m is an even multiple of n; and in the second, $m - 2pn = \pm n$, with m an odd multiple of n. So, F_n divides F_m if and only if n divides m, $n > 2$.

That F_n divides F_m only if n divides m can also be proved through use of the Euclidean Algorithm [2] or as the solution to a Diophantine equation [6] to establish that

$$(F_m, F_n) = F_{(m,\ n)} \qquad (m \geq n > 2),$$

or, that the greatest common divisor of two Fibonacci numbers is a Fibonacci number whose subscript is the greatest common divisor of the subscripts of the other two Fibonacci numbers.

4. THE GENERAL CASE

A second proof that L_n divides L_m if and only if $m = (2k+1)n$, $n \geq 2$, provides a springboard for studying the general case. The identity

(8) $$L_{m+n} = F_{m+1}L_n + F_mL_{n-1}$$

indicates that L_n divides L_{m+n} if L_n divides F_m. Since $F_{2p} = L_pF_p$, L_p divides F_{2p}. But since

$$F_{2(k+1)p} = F_{2kp+2p} = F_{2kp}F_{2p+1} + F_{2kp-1}F_{2p} \quad ,$$

whenever L_p divides F_{2kp}, it must divide $F_{2(k+1)p}$, and we have proved by mathematical induction that L_p divides F_{2kp} for all positive integers k. Then, returning to (8), if $m = 2kn$, L_n divides L_{m+n}, or,

$$L_{2kn+n} = L_{(2k+1)n} = F_{2kn+1}L_n + F_{2kn}L_{n-1} \quad ,$$

so that L_n divides $L_{(2k+1)n}$.

To prove that L_n divides L_m only if $m = (2k + 1)n$, $n \geq 2$, we prove that L_n divides F_m only if $m = 2kn$, $n \geq 2$. We use the identity

$$F_{2n-j} = L_nF_{n-j} + (-1)^{n+1}F_{-j} \, , \qquad j = 1, 2, \ldots, n-1 \quad ,$$

to show that L_n cannot divide F_{2n-j}. If L_n divides F_{2n-j}, then L_n must divide F_{-j}, but $L_n > F_n > |F_{-j}|$, clearly a contradiction. Thus, L_n divides L_m if and only if $m = (2k+1)n$. A proof of this same theorem using algebraic numbers is given by Carlitz in [7].

Now we consider the general case. Given a Fibonacci sequence defined by

$$H_1 = p, \qquad H_2 = q, \qquad H_{n+2} = H_{n+1} + H_n \quad ,$$

under what circumstances does H_n divide H_m?

Studying a sequence such as

1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, 665, 1076, ...

quickly convinces one that each member divides other members of the sequence in a regular fashion. For example, 5 divides itself and every fifth member thereafter, while 4 divides itself and every sixth member thereafter.

The mystery is resolved by the identity

$$H_{m+n} = F_{m+1}H_n + F_mH_{n-1} .$$

If H_n divides F_m, then H_n divides every m^{th} term of the sequence thereafter. Further, divisibility of terms of H_n by an arbitrary integer p can be predicted using tables of Fibonacci entry points. If H_k is divisible by p, then H_{k+e} is the next member of the sequence divisible by p, where e is the entry point of p for the Fibonacci sequence. For example, if 41 divides H_n, then 41 divides H_{n+20} and 41 divides H_{n+20k} since 20 is the subscript of the first Fibonacci number divisible by 41, but 41 will divide no member of the sequence between H_n and H_{n+20}.

While any member of the Lucas sequence divides some Fibonacci number and hence many Fibonacci numbers (obvious by the identity $F_{2k} = L_kF_k$), it can be proved that no Fibonacci number greater than or equal to 5 divides any Lucas number. Also, it can be proved that every integer divides some Fibonacci number, which is false for generalized Fibonacci numbers and for Lucas numbers.

REFERENCES

1. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers (Boston: Houghton Mifflin Co., 1969), pp. 37-39.
2. N. N. Vorobyov, Fibonacci Numbers (Boston: D. C. Heath, 1963), pp. 22-24.
3. Ken Siler, "Fibonacci Summations," Fibonacci Quarterly, Vol. 1, No. 3, October, 1963, pp. 67-69.
4. V. E. Hoggatt, Jr., and D. A. Lind, "A Primer for the Fibonacci Numbers, Part VI." Fibonacci Quarterly, Vol. 5, No. 5, Dec., 1967, pp. 445-460.
5. Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII," Fibonacci Quarterly, Vol. 8, No. 4, October, 1970, pp. 407-420.
6. Glen Michael, "A New Proof for an Old Property," Fibonacci Quarterly, Vol. 2, No. 1, February, 1964, pp. 57-58.
7. Leonard Carlitz, "A Note on Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 1, February, 1964, pp. 15-28.

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That F_n divides F_{nk} also follows from the identity

$$F_{kn}/F_n = \sum_{t=1}^{[(k+1)/2]} (-1)^{(n+1)(t+1)} \binom{k-t}{t-1} L_n^{k-2t+1}$$

where [x] denotes the greatest integer function. (Problem H-172, David Englund, and Problem H-135, James E. Desmond, FQ, Dec., 1969, pp. 518-519.)

A PRIMER FOR THE FIBONACCI NUMBERS: PART X

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ON THE REPRESENTATION OF INTEGERS

The representation of integers is a topic that has been implicit in our mathematics education from our earliest years due to the fact that we employ a positional system of notation. A number such as 35864 in base ten assumes the existence of a sequence 1, 10, 100, 1000, 10000, ... , running from right to left. The digits multiplied by the members of the sequence taken in order give the indicated integer. In this case, the representation means

$$3 \cdot 10000 + 5 \cdot 1000 + 8 \cdot 100 + 6 \cdot 10 + 4 .$$

Another way of thinking of these multipliers is this: they are the number of times various members of the sequence are being used.

It is instructive to see that such a sequence used as a base for representing integers arises naturally. Suppose we allow multipliers 0, 1, or 2. We wish to have a sequence that will enable us to represent all the positive integers and furthermore we want this sequence with the multipliers to do this uniquely; that is, for each integer there is one and only one representation by means of the sequence and the multipliers. Clearly, the first member of the sequence will have to be 1; otherwise, we could never represent the first integer 1. With this, we can represent 0, 1, or 2. Hence, the next integer we need is 3. The following table shows how at each step we are able to represent additional integers and likewise what is the next integer that is needed.

Sequence	Representations added
1	0, 1, 2
3	3, 4, 5, 6, 7, 8
9	9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, ..., 26
27	27, 28, 29, ..., 79, 80
81	81, 82, 83, ..., 241, 242

Note that, as far as we have gone, the representation is unique. Assume that we have unique representation when the sequence goes to 3^n and that this representation extends to $3^{n+1} - 1$. Adding 3^{n+1} to the sequence enables us to go from 3^{n+1} to $2 \cdot 3^{n+1} + 3^{n+1} - 1$ in a unique manner, but this sum is $3^{n+2} - 1$.

Thus, the base three representation of integers using the sequence 1, 3, 9, 27, 81, ... arises naturally in the case of allowed multipliers 0, 1, 2, and the requirements of complete and unique representation.

Perhaps the most interesting case of representation is that in which the allowed multipliers are 0, 1. We build up the sequence that goes with these multipliers giving complete and unique representation.

Sequence	Representations added
1	1
2	2, 3
4	4, 5, 6, 7
8	8, 9, 10, 11, 12, 13, 14, 15
16	16, 17, 18, ..., 30, 31
32	32, 33, 34, ..., 62, 63

Thus far the representation is unique. If we have unique and complete representation when the largest term of the sequence is 2^n and the representation extends to $2^{n+1} - 1$, then on adding 2^{n+1} to the sequence, we extend complete and unique representation to $2^{n+1} + 2^{n+1} - 1 = 2^{n+2} - 1$.

Another way of thinking of representation when the multipliers are 0 and 1 is this: We have a sequence where integers are represented by distinct members of the sequence. Thus the base two integer 110111010 says that the number in question is the sum of 2^8, 2^7, 2^5, 2^4, 2^3, and 2. The powers of two along with 1 enable us to represent all integers uniquely by combining different powers of two.

INCOMPLETE AND NON-UNIQUE SEQUENCES

Let us return to the representation with multipliers 0, 1, and 2. Clearly, if instead of taking 1, 3, 9, 27, 81, ..., we take some larger numbers such as 1, 3, 10, 28, 82, 244, ..., it will not be possible to represent all integers.

Sequence	Representations added
1	1, 2
3	3, 4, 5, 6, 7, 8
10	10-18, 20-28
28	28-36, 38-46, 48-56, 56-64, 66-74, 76-84
82	82-90, 92-100, etc.

Below 100, the numbers that cannot be represented are 9, 19, 37, 47, 65, 75, and 91. On the other hand, 28, 56, 82, 83, and 84 have two representations.

Suppose that instead of making the numbers of the sequence slightly larger we make them a bit smaller. Let us take the sequence 1, 3, 8, 26, 80, 242, ..., as before:

Sequence	Representations added
1	1, 2
3	3, 4, 5, 6, 7, 8
8	8-16, 16-24
26	26-34, 34-42, 42-50, 52-60, 60-68, 68-76
80	80-88, 88-96, 96-104, 106-114, 114-122, 122-130, 132-140, 140-148, 148-156, 160 etc.

Up to 160, the missing integers are 25, 51, 77, 78, 79, 105, 131, 157, 158, and 159. Duplicated integers are 8, 16, 34, 42, 60, 68, 88, 96, 114, 122, 140, and 148.

The sequence 1, 3, 8, 23, 68, 203, ..., gives complete but not unique representation.

Sequence	Representations added
1	1, 2
3	3-8
8	8-16, 16-24
23	23-31, 31-39, 39-47, 46-54, 54-62, 62-70
68	68-76, 76-84, 84-92, 91-99, 99-107, 107-115, 114-122, 122-130, 130-138, 136-144, 144-152, etc.

Up to 140 there is complete representation but duplicate representation for the following: 8, 16, 23, 24, 31, 39, 46, 47, 54, 62, 68, 69, 70, 76, 84, 91, 92, 99, 107, 114, 115, 122, 130, 136, 137, and 138.

FIBONACCI REPRESENTATIONS

Let us now consider the case in which the multipliers are 0, 1 and the basic sequence is the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... That this sequence gives complete representation is not difficult to prove. In fact, the representation is still complete if we eliminate the first 1 and use the sequence 1, 2, 3, 5, 8, 13, ... In the table following, note that the representation at each stage gives complete representation up to and including $F_{n+2} - 2$. Assume this to be so up to a certain F_n. Then upon adjoining F_{n+1} to the sequence the representation will be complete to $F_{n+1} + F_{n+2} - 2$, which is much beyond F_{n+2}, the next term to be added. Thus the representation is complete, but it is evidently not unique.

Sequence	Representations added
1	1
2	2, 3
3	3, 4, 5, 6
5	5-8, 8-11
8	8-11,11-14, 13-16, 16-19
13	13-16, 16-19, 18-21. 21-24, 21-24, 24-27, 26-29, 29-32

AN INTERESTING THEOREM

To get a new perspective on representation by this Fibonacci sequence we write down the representations of the integers in their various possible forms. (Read 10110 as $8 + 3 + 2$ or $1 \cdot F_6 + 0 \cdot F_5 + 1 \cdot F_4 + 1 \cdot F_3 + 0 \cdot F_2$)

INTEGER	REPRESENTATIONS	INTEGER	REPRESENTATIONS
1	1	11	10100, 10011, 1111
2	10	12	10101
3	11, 100	13	11000, 10110, 100000
4	101	14	100001, 11001, 10111
5	110, 1000	15	100010, 11010
6	111, 1001	16	100100, 100011, 11100, 11011
7	1010	17	100101, 11101
8	1100, 1011, 10000	18	101000, 100110, 11110
9	10001, 1101	19	101001, 100111, 11111
10	10010, 1110	20	101010

Now the Fibonacci sequence has the property that the sum of two consecutive members of the sequence gives the next member of the sequence. Accordingly, one might argue, it is superfluous to have two successive members of the sequence in a representation since they can be combined to give the next member. If this is done, we arrive at representations in which there are no two consecutive ones in the representation. Looking over the list of integers that we have represented thus far, it appears that there is just one such representation for each integer in this form.

Suppose we go at this from another direction. We are building up a sequence that will represent the integers uniquely with multipliers 0 and 1. However, we stipulate that no two consecutive members of the sequence may be found in any representation. We form a table as before.

Sequence	Representations added
1	1
2	2
3	3, 4
5	5, 6, 7
8	8, 9, 10, 11, 12
13	13, 14, 15, 16, 17, 18, 19, 20

To this point the representation is unique and the sequence that is emerging is the Fibonacci sequence 1, 2, 3, 5, 8, 13, ... Assume that up to F_n there is unique representation to $F_{n+1} - 1$. On adding F_{n+1} to the sequence, we cannot use F_n in conjunction with it but only terms up to F_{n-1}. But by supposition these may represent all integers up to $F_n - 1$ in a unique way. Hence with F_{n+1} we can represent uniquely all integers from F_{n+1} to $F_{n+1} + F_n - 1 = F_{n+2} - 1$. Hence the uniqueness and completeness of this type of representation are established, which is known as Zeckendorf's Theorem.

MORE ZEROES IN THE REPRESENTATION

A natural question to ask is: Would it be possible to require that there be at least two zeroes between 1's in the representation and obtain unique representation? We can build up the sequence as before taking into account this requirement.

Sequence	Representations added
1	1
2	2
3	3
4	4, 5
6	6, 7, 8
9	9, 10, 11, 12
13	13, 14, 15, 16, 17, 18
19	19, 20, 21, 22, 23, 24, 25, 26, 27
28	28-40

Up to this point, the representation is complete and unique. We have a sequence, but it would be difficult to operate with it unless we knew the way it builds up according to some recursion relation. The relation appears as

$$T_{n+1} = T_n + T_{n-2} .$$

Now assume that up to T_n we have unique representation to $T_{n+1} - 1$, where T_{n+1} is given by the recursion relation in terms of previous members of the sequence. Then on adding T_{n+1} to the sequence we may not use T_n or T_{n-1} in conjunction with it but only terms up to T_{n-2}. But these give unique and complete representation to $T_{n-1} - 1$. Hence upon adding T_{n+1} to the sequence we have extended unique and complete representation from T_{n+1} to $T_{n+1} + T_{n-1} - 1 = T_{n+2} - 1$. Thus, the uniqueness and completeness are established in general.

The sequences required for unique and complete representation when three, four, or more zeroes are required between 1's in the representation can be built up in the same way. Some are listed below.

Zeroes	Sequence derived	Recursion relation
3	1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, 69, 95, 131, 181, 250, ...	$T_{n+1} = T_n + T_{n-3}$
4	1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, 45, 60, 80, 106, 140, 185, ...	$T_{n+1} = T_n + T_{n-4}$
5	1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 27, 34, 43, 55, 71, 92, 119, ...	$T_{n+1} = T_n + T_{n-5}$
6	1, 2, 3, 4, 5, 6, 7, 8, 10, 13, 17, 22, 28, 35, 43, 53, 66, 83, 105, 133, ...	$T_{n+1} = T_n + T_{n-6}$

For k zeroes, the sequence is 1, 2, 3, 4, ..., k, k + 1, k + 2, which enables us to get k + 3; then k + 4 which gives k + 5, k + 6; and so on. Up to this point the representation is unique and complete; the recursion relation beginning with k + 2 is $T_{n+1} = T_n + T_{n-k}$. Assume that the sequence up to T_n gives unique and complete representation to $T_{n+1} - 1$. Then upon adding T_{n+1} the highest term we can use in conjunction with it is $T_{n+1-k-1} = T_{n-k}$ which gives unique representation to $T_{n-k+1} - 1$ by hypothesis. Hence upon adding T_{n+1} we have unique representation from T_{n+1} to $T_{n+1} + T_{n-k+1} - 1 = T_{n+2} - 1$.

MULTIPLIERS 0, 1, 2

We know that we obtain unique and complete representation using multipliers 0, 1, 2 when we have the geometric progression 1, 3, 9, 27, ...

Can we find a unique and complete representation if we demand that there be a zero between any two non-zero digits in the representation? Let us build this up as before.

Sequence	Representations added
1	1, 2
3	3, 6
4	4, 5, 6, 8, 9, 10
7	7,8, 9, 10, 13, 14, 15, 16, 17, 20
11	11-14, 17, 15-17, 19-25, 28, 26-28, 30-32
18	18-21, 24, 22-24, 26-28, 25-28, 31-35, 38, 36-39, 42, 40-42, 44-46, 43-46, 49-53, 56

It appears that the sequence is the Lucas numbers. The representation is not unique. But a Lucas number L_n allows complete representation to the next Lucas number L_{n+1}(and beyond) without any additional Lucas numbers being represented. Assume that this is the case up to a certain n. Upon adding L_{n+1} we may not use L_n. Going back to L_{n-1} and preceding terms we can represent all integers up to $L_n - 1$ without being able to represent any Lucas numbers L_n, L_{n+1}, ... Thus adding L_{n+1} allows the representation of numbers L_{n+1} to $L_{n+1} + L_n - 1 = L_{n+2} - 1$, but does not give L_{n+2} since this would require L_n. If we use $2L_{n+1}$ we would need L_n to get L_{n+3}, but since we do not have L_n it is not possible to arrive at this Lucas number. To dispose of L_{n+4} and higher Lucas numbers, we have to set a bound on the highest number at which we may arrive. Starting with L_{n-1} and working backward, the highest sum we can have is twice the sum of alternate terms beginning with L_{n-1}. If n - 1 is odd, this sum is $2(L_n - 2)$, and if n - 1 is even, this sum is $2(L_n - 1)$. In either case, the sum is less than $2L_n$. Hence an upper bound for terms when L_{n+1} is added to the sequence is $2L_{n+1} + 2L_n = 2L_{n+2}$. But $L_{n+4} = 2L_{n+2} + L_{n+1}$ which is greater than $2L_{n+2}$. Hence it is not possible to arrive at L_{n+4} or higher Lucas numbers.

This result was very encouraging and led to an investigation of cases with multipliers 0, 1, 2, 3; then 0, 1, 2, 3, 4; etc., where we still require one zero between non-zero digits. The first few terms looked interesting.

Multipliers 0, 1, 2, 3: 1, 4, 5, 9, 14, ...
Multipliers 0, 1, 2, 3, 4: 1, 5, 6, 11, 17, ...
Multipliers 0, 1, 2, 3, 4, 5: 1, 6, 7, 13, 20, ...

Unfortunately, in the sequence 1, 4, 5, 9, 14, ..., if we continue with the terms 23, 37, 60, we find that 60 is already represented by 14 and lower terms. In the sequence 1, 5, 6, 11, 17, 28, ..., the 28 is represented by earlier terms. We have run into a DRY HOLE.

Next, keeping the multipliers 0, 1, 2, the case of two zeroes between non-zero digits was investigated. This led to the sequence 1, 3, 4, 5, 9, 13, 22, 31, 53, 75, 128, 181, ..., where there are two apparent laws of formation, one for odd-numbered terms, and a second for even-numbered terms,

(1) $$T_{2n+1} = T_{2n} + T_{2n-1} ,$$

(2) $$T_{2n+2} = T_{2n+1} + T_{2n-1} .$$

There are equivalent representations of these relations. By (1) and (2),

(3) $$T_{2n+1} = (T_{2n-1} + T_{2n-3}) + T_{2n-1} = 2T_{2n-1} + T_{2n-3} ,$$

(4) $$T_{2n+2} = (T_{2n} + T_{2n-1}) + T_{2n-1} = T_{2n} + 2T_{2n-1} .$$

Since by (1) $T_{2n-1} = T_{2n+1} - T_{2n}$, we have from (4) $T_{2n+2} = 2T_{2n+1} - T_{2n}$, or,

(5) $$2T_{2n+1} = T_{2n+2} + T_{2n} .$$

Therefore, by using (5) to express $2T_{2n-1}$ in (4),

(6) $$T_{2n+2} = 2T_{2n} + T_{2n-2} .$$

Hence, combining (3) and (6), there is one recursion relation for the entire sequence,

(7) $$T_{n+1} = 2T_{n-1} + T_{n-3} .$$

The manner in which the sequence builds up is shown by the following table.

Sequence	Representations added
1	1, 2
3	3, 6
4	4, 8
5	5, 6, 7, 10, 11, 12
9	9-12, 15, 18-21, 24
13	13-16, 19, 17, 21, 26-29, 32, 30, 34
22	22-25, 28,26,30, 27-29, 32-34, 44-47, 50,48,52, 49-51, 54-56

To show that the sequence will continue to be built up in this way we note the following as a basis for our induction:

(1) Adding a term T_k covers all representations up to $T_{k+1} - 1$.

(2) Adding another term of the sequence does not give additional terms of the representing sequence.

(3) The largest term that can be represented by adding T_k is less than T_{k+3}.

Now, if the above is true to T_n, add the term T_{n+1}. We can use only terms to T_{n-2} and smaller in the sequence in conjunction with T_{n+1}. Such terms can represent values up to $T_{n-1} - 1$. Hence adding T_{n+1} enables us to represent values from T_{n+1} to $T_{n+1} + T_{n-1} - 1$, which gives $T_{n+2} - 1$ if $n + 1$ is odd. If $n + 1$ is even, $T_{n+1} + T_{n-1} - 1 = 2T_n - 1$, but using terms up to T_{n-2} we can represent values to $2T_n + T_{n-2} - 1 = T_{n+2} - 1$. Hence all representations up to $T_{n+2} - 1$ are covered.

On adding T_{n+1} to the sequence we do not obtain any other sequence terms. For $T_{n+2} = T_{n+1} + T_{n-1}$ and $T_{n+3} = 2T_{n+1} + T_{n-1}$ if $n + 1$ is odd, and T_{n-1} is not available in conjunction with T_{n+1}. Similarly, if $n + 1$ is even, $T_{n+2} = T_{n+1} + T_n$ and $T_{n+3} = 2T_{n+1} + T_{n-1}$ where neither T_n nor T_{n-1} is available. Finally, T_{n+4} is larger than any term that can be formed using T_{n+1} and smaller terms.

CONCLUSION

A great deal of work has been done on representations of integers in recent years. Much of this has appeared in the Fibonacci Quarterly which has published some two dozen articles totalling approximately 300 pages by such mathematicians as Carlitz, Brown, Hoggatt, Ferns, Klarner, Daykin, and others. The number of byways that may be investigated is great. It could be the project of a lifetime.

A PRIMER FOR THE FIBONACCI NUMBERS: PART XII

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ON REPRESENTATIONS OF INTEGERS USING FIBONACCI NUMBERS

In how many ways may a given positive integer p be written as the sum of distinct Fibonacci numbers, order of the summands not being considered? The Fibonacci numbers are 1, 1, 2, 3, 5, ..., F_n, ..., where $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. For example, $10 = 8 + 2 = 2 + 3 + 5$ is valid, but $10 = 5 + 5 = 1 + 1 + 8$ would not be valid. The original question is an example of a representation problem, which we do not intend to answer fully here. We will explore representations using the least possible number or the greatest possible number of Fibonacci numbers.

1. THE ZECKENDORF THEOREM

First we prove by mathematical induction a lemma which has immediate application.

Lemma: The number of subsets of the set of the first n integers, subject to the constraint that no two consecutive integers appear in the same subset, is F_{n+2}, $n \geq 0$.

Proof: The theorem holds for n = 0, for when we have a set of no integers the only subset is $\emptyset$, the empty set. We thus have one subset and $F_{0+2} = F_2 = 1$.

For n = 1,	2 subsets:	$\{1\}, \emptyset$;	$F_{1+2} = F_3 = 2$
n = 2,	3 subsets:	$\{1\}, \{2\}, \emptyset$;	$F_{2+2} = F_4 = 3$
n = 3,	5 subsets:	$\{1,3\}, \{3\}, \{2\}, \{1\}, \emptyset$;	$F_{3+2} = F_5 = 5$

Assume that the lemma holds for $n \leq k$. Then notice that the subsets formed from the first (k + 1) integers are of two kinds--those containing (k + 1) as an element and those which do not contain (k + 1) as an element. All subsets which contain (k + 1) cannot contain element k and can be formed by adding (k + 1) to each subset, made up of the (k - 1) integers, which satisfies the constraint. By the inductive hypothesis there are F_{k+2} subsets satisfying the constraint and using only the first k integers, and there are F_{k+1} subsets satisfying the constraints and using the first (k - 1) integers. Thus there are precisely

$$F_{k+2} + F_{k+1} = F_{k+3} = F_{(k+1)+2}$$

subsets satisfying the constraint and using the first (k + 1) integers. The proof is complete by mathematical induction.

Now, for the application. The number of ways in which n boxes can be filled with zeroes or ones (every box containing exactly one of those numbers) such that no two "ones" appear in adjacent boxes is F_{n+2}. (To apply the lemma simply number the n boxes.) Since we do not wish to use all zeroes ($\emptyset$, the empty set in the lemma) the number of logically useable arrangements is $F_{n+2} - 1$. Now, to use the distinctness of the Fibonacci numbers in our representations, we must omit the initial $F_1 = 1$, so that to the n boxes we assign in order the Fibonacci numbers $F_2, F_3, \ldots, F_{n+1}$. This gives us a binary form for the <u>Fibonacci positional notation</u>. The interpretation to give the "zero" or "one" designation is whether or not one uses that particular Fibonacci number in the given representation. If a one appears in the box allocated to F_k, then F_k is used in this particular representation. Notice that since no two adjacent boxes can each contain a "one", no two consecutive Fibonacci numbers may occur in the same representation.

Since the following are easily established identities,

$$F_2 + F_4 + \cdots + F_{2k} = F_{2k+1} - 1 ,$$

$$F_3 + F_5 + \cdots + F_{2k+1} = F_{2k+2} - 1 ,$$

using the Fibonacci positional notation the largest number representable under the constraint with our n boxes is $F_{n+2} - 1$. Also the number F_{n+1} is in the n^{th} box, so we must be able to represent at most $F_{n+2} - 1$ distinct numbers with $F_2, F_3, \ldots, F_{n+1}$ subject to the constraint that no two adjacent Fibonacci numbers are used. Since there are $F_{n+2} - 1$ different ways to distribute ones and zeroes in our n boxes, there are $F_{n+2} - 1$ different representations which could represent possibly $F_{n+2} - 1$ different integers. That each integer p has a unique representation is the <u>Zeckendorf Theorem:</u> [1]

Theorem: Each positive integer p has a unique representation as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are used in the representation.

We shall defer the proof of this until a later section. Now, a minimal representation of an integer p uses the least possible number of Fibonacci numbers in the sum. If both F_k and F_{k-1} appeared in a representation, they could both be replaced by F_{k+1}, thereby reducing the number of Fibonacci numbers used. It follows that a representation that uses no two consecutive Fibonacci numbers is a minimal representation and a Zeckendorf representation.

2. ENUMERATING POLYNOMIALS

Next, we use enumerating polynomials to establish the existence of at least one minimal representation for each integer.

An enumerating polynomial counts the number of Fibonacci numbers necessary in the representation of each integer p in a given interval $F_m \leq p < F_{m+1}$ in the following way. Associated with this interval is a polynomial $P_{m-1}(x)$. A term ax^j belongs to $P_{m-1}(x)$ if in the interval $F_m \leq p < F_{m+1}$, there are $\underline{a}$ integers p whose minimal representation requires $\underline{j}$ Fibonacci numbers. For example, consider the interval $F_6 = 8 \leq p < 13 = F_7$. Here, we can easily determine the minimal representations

$$\begin{aligned} 8 &= 8 \\ 9 &= 1 + 8 \\ 10 &= 2 + 8 \\ 11 &= 3 + 8 \\ 12 &= 1 + 3 + 8 \end{aligned}$$

Thus, $P_5(x) = x^3 + 3x^2 + x$ because one integer required 3 Fibonacci numbers, 3 integers required 2 Fibonacci numbers, and one integer required one Fibonacci number in its minimal representation. We note in passing that all the minimal representations in this interval contain 8 but not 5. We now list the first nine enumerating polynomials.

	$F_m \leq p < F_{m+1}$	$P_{m-1}(x)$
$m = 1$	$1 \leq p < 1$	$0 = P_0(x)$
$m = 2$	$1 \leq p < 2$	$x = P_1(x)$
$m = 3$	$2 \leq p < 3$	$x = P_2(x)$
$m = 4$	$3 \leq p < 5$	$x^2 + x = P_3(x)$
$m = 5$	$5 \leq p < 8$	$2x^2 + x = P_4(x)$
$m = 6$	$8 \leq p < 13$	$x^3 + 3x^2 + x = P_5(x)$
$m = 7$	$13 \leq p < 21$	$3x^3 + 4x^2 + x = P_6(x)$
$m = 8$	$21 \leq p < 34$	$x^4 + 6x^3 + 5x^2 + x = P_7(x)$
$m = 9$	$34 \leq p < 55$	$4x^4 + 10x^3 + 6x^2 + x = P_8(x)$

We shall now proceed by mathematical induction to derive a recurrence relation for the enumerating polynomials $P_m(x)$. It is evident from the definitions that an enumerating polynomial for $F_m \leq p < F_{m+2}$ is the sum of the enumerating polynomials for $F_m \leq p < F_{m+1}$ and $F_{m+1} \leq p < F_{m+2}$. Also it will be proved that the minimal representation of any integer p in the interval $F_m \leq p < F_{m+1}$ contains F_m but not F_{m-1}. If we added F_{m+2} to each such minimal representation of p in $F_m \leq p < F_{m+1}$ we would get a minimal representation of an integer in the interval

$$L_{m+1} = F_m + F_{m+2} \leq p < F_{m+1} + F_{m+2} = F_{m+3} .$$

Clearly the enumerating polynomial for this interval is $xP_{m-1}(x)$ since each integer p in this interval has one more Fibonacci number in its minimal representation than did the corresponding integer p in the interval $F_m \leq p < F_{m+1}$.

Next, the integers p in the interval $F_{m+2} \leq p < F_{m+3}$ require an F_{m+2} in this minimal representation while all the numbers in the interval $F_{m+1} \leq p < F_{m+2}$ have F_{m+1} in their minimal representation. In each of these minimal representations remove the F_{m+1} and put in an F_{m+2}. The resulting integer will have a minimal representation with the same number of Fibonacci numbers as was required before. In other words, the enumerating polynomial $P_{m-1}(x)$ is also the enumerating polynomial for

$$F_{m+2} - F_{m+1} + F_{m+1} \leq p' < F_{m+2} - F_{m+1} + F_{m+2} = L_{m+1} .$$

Now, the intervals $F_{m+2} \leq p' < L_{m+1}$ and $L_{m+1} \leq p' < F_{m+3}$ are not overlapping and exhaust the interval $F_{m+2} \leq p < F_{m+3}$. Thus, the enumerating polynomial for this interval is

$$P_{m+1}(x) = P_m(x) + xP_{m-1}(x) , \quad P_0(x) = 0 , \quad P_1(x) = x ,$$

which is the required recurrence relation.

Now, to show by mathematical induction that the minimal representation of any integer p in the interval $F_m \leq p < F_{m+1}$ contains F_m but not F_{m-1}, re-examine the preceeding steps. Each minimal representation in the interval $F_{m+2} \leq p < F_{m+3}$ contains F_{m+2} explicitly since we added F_{m+2} to a representation from the interval $F_m \leq p < F_{m+1}$ and by the inductive hypothesis those representations did not contain F_{m+1} but all contained F_m. Next, for the representations from $F_{m+1} \leq p < F_{m+2}$, all of which used F_{m+1} explicitly by inductive assumption, we removed the F_{m+1} and replaced it by F_{m+2} so that each representation in $F_{m+2} \leq p < F_{m+3}$ contains F_{m+2} but not F_{m+1}. Thus, if the integers p in the previous two intervals, namely, $F_m \leq p < F_{m+1}$ and $F_{m+1} \leq p < F_{m+2}$, had Zeckendorf representations, then the representations of the integers p in the interval $F_{m+2} \leq p < F_{m+3}$ are also Zeckendorf representations.

Now, notice that $P_m(1)$ is the sum of the coefficients of $P_m(x)$, or the count of the numbers for which a minimal representation exists in the interval $F_{m+1} \leq p < F_{m+2}$. But, $P_m(1) = F_m$ because $P_1(1) = P_2(1) = 1$ and $P_{m+1}(1) = P_m(1) + 1 \cdot P_{m-1}(1)$, so that the two sequences have the same beginning values and the same recursion formula. The number of integers in the interval $F_{m+1} \leq p < F_{m+2}$ is $F_{m+2} - F_{m+1} = F_m$, so that every integer is represented. Thus, at least one minimal representation exists for each integer, and we have established Zeckendorf's theorem, that each integer has a unique minimal representation in Fibonacci numbers. Notice that this means that it is possible to express any integer as a sum of distinct Fibonacci numbers. Also, notice that the coefficients of $P_m(x)$ are the summands along the diagonals of Pascal's triangle summing to F_m with increasing powers as one proceeds up the diagonals beginning with x.

3. THE DUAL ZECKENDORF THEOREM

Suppose that, instead of a minimal representation, we wished to write a maximal representation, or, to use as many distinct Fibonaçci numbers as possible in a sum to represent an integer. Then, we want no two consecutive Fibonacci numbers to be missing in the representation. Returning to our n non-empty boxes, for this case we wish to fill the boxes with zeroes and ones with no two consecutive zeroes. Here we consider n ones interposed by at most one zero. Thus, we have boxes to zero or not to zero. These zeroes can occur between the left-most one and the next on the right, between any adjacent pair of ones, and on the right of the last one if necessary. Thus, there are precisely 2^n possibilities, or, 2^n maximal representations can be written using the n Fibonacci numbers from among $F_2, F_3, \ldots, F_{2n+1}$.

Now, associate with integers p in the interval $F_n - 1 \leq p < F_{n+1} - 1$ an enumerating maximal polynomial $P^*_{n-1}(x)$ which has a term ax^j if a of the integers p require j Fibonacci numbers in their maximal representation. For example, in the interval $F_6 - 1 = 7 \leq p < 12 = F_7 - 1$, the maximal representations are

$$7 = 5 + 2$$
$$8 = 5 + 2 + 1$$
$$9 = 5 + 3 + 1$$
$$10 = 5 + 3 + 2 + 1$$
$$11 = 5 + 3 + 2 + 1$$

Thus, $P_5(x) = x^4 + 3x^3 + x^2$ because one integer requires 4 Fibonacci numbers, 3 integers require 3 Fibonacci numbers, and one integer requires 2 Fibonacci numbers in its maximal representation. Notice that all maximal representations above use 5 but none use 8. The first eight enumerating maximal polynomials are:

	$F_m - 1 \leq p < F_{m+1} - 1$	$P^*_{m-1}(x)$
$m = 2$	$0 \leq p < 1$	$1 = P^*_1(x)$
$m = 3$	$1 \leq p < 2$	$x = P^*_2(x)$
$m = 4$	$2 \leq p < 4$	$x^2 + x = P^*_3(x)$
$m = 5$	$4 \leq p < 7$	$x^3 + 2x^2 = P^*_4(x)$
$m = 6$	$7 \leq p < 12$	$x^4 + 3x^3 + x^2 = P^*_5(x)$
$m = 7$	$12 \leq p < 20$	$x^5 + 4x^4 + 3x^3 = P^*_6(x)$
$m = 8$	$20 \leq p < 33$	$x^6 + 5x^5 + 6x^4 + x^3 = P^*_7(x)$
$m = 9$	$33 \leq p < 54$	$x^7 + 6x^6 + 10x^5 + 4x^4 = P^*_8(x)$

As before, we now derive the recurrence relation for the polynomials $P_n^*(x)$.

Lemma: Each maximal representation for integers p in the interval $F_m - 1 \le p < F_{m+1} - 1$ contains explicitly F_{m-1}.

Proof: We can add F_m to each maximal representation in the interval $F_m - 1 \le p < F_{m+1} - 1$ and these numbers fall in the interval

$$2F_m - 1 \le p' < F_{m+2} - 1 \ .$$

We can also add F_m to each maximal representation in the interval $F_{m-1} - 1 \le p < F_m - 1$ and these numbers fall in the interval

$$F_{m+1} - 1 \le p' < 2F_m - 1 \ .$$

These two intervals are non-overlapping and exhaustive of the interval

$$F_{m+1} - 1 \le p < F_{m+2} - 1 \ .$$

Thus, each maximal representation in this interval contains explicitly F_m.

Thus, the enumerating polynomials $P_n^*(x)$ for maximal representations satisfy

$$P_n^*(x) = x[P_{n-1}^*(x) + P_{n-2}^*(x)], \quad P_1^*(x) = 1, \quad P_2^*(x) = x \ ,$$

and again $P_n^*(1) = F_n$. This establishes that each non-negative integer has at least one maximal representation.

Returning to the table of the first eight polynomials $P_n^*(x)$, by laws of polynomial addition, adding the enumerating maximal polynomials yields a count of how many numbers require k Fibonacci numbers in their maximal representation. So, it appears that

$$\begin{aligned}\sum_{n=1}^{\infty} P_n^*(x) &= P_1^*(x) + P_2^*(x) + P_3^*(x) + P_4^*(x) + P_5^*(x) + \dots + P_n^*(x) + \dots \\ &= 1 + x + (x^2 + x) + (x^3 + 2x^2) + (x^4 + 3x^3 + x^2) + \dots \\ &= 1 + 2x + 4x^2 + 8x^3 + \dots + 2^k x^k + \dots\end{aligned}$$

(That this is indeed the case is proved in the two lemmas following the Dual Zeckendorf Theorem.) In other words, 2^k non-negative integers require k Fibonacci numbers in their maximal representation. But requiring that each integer has at least one maximal representation exhausts the logical possibilities. Thus, each integer has a unique maximal representation in distinct Fibonacci numbers, which proves the Dual Zeckendorf Theorem [2]:

Theorem: Each positive integer has a unique representation as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are omitted in the representation.

Lemma: Let $f_1(x) = 1$, $f_2(x) = x$, and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$ be the Fibonacci polynomials. Then

$$P_n^*(x^2) = x^{n-1}f_n(x), \quad n \geq 0.$$

Proof: We proceed by mathematical induction. Observe that

$$P_1^*(x^2) = 1 = x^0 f_1(x),$$

$$P_2^*(x^2) = x^2 = x^1 f_2(x),$$

$$P_n^*(x^2) = x^2[P_{n-1}^*(x^2) + P_{n-2}^*(x^2)] .$$

Assume that

$$P_{n-1}^*(x^2) = x^{n-2}f_{n-1}(x) ,$$

$$P_{n-2}^*(x^2) = x^{n-3}f_{n-2}(x) .$$

Thus,

$$P_n^*(x^2) = x^2[x^{n-2}f_{n-1}(x) + x^{n-3}f_{n-2}(x)]$$

$$= x^{n-1}[xf_{n-1}(x) + f_{n-2}(x)] = x^{n-1}f_n(x).$$

Lemma: $$\sum_{n=1}^{\infty} P_n^*(x) = \frac{1}{1-2x}$$

Proof: The Fibonacci polynomials have the generating function

$$\frac{1}{1 - xt - t^2} = \sum_{n=1}^{\infty} f_n(x)\, t^{n-1}$$

Now let $x = t$, and then by the previous lemma,

$$\frac{1}{1 - x^2 - x^2} = \sum_{n=1}^{\infty} f_n(x)\, x^{n-1} = \sum_{n=1}^{\infty} P_n^*(x^2) = \frac{1}{1 - 2x^2} .$$

Therefore,

$$\sum_{n=1}^{\infty} P_n^*(x) = \frac{1}{1 - 2x} = 1 + 2x + 4x^2 + \ldots + 2^n x^n + \ldots .$$

Notice that the polynomials $P_n^*(x)$ have as their coefficients the summands along the rising diagonals of Pascal's triangle whose sums are the Fibonacci numbers but in the reverse order of those for $P_n(x)$. In fact, the minimal enumerating polynomials $P_n(x)$ and the maximal enumerating polynomials $P_n^*(x)$ are related as in the following lemma.

Lemma: $$P_m(x) = x^m P_m^*(1/x) \quad \text{for } m \geq 1 .$$

Proof: This relationship will be proved by mathematical induction.

$m = 1$: $P_1(x) = x = x^1[P_1^*(1/x)]$

$m = 2$: $P_2(x) = x = x^2(1/x) = x^2[P_2^*(1/x)]$

$m = 3$: $P_3(x) = x^2 + x = x^3(1/x + 1/x^2) = x^3[P_3^*(1/x)]$

Assume that

$$P_{k-1}(x) = x^{k-1} P_{k-1}^*(1/x) ,$$

$$P_k(x) = x^k P_k^*(1/x) .$$

Then, by the recurrence relations for the polynomials $P_n(x)$ and $P_n^*(x)$,

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + x\, P_{k-1}(x) \\ &= x^k P_k^*(1/x) + x x^{k-1} P_{k-1}^*(1/x) \\ &= x^{k+1}(1/x)[P_k^*(1/x) + P_{k-1}^*(1/x)] \\ &= x^{k+1} P_{k+1}^*(1/x) , \end{aligned}$$

which establishes the lemma by mathematical induction.

Then, both the minimal and maximal representations of an integer are unique. Then, an integer has a unique representation in Fibonacci numbers if and only if its minimal and maximal representations are the same, which condition occurs only for integers of the form $F_n - 1$, $n \geq 3$ [3].

In general, the representation of an integer in Fibonacci numbers is not unique, and, from the above remarks, unless the number is one less than a Fibonacci number, it will have at least two representations in Fibonacci numbers. But, one need not stop here. The Fibonacci numbers F_{2n} and F_{2n+1} can each be written as the sum of distinct Fibonacci numbers 1, 2, 3, 5, 8, ..., in n different ways. For other integers p, the reader is invited to experiment to see what theorems he can produce.

We now turn to representations of integers using Lucas numbers.

4. THE LUCAS CASE

If we change our representative set from Fibonacci numbers to Lucas numbers, we can find minimal and maximal representations of integers as sums of distinct Lucas numbers. The Lucas numbers are 2, 1, 3, 4, 7, 11, ..., defined by $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$, $n \geq 1$. (See Brown [6a].)

The derivation of a recursion formula for the enumerating minimal polynomials $Q_n(x)$ for Lucas numbers is very similar to that for the polynomials $P_n(x)$ for Fibonacci numbers. Details of the proofs are omitted here. Now, for integers p in the interval $L_n \leq p < L_{n+1}$, the enumerating minimal polynomial $Q_{n-1}(x)$ has a term dx^j if d of the integers p require j Lucas numbers in their minimal representation. For example, the minimal representation in Lucas numbers for integers p in the interval $11 = L_5 \leq p < L_6 = 18$ are:

$$\begin{aligned} 11 &= 11 \\ 12 &= 11 + 1 \\ 13 &= 11 + 2 \\ 14 &= 11 + 3 \\ 15 &= 11 + 4 \\ 16 &= 11 + 4 + 1 \\ 17 &= 11 + 4 + 2 \end{aligned}$$

so that $Q_4(x) = 2x^3 + 4x^2 + x$ since 2 integers require 3 Lucas numbers, 4 integers require 2 Lucas numbers, and one integer requires one Lucas number. Notice that $L_5 = 11$ is included in each representation, but that

$L_4 = 7$ does not appear in any representation in this interval. Also notice that we could have written $16 = 11 + 3 + 2$. To make the minimal representation unique, it is necessary to avoid one of the combinations $L_0 + L_1$ or $L_1 + L_3$; we agree not to use the combination $L_0 + L_2 = 2 + 3$ in any minimal representation unless one or both of L_1 and L_3 also appear. The first nine Lucas enumerating minimal polynomials follow.

	$L_m \le p < L_{m+1}$	$Q_{m-1}(x)$
$m = 1$	$1 \le p < 3$	$2x = Q_0(x)$
$m = 2$	$3 \le p < 4$	$x = Q_1(x)$
$m = 3$	$4 \le p < 7$	$2x^2 + x = Q_2(x)$
$m = 4$	$7 \le p < 11$	$3x^2 + x = Q_3(x)$
$m = 5$	$11 \le p < 18$	$2x^3 + 4x^2 + x = Q_4(x)$
$m = 6$	$18 \le p < 29$	$5x^3 + 5x^2 + x = Q_5(x)$
$m = 7$	$29 \le p < 47$	$2x^4 + 9x^3 + 6x^2 + x = Q_6(x)$
$m = 8$	$47 \le p < 76$	$7x^4 + 14x^3 + 7x^2 + x = Q_7(x)$
$m = 9$	$76 \le p < 123$	$2x^5 + 16x^4 + 20x^3 + 8x^2 + x = Q_8(x)$

Similarly to $P_n(x)$, by the rules of polynomial addition and because of the way the polynomials $Q_n(x)$ are defined,

$$Q_{n+1}(x) = Q_n(x) + xQ_{n-1}(x)\ , \qquad Q_0(x) = 2x\ , \quad Q_1(x) = x\ ,$$

is the recursion relation satisfied by the polynomials $Q_n(x)$. Here we have the same recursion formula satisfied by the polynomials $P_n(x)$, but with different starting values. Notice that $Q_n(1) = L_n$. As before, $Q_{n-1}(1)$ is the sum of the coefficients of $Q_{n-1}(x)$, or, the count of the numbers for which a minimal representation exists in the interval $L_n \le p < L_{n+1}$, which contains exactly $L_{n+1} - L_n = L_{n-1}$ integers. Thus, each integer has at least one minimal representation in distinct Lucas numbers.

Now, let us reconsider the n boxes. To have a minimal representation, we wish to fill the n boxes with zeroes or ones such that no two ones are adjacent and to discard the arrangement using all zeroes. As before, there are $F_{n+2} - 1$ such arrangements. Now, establish a Lucas number positional notation by putting the Lucas numbers $L_0, L_1, L_2, L_3, \ldots, L_{n-1}$ into the n boxes. Again, the significance of the ones and zeroes is determination of which Lucas numbers are used in the sum. But, notice that $L_0 + L_2 = L_1 + L_3$, which would make more than one minimal representation of an integer possible. To avoid this problem, we consider the first four boxes and reject $L_0 + L_2$ whenever that combination occurs without L_1 or L_3. If such four boxes hold

0	1	0	1
L_3	L_2	L_1	L_0

then there are (n - 4) remaining boxes which can hold F_{n-2} compatible arrangements. Thus, rejecting these endings eliminates F_{n-2} arrangements, making the number of admissible arrangements $F_{n+2} - F_{n-2} - 1 = L_n - 1$. But the Lucas sequence begins with $L_0 = 2$, so that the number L_n is in the box numbered (n + 1). Therefore, using the first n Lucas numbers and the two constraints, we can have at most $L_n - 1$ different numbers represented, for

$$L_1 + L_3 + \ldots + L_{2k-1} = L_{2k} - 2\ ,$$

$$L_2 + L_4 + \ldots + L_{2k-2} = L_{2k-1} - 1\ ,$$

and the $L_0 + L_2$ ending was rejected.

Then, we can have at most $L_n - 1$ different numbers represented using $L_0, L_1, \ldots, L_{n-1}$, but the enumerating minimal polynomial guarantees that each of the numbers $1, 2, 3, \ldots, L_n - 1$, has at least one minimal representation. Thus, the minimal representation of an integer in Lucas numbers, subject to the two constraints given, is unique. This is the Lucas Zeckendorf Theorem.

For the maximal representation of an integer using distinct Lucas numbers, again we will need to use adjacent Lucas numbers whenever possible. In our n boxes, then, we will want to place the ones and zeroes so that there never are two consecutive zeroes. Also, we need to exclude the ending $L_1 + L_3$ in our representations to exclude the possibility of two maximal representations for an integer, one using $L_0 + L_2 = 5$ and the other $L_1 + L_3 = 5$. We will use the combination $L_1 + L_3$ only when one of L_0 or L_2 occurs in the same maximal representation.

Now, let the enumerating maximal polynomials for the Lucas case for the interval $L_n \le p < L_{n+1}$ be $Q^*_{n-1}(x)$, where dx^j is a term of $Q^*_{n-1}(x)$ if d of the integers p require j Lucas numbers in their maximal representation. For example, the maximal representation in Lucas numbers for integers p in the interval $11 = L_5 \le p < L_6 = 18$ are:

$$11 = 7 + 3 + 1$$
$$12 = 7 + 3 + 2$$
$$13 = 7 + 3 + 2 + 1$$
$$14 = 7 + 4 + 2 + 1$$
$$15 = 7 + 4 + 3 + 1$$
$$16 = 7 + 4 + 3 + 2$$
$$17 = 7 + 4 + 3 + 2 + 1$$

so that $Q^*_4(x) = x^5 + 4x^4 + 2x^3$, since one integer requires 5 Lucas numbers, 4 integers require 4 Lucas numbers, and 2 integers require 3 Lucas numbers in their maximal representation. The first nine polynomials $Q^*_n(x)$ follow.

	$L_m \le p < L_{m+1}$	$Q^*_{m-1}(x)$
$m = 1$	$1 \le p < 3$	$2x = Q^*_0(x)$
$m = 2$	$3 \le p < 4$	$x^2 = Q^*_1(x)$
$m = 3$	$4 \le p < 7$	$x^3 + 2x^2 = Q^*_2(x)$
$m = 4$	$7 \le p < 11$	$x^4 + 3x^3 = Q^*_3(x)$
$m = 5$	$11 \le p < 18$	$x^5 + 4x^4 + 2x^3 = Q^*_4(x)$
$m = 6$	$18 \le p < 29$	$x^6 + 5x^5 + 5x^4 = Q^*_5(x)$
$m = 7$	$29 \le p < 47$	$x^7 + 6x^6 + 9x^5 + 2x^4 = Q^*_6(x)$
$m = 8$	$47 \le p < 76$	$x^8 + 7x^7 + 14x^6 + 7x^5 = Q^*_7(x)$
$m = 9$	$76 \le p < 123$	$x^9 + 8x^8 + 20x^7 + 16x^6 + 2x^5 = Q^*_8(x)$

The recursion relation for the polynomials $Q^*_n(x)$ can be derived in a similar fashion to $P^*_n(x)$, becoming

$$Q^*_{n+1}(x) = x[Q^*_n(x) + Q^*_{n-1}(x)] , \quad Q^*_0(x) = 2x , \quad Q^*_1(x) = x^2.$$

Notice that the same coefficients occur in the enumerating minimal Lucas polynomial $Q_n(x)$ and in the enumerating maximal Lucas polynomial $Q^*_n(x)$. The relationship in the lemma below could be proved by mathematical induction, paralleling the proof of the similar property of $P_n(x)$ and $P^*_n(x)$ given in the preceding section.

<u>Lemma</u>: $$Q_m(x) = x^{m+1} Q^*_m(1/x) \quad \text{for } m \geq 1 .$$

Also, the polynomials $P^*_n(x)$ and $Q^*_n(x)$ are related as follows:

<u>Lemma</u>: $$Q^*_{n-1}(x) = x P^*_n(x) + x^2 P^*_{n-2}(x) , \quad n \geq 1 ,$$

which could be proved by mathematical induction. Notice that the lemma above becomes the well-known identity, $L_{n-1} = F_n + F_{n-2}$, when $x = 1$.

Now we return to our main problem.

By laws of polynomial addition, if we add all polynomials $Q^*_n(x)$, the coefficients in the sum will provide a count of how many integers require k Lucas numbers in their maximal representation. Then, it would appear that

$$\begin{aligned}\sum_{n=0}^{\infty} Q^*_n(x) &= Q^*_0(x) + Q^*_1(x) + Q^*_2(x) + Q^*_3(x) + Q^*_4(x) + \dots + Q^*_k(x) + \dots \\ &= 2x + x^2 + (x^3 + 2x^2) + (x^4 + 3x^3) + (x^5 + 4x^4 + 2x^3) + \dots \\ &= 2x + 3x^2 + 6x^3 + 12x^4 + 24x^5 + \dots + 3\cdot 2^{k-2}x^k + \dots ,\end{aligned}$$

so that $3\cdot 2^{k-2}$ integers require k Lucas numbers in their maximal representation, $k \geq 2$. A proof that this is the correct computation of the sum of the polynomials $Q^*_n(x)$ follows.

<u>Lemma</u>: If $Q^*_0(x) = 2x$, $Q^*_1(x) = x^2$, and $Q^*_n(x) = x[Q^*_{n-1}(x) + Q^*_{n-2}(x)]$, then

$$Q^*_{n-1}(x^2) = x^{n+1}[f_n(x) + f_{n-2}(x)]$$

where $f_n(x)$ are the Fibonacci polynomials.

<u>Proof</u>: To begin a proof by mathematical induction, observe that

$$n = 1: \quad Q_0^*(x^2) = 2x^2 = x^2(1 + 1) = x^{1+1}[f_1(x) + f_{-1}(x)]$$

$$n = 2: \quad Q_1^*(x^2) = x^4 = x^3(x + 0) = x^{2+1}[f_2(x) + f_0(x)]$$

Assume that the lemma holds for (n - 1) and (n - 2). Then

$$\begin{aligned} Q_n^*(x^2) &= x^2[Q_{n-1}^*(x^2) + Q_{n-2}^*(x^2)] \\ &= x^2\{x^{n+1}[f_n(x) + f_{n-2}(x)] + x^n[f_{n-1}(x) + f_{n-3}(x)]\} \\ &= x^{n+2}\{[xf_n(x) + f_{n-1}(x)] + [xf_{n-2}(x) + f_{n-3}(x)]\} \\ &= x^{n+2}[f_{n+1}(x) + f_{n-1}(x)], \end{aligned}$$

establishing the lemma by mathematical induction for $n \geq 1$.

Using known generating functions for the Fibonacci polynomials as before,

$$\sum_{n=1}^{\infty} f_n(x)\, t^{n+1} = \frac{t^2}{1 - xt - t^2},$$

$$\therefore \sum_{n=1}^{\infty} f_{n-2}(x) t^{n+1} = \frac{t^2(1 - xt)}{1 - xt - t^2}.$$

Adding,

$$\frac{t^2(2 - xt)}{1 - xt - t^2} = \sum_{n=1}^{\infty} [f_n(x) + f_{n-2}(x)] t^{n+1}.$$

Setting t = x,

$$\frac{2x^2 - x^4}{1 - 2x^2} = \sum_{n=1}^{\infty} x^{n+1}[f_n(x) + f_{n-2}(x)] = \sum_{n=1}^{\infty} Q_{n-1}^*(x^2).$$

Therefore,

$$\sum_{n=0}^{\infty} Q_n^*(x) = \frac{2x - x^2}{1 - 2x} = 2x + \frac{3x^2}{1 - 2x} = 2x + \sum_{n=2}^{\infty} 3 \cdot 2^{n-2} x^n .$$

To see the reason for the peculiar coefficients $3 \cdot 2^{k-1}$, examine the eight possible ways to fill the first four boxes with zeroes and ones. Then see how many numbers requiring n Lucas numbers in their maximal representation could be written. In other words, consider how to distribute n ones without allowing two consecutive zeroes. The eight cases follow.

L_3	L_2	L_1	L_0	Count of Possibilities ($n \geq 4$)
1	1	1	1	2^{n-4}
1	0	1	0	excluded
0	1	0	1	2^{n-3}
1	1	0	1	2^{n-3}
1	0	1	1	2^{n-3}
1	1	1	0	2^{n-3}
0	1	1	1	2^{n-4}
0	1	1	0	2^{n-3}

Summing the seven useable cases gives

$$5 \cdot 2^{n-3} + 2 \cdot 2^{n-4} = 6 \cdot 2^{n-3} = 3 \cdot 2^{n-2}, \quad n \geq 4,$$

possible maximal representations. The endings with a zero in the left-most box would require that the L_4 box contain a one, while all would have either a L_4 or a L_5 appearing in the representation. The endings listed above do not give the numbers requiring 1, 2, or 3 Lucas numbers in their maximal representation. So, the endings given above do not include the representations of 1 through 9, 11 and 12, which give the first three terms $2x + 3x^2 + 6x^3$ of the enumerating maximal Lucas polynomial sum and explain the irregular first term in the sum of the polynomials $Q_n^*(x)$. The numbers not included in the count of possibilities above follow.

L_4	L_3	L_2	L_1	L_0	representing:	
	0	0	1	0	1	$2x$
	0	0	0	1	2	
	0	0	1	1	3	$3x^2$
	0	1	1	0	4	
	0	1	0	1	5	
	0	1	1	1	6	$6x^3$
	1	0	1	1	7	
	1	1	1	0	8	
	1	1	0	1	9	
1	0	1	1	0	11	
1	0	1	0	1	12	

Now, the enumerating maximal polynomial guarantees that $3\cdot 2^{k-2}$ integers require k Lucas numbers in their maximal representation, but examining the possible maximal representations which could be written using k Lucas numbers shows that at most $3\cdot 2^{k-2}$ different representations could be formed. That is exactly one apiece, so the maximal representation of an integer using Lucas numbers subject to the two constraints, that no two consecutive Lucas numbers are omitted and that the combination $L_3 + L_1$ is not used unless L_0 or L_2 also appear, is unique.

5. CONCLUDING REMARKS

Much interest has been shown in the subject of representations of integers in recent years. Some of the many diverse new results which arise naturally from this paper are recorded here with references for further reading.

That the Fibonacci and Lucas sequences are complete has been shown in this paper, although the property was not named. A sequence of positive integers, $a_1, a_2, \ldots, a_n, \ldots$, is complete with respect to the positive integers if and only if every positive integer m is the sum of a finite number of the members of the sequence, where each member is used at most once in any given representation. (See [4], [5].) For example, the sequence of powers of two is complete; any positive integer can be represented in the binary system of numeration. However, if any power of 2, for example, $1 = 2^0$, is omitted, the new sequence is not complete. It is surprising that, for the Fibonacci sequence where $a_n = F_n$, $n \geq 1$, if any one arbitrary number F_k is missing, the sequence is still complete, but if any two arbitrary Fibonacci numbers F_p and F_q are missing, the sequence is incomplete [4].

The Dual Zeckendorf Theorem has an extension that characterizes the Fibonacci numbers. Brown in [2] proves that, if each positive integer has a unique representation as the sum of distinct members of a given sequence when no two consecutive members of the sequence are omitted in the representation, then the given sequence is the sequence of Fibonacci numbers.

Generalized Fibonacci numbers can be studied in a manner similar to the Lucas case. A set of particularly interesting sequences arising in Pascal's triangle appears in [6]: the sequences formed as the sums of elements of the diagonals of Pascal's left-justified triangle, beginning in the left-most column and going right one and up p throughout the array. (The Fibonacci numbers occur when p = 1.) Or, the squares of Fibonacci numbers may be used (see [7]), which gives a complete sequence if members of the sequence can be used twice. Other ways of studying generalized Fibonacci numbers include those given in [8], [9], [10], and [11].

To return to the introduction, Carlitz [12] and Klarner [13] have studied the problem of counting the number of representations possible for a given integer. Tables of the number of representations of integers as sums of distinct elements of the Fibonacci sequence as well as other related tables appear in [14]. The general problem of representations of integers using the Fibonacci numbers, enumerating intervals, and positional binary notation for the representations were given by Ferns [15] while [16] is one of the earliest references following Daykin [8]. The suggested readings and the references given here are by no means exhaustive. The range of representation problems is bounded only by the imagination.

REFERENCES

1. John L. Brown, Jr., "Zeckendorf's Theorem and Some Applications," Fibonacci Quarterly, Vol. 2, No. 3, October, 1964, pp. 163-168.

2. John L. Brown, Jr., "A New Characterization of the Fibonacci Numbers," Fibonacci Quarterly, Vol. 3, No. 1, February, 1965, pp. 1-8.

3. David A. Klarner, "Representations of N as a Sum of Distinct Elements from Special Sequences," Fibonacci Quarterly, Dec., 1966, pp. 289-306.

4. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin, Boston, 1969. Chapter 12.

5. John L. Brown, Jr., "Note on Complete Sequences of Integers," American Mathematical Monthly, June-July, 1961, pp. 557-561.

6. Nannette S. Cox, "Unique Integer Representations by Generalized Fibonacci Sequences Derived from Pascal's Triangle," San Jose State College Master's Thesis, July, 1970.

6a. John L. Brown, Jr., "Unique Representations of Integers as Sums of Distinct Lucas Numbers," Fibonacci Quarterly, October, 1969, pp. 243-252.

7. Roger O'Connell, "Representations of Integers as Sums of Fibonacci Squares," The Fibonacci Quarterly, Vol. 10, No. 1, January, 1972, pp. 103-112.

8. D. E. Daykin, "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers, " Journal London Math. Soc., Vol. 35 (1960), pp. 143-161.

9. D. E. Daykin, "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers--II," Fibonacci Quarterly, December, 1969, pp. 494-510.

10. J. W. Gootherts, Representations of Integers as Sums of Elements from Fibonacci-Type Sequences, Master's Thesis, San Jose State College, June, 1966.

11. John W. Phillips, Integer Representations as Sums of Generalized Fibonacci Numbers, Master's Thesis, San Jose State College, June, 1970.

12. L. Carlitz, "Fibonacci Representations," Fibonacci Quarterly, October, 1968, pp. 193-220.

13. David A. Klarner, "Partitions of N into Distinct Fibonacci Numbers," Fibonacci Quarterly, October, 1968, pp. 235-244.

14. Brother Alfred Brousseau, Fibonacci and Related Number Theoretic Tables, The Fibonacci Association, San Jose State College, San Jose, Calif., 1972.

15. H. H. Ferns, "On the Representation of Integers as Sums of Distinct Fibonacci Numbers," Fibonacci Quarterly, February, 1965, pp. 21-30.

16. V. E. Hoggatt, Jr., and S. L. Basin, "Representations by Complete Sequences: Part I (Fibonacci)", Fibonacci Quarterly, October, 1963, pp. 1-14.

* * * NINE FIBONACCI PUZZLERS * * *

B-39 (Proposed by J. A. Fuchs) Prove that $F_{n+2} < 2^n$ for $n \geq 3$.

B-41 (Proposed by David L. Silverman) Do there exist four positive Fibonacci numbers in arithmetic progression?

B-42 (Proposed by S. L. Basin) Express the $(n + 1)$-st Fibonacci number F_{n+1} as a function of F_n. Solve the same problem for Lucas numbers.

B-44 (Proposed by Douglas Lind) Prove that for every positive integer k there are no more than n Fibonacci numbers between n^k and n^{k+1}.

B-47 (Proposed by Barry Litvack) Prove that for every positive integer k there are k consecutive Fibonacci numbers, each of which is composite.

B-58 (Proposed by Sidney Kravitz) Show that no Fibonacci number other than 1, 2, or 3 is equal to a Lucas number.

B-62 (Proposed by Brother Alfred Brousseau) Prove that a Fibonacci number with odd subscript cannot be represented as the sum of squares of two Fibonacci numbers in more than one way.

B-95 (Proposed by Brother Alfred Brousseau) What is the highest power of 2 that exactly divides $F_1F_2F_3 \cdots F_{100}$?

H-2 (Proposed by L. Moser and L. Carlitz) Resolve the conjecture: There are no Fibonacci numbers which are integral squares except 0, 1, and 144.

FIBONACCI NUMBERS AND GEOMETRY

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The Fibonacci relations we are going to develop represent a special case of algebra. If we are able to relate them to geometry we should take a quick look at the way algebra and geometry can be tied together.

One use of geometry is to serve as an illustration of an algebraic relation. Thus

$$(a + b)^2 = a^2 + 2ab + b^2$$

is exemplified by Figure 1.

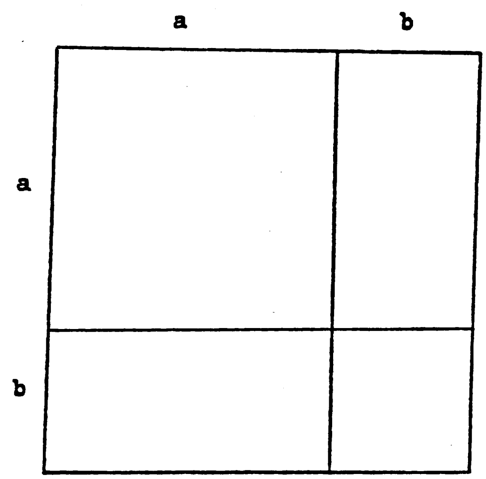


Figure 1

A second use of geometry is to provide a PROOF of an algebraic relation. As we ordinarily conceive the Pythagorean Theorem (though this was not the original thought of the Greeks) we tend to think of it as an algebraic relation on the sides of the triangle, namely,

$$c^2 = a^2 + b^2 .$$

One proof by geometry of this algebraic relation is shown in Figure 2.

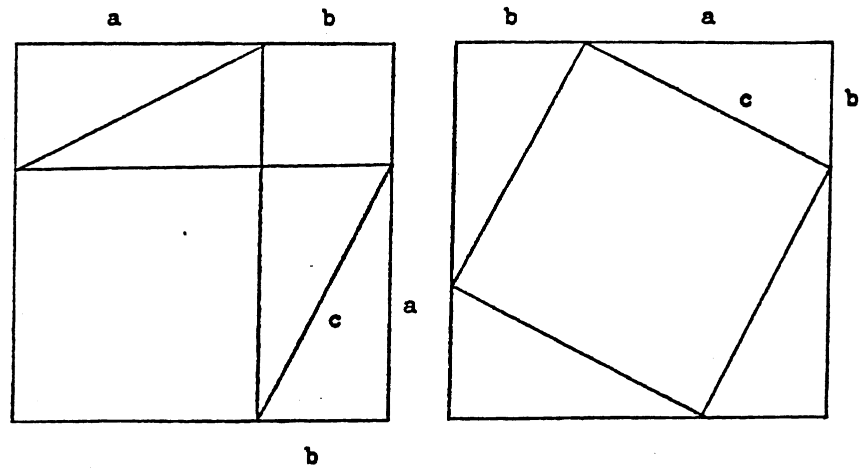
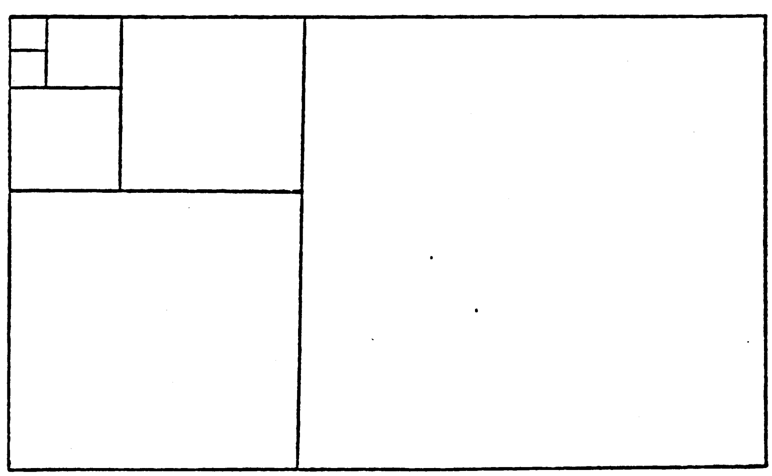


Figure 2

In summary, geometric figures may illustrate algebraic relations or they may serve as proofs of these relations. In our development, the main emphasis will be on proof though obviously illustration occurs simultaneously as well.

SUM OF FIBONACCI SQUARES

In the standard treatment of the Fibonacci sequence, geometry enters mainly at one point: summing the squares of the first n Fibonacci numbers. Algebraically, it can be shown by intuition and proved by induction that the sum of the squares of the first n Fibonacci numbers is F_nF_{n+1}. But there is a geometric pattern which ILLUSTRATES this fact beautifully as shown in Figure 3.



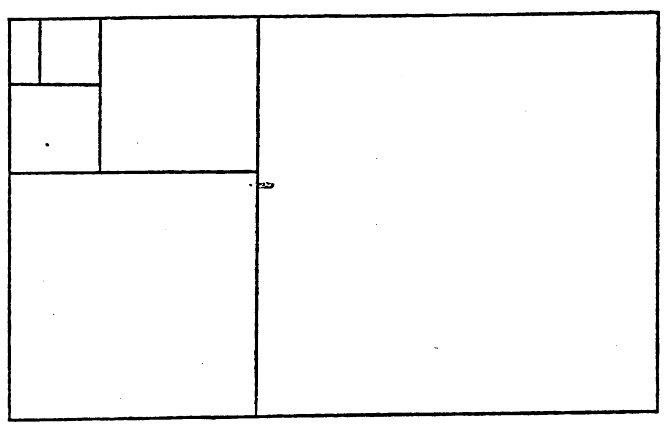
$$\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$$

Figure 3

The figure is built up as follows. We put down two unit squares which are the squares of F_1 and F_2. Now we have a rectangle of dimensions 1 by 2. On the right of this can be placed a square of side 2 (F_3) which gives a 2 by 3 rectangle. Then below can be set a square of side 3 (F_4) which produces a rectangle of sides 3 and 5. To the right of this can be placed a square of side 5 (F_5) which gives a 5 by 8 (F_5 by F_6) rectangle, and so on.

This is where geometry begins and ends in the usual treatment of Fibonacci sequences. For if one tries to produce a similar pattern for the sum of the squares of any other Fibonacci sequence, there is an impasse. To meet this road block the following detour was conceived.

Suppose we are trying to find the sum of the squares of the first n Lucas numbers. Instead of starting with a square, we put down a rectangle whose sides are 1 and 3, the first and second Lucas numbers. (Figure 4 illustrates the general procedure.) Then on the side of length 3 it is possible to place a square of side 3: this gives a 3 by 4 rectangle.



$$\sum_{k=1}^{n} T_k^2 = T_n T_{n+1} - T_1(T_2 - T_1)$$

Figure 4

Against this can be set a square of side 4 thus producing a 4 by 7 rectangle. On this a square of side 7 is laid giving a 7 by 11 rectangle. Thus the same process that operated for the Fibonacci numbers is now operating for the Lucas numbers. The only difference is that we began with a 1 by 3 rectangle instead of a 1 by 1 square. Hence, if we subtract 2 from the sum we should have the sum of the squares of the first n Lucas numbers. The formula for this sum is thus:

(1) $$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 .$$

Using a direct geometric approach it has been possible to arrive at this algebraic formula with a minimum of effort. By way of comparison it may be noted that the intuitional algebraic route usually leads to difficulties for students.

Still more striking is the fact that by using the same type of procedure it is possible to determine the sum of the squares of the first n terms of ANY Fibonacci sequence. We start again by drawing a rectangle of sides T_1 and T_2 (see Fig. 4). On the side T_2 we place a square of side T_2 to give a rectangle of sides T_2 and T_3. Against the T_3 side we set a square of side T_3 to produce a rectangle of sides T_3 and T_4. The operation used in the Fibonacci and Lucas sequences is evidently working again in this general case, the sum being $T_n T_{n+1}$ if we end with the nth term squared. But instead of having the square of T_1 as the first term, we used instead $T_1 T_2$. Thus it is necessary to subtract $T_1 T_2 - T_1^2$ from the sum to arrive at the sum of the squares of the first n terms of the sequence. The formula that results is:

(2) $$\sum_{k=1}^{n} T_k^2 = T_n T_{n+1} - T_1(T_2 - T_1) = T_n T_{n+1} - T_1 T_0 .$$

ILLUSTRATIVE FORMULAS

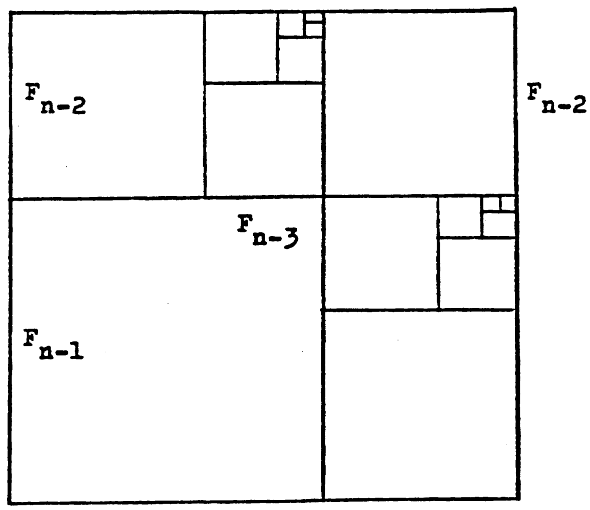
The design in Figure 1 for $(a + b)^2 = a^2 + 2ab + b^2$ can be used to illustrate Fibonacci relations that result from this algebraic identity. For example,

$$L_n^2 = (F_{n+1} + F_{n-1})^2 = F_{n+1}^2 + 2F_{n+1}F_{n-1} + F_{n-1}^2 .$$

This evidently leads to nothing new but the algebraic relations can be exemplified in this way as special cases of a general algebraic relation which is depicted by geometry.

LARGE SQUARE IN ONE CORNER

We shall deal with a number of geometric patterns which can be employed in a variety of ways in many cases. In the first type we place in one corner of a given figure the largest possible Fibonacci (or Lucas) square that will fit into it. Take, for example, a square whose side is F_n. (See Figure 5.)



$$F_n^2 = F_{n-1}^2 + 3F_{n-2}^2 + 2\sum_{k=1}^{n-3} F_k^2$$

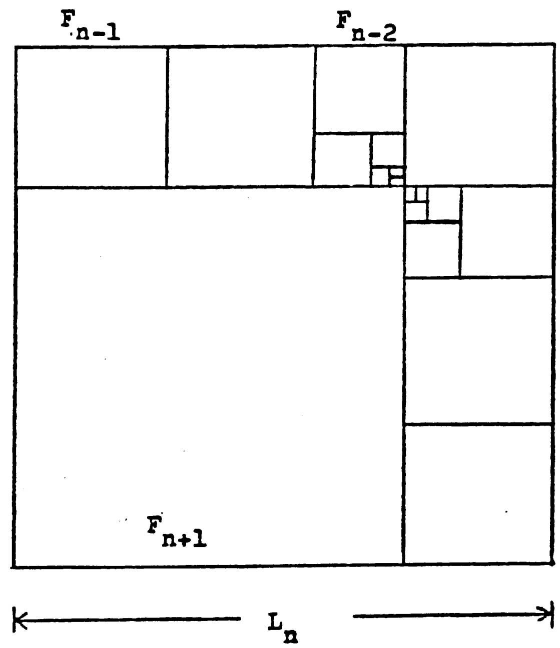
Figure 5

This being the sum of F_{n-1} and F_{n-2}, a square of side F_{n-1} can be put into one corner and its sides extended. In the opposite corner is a square of side F_{n-2}. From the two rectangles can be taken squares of side F_{n-2} leaving two smaller rectangles of dimensions F_{n-2} and F_{n-3}. But by what was found in the early part of this discussion, such a rectangle can be represented as the sum of the first n - 3 Fibonacci squares. We thus arrive at the formula:

$$(3) \qquad F_n^2 = F_{n-1}^2 + 3F_{n-2}^2 + 2\sum_{k=1}^{n-3} F_k^2 .$$

As a second example, take a square of side $L_n = F_{n+1} + F_{n-1}$. (See Figure 6.) In one corner is a square of side F_{n+1} and in the opposite a square of side F_{n-1}. The rectangles have dimensions F_{n+1} and F_{n-1}. But F_{n+1} equals $2F_{n-1} + F_{n-2}$, so that each rectangle contains two squares of side F_{n-1} and a rectangle of sides F_{n-1} and F_{n-2}. Thus the following formula results:

(4) $$L_n^2 = F_{n+1}^2 + 5F_{n-1}^2 + 2\sum_{k=1}^{n-2} F_k^2 \, .$$



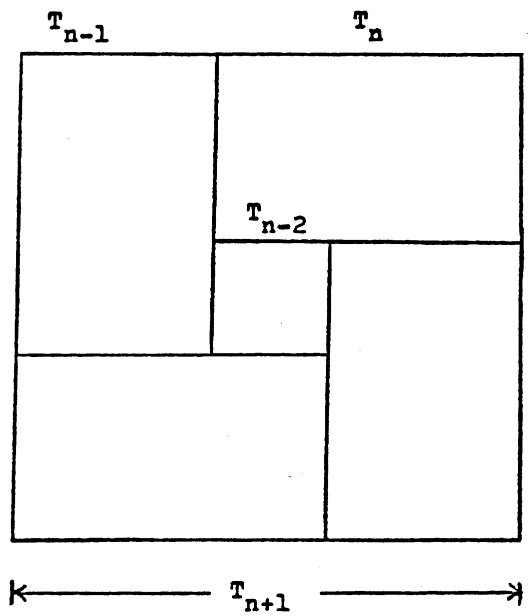
$$L_n^2 = F_{n+1}^2 + 5F_{n-1}^2 + 2\sum_{k=1}^{n-2} F_k^2$$

Figure 6

CYCLIC RECTANGLES

A second type of design leading to Fibonacci relations is one that may be called cyclic rectangles. Take a square of side T_{n+1}, a general Fibonacci number. Put in one corner a rectangle of sides T_n and T_{n-1} (Figure 7). The process can be continued until there are four such rectangles in a sort of whorl with a square in the center. This square has side $T_n - T_{n-1}$ or T_{n-2}. Accordingly the general relation for all Fibonacci sequences results:

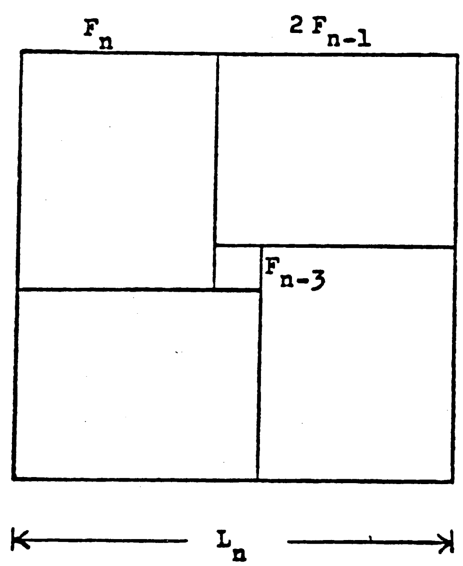
(5) $$T_{n+1}^2 = 4T_nT_{n-1} + T_{n-2}^2 \, .$$



$$T_{n+1}^2 = 4T_nT_{n-1} + T_{n-2}^2$$

Figure 7

As another example of this type of configuration consider a square of side L_n and put in each corner a rectangle of dimensions $2F_{n-1}$ by F_n. (See Fig. 8.)



$$L_n^2 = 8F_nF_{n-1} + F_{n-3}^2$$

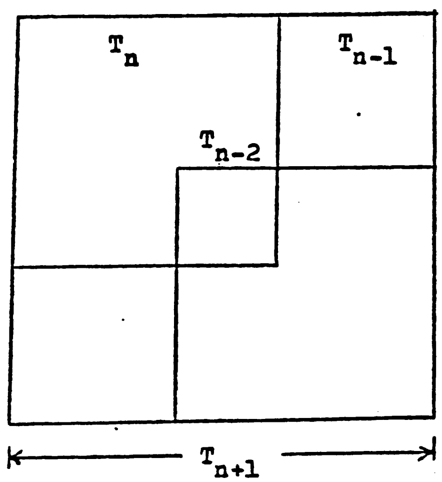
Figure 8

Again, there is a square in the center with side $2F_{n-1} - F_n$ or $F_{n-1} - F_{n-2}$ or F_{n-3}. Hence:

(6) $$L_n^2 = 8F_nF_{n-1} + F_{n-3}^2 .$$

OVERLAPPING SQUARES IN TWO OPPOSITE CORNERS

Construct a square whose side is T_{n+1} which equals $T_n + T_{n-1}$. In two opposite corners place squares of side T_n (Fig. 9). Since T_n is greater than



$$T_{n+1}^2 = 2T_n^2 + 2T_{n-1}^2 - T_{n-2}^2$$

Figure 9

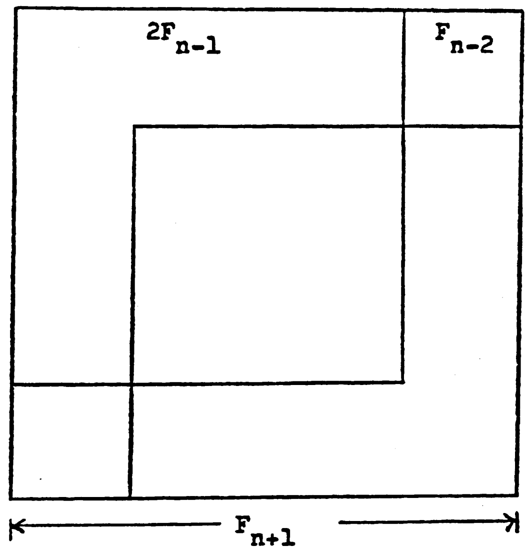
half of T_{n+1} it follows that these squares must overlap in a square. The side of this square is $T_n - T_{n-1} = T_{n-2}$. The entire square is composed of two squares of side T_n and two squares of side T_{n-1}. But since the area of the central square of side T_{n-2} has been counted twice, it must be subtracted once to give the proper result. Thus:

(7) $$T_{n+1}^2 = 2T_n^2 + 2T_{n-1}^2 - T_{n-2}^2 ,$$

a result applying to ALL Fibonacci sequences.

Example 2. Take a square of side $F_{n+1} = 2F_{n-1} + F_{n-2}$. In opposite corners, place squares of side $2F_{n-1}$. (See Figure 10.) Then the overlap square in the center has side $2F_{n-1} - F_{n-2} = F_{n-1} + F_{n-3} = L_{n-2}$. Thus:

(8) $$F_{n+1}^2 = 8F_{n-1}^2 + 2F_{n-2}^2 - L_{n-2}^2 .$$



$$F_{n+1}^2 = 8F_{n-1}^2 + 2F_{n-2}^2 - L_{n-2}^2$$

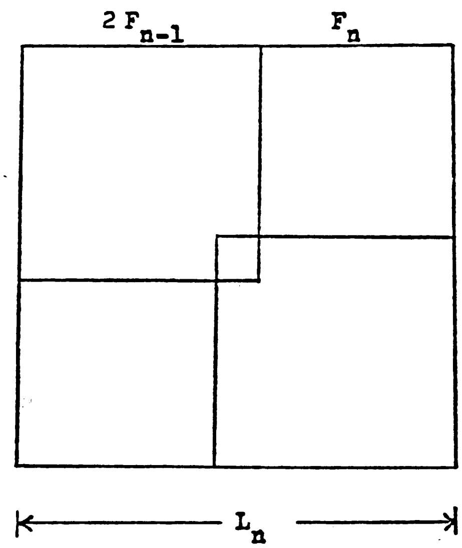
Figure 10

<u>Third example.</u> A square of side $L_n = F_{n+1} + F_{n-1}$ has a central overlap square of side $F_{n+1} - F_{n-1} = F_n$. Accordingly,

(9) $$L_n^2 = 2F_{n+1}^2 + 2F_{n-1}^2 - F_n^2 .$$

<u>Final example.</u> In a square of side $L_n = 2F_{n-1} + F_n$, place in two opposite corners squares of side $2F_{n-1}$. The overlap square in the center has side $2F_{n-1} - F_n = F_{n-1} - F_{n-2} = F_{n-3}$. (See Figure 11.) Hence:

(10) $$L_n^2 = 8F_{n-1}^2 + 2F_n^2 - F_{n-3}^2 .$$

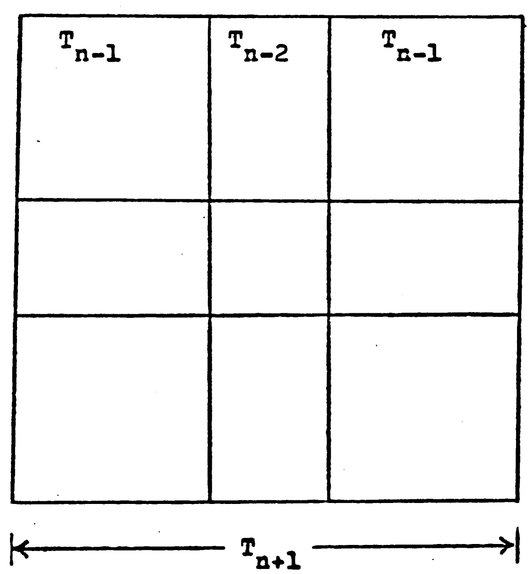


$$L_n^2 = 8F_{n-1}^2 + 2F_n^2 - F_{n-3}^2$$

Figure 11

NON-OVERLAPPING SQUARES IN FOUR CORNERS

Consider the relation $T_{n+1} = 2T_{n-1} + T_{n-2}$. Each side of the square can be divided into segments T_{n-1}, T_{n-2}, T_{n-1} in that order (Fig. 12).



$$T_{n+1}^2 = 4T_{n-1}^2 + 4T_{n-1}T_{n-2} + T_{n-2}^2$$

Figure 12

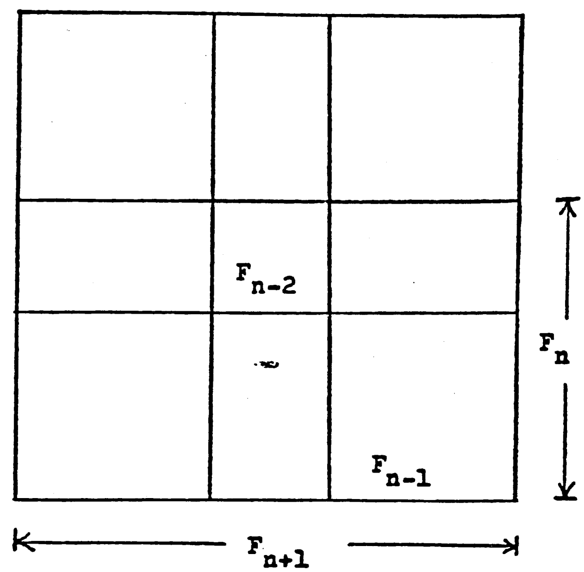
There are now four squares of side T_{n-1} in the corners, a square of side T_{n-2} in the center and four rectangles of dimensions T_{n-1} and T_{n-2}. From this results the formula

(11) $$T_{n+1}^2 = 4T_{n-1}^2 + 4T_{n-1}T_{n-2} + T_{n-2}^2$$

which applies to ALL Fibonacci sequences.

OVERLAPPING SQUARES IN FOUR CORNERS

We start with $F_{n+1} = F_n + F_{n-1}$ and put four squares of side F_n in the corners (Fig. 13). Clearly there is a great deal of overlapping. The



$$F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$$

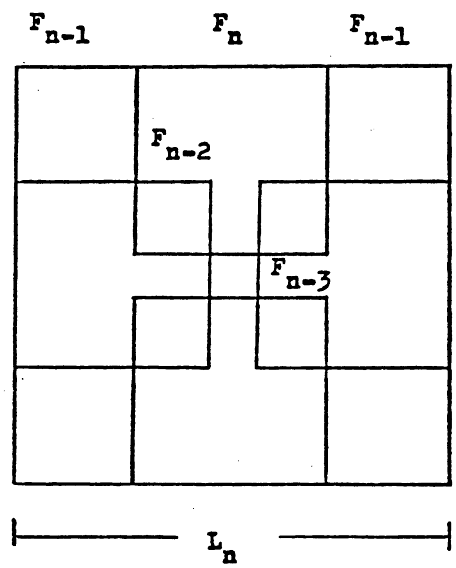
Figure 13

square at the center of side $F_n - F_{n-1} = F_{n-2}$ is covered four times; the four rectangles are each found in two of the corner squares so that this rectangle must be subtracted out four times. The central square being covered four times must be subtracted out three times. As a result the following formula is obtained:

(12) $$F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2 \ .$$

OVERLAPPING SQUARES PROJECTING FROM THE SIDES

We start with the relation $L_n = F_n + 2F_{n-1}$ and divide the side into segments F_{n-1}, F_n, F_{n-1} in that order (Fig. 14). On the F_n segments build squares which evidently overlap as shown. The overlap squares in the corners of these four squares have a side $F_n - F_{n-1} = F_{n-2}$ while the central square



$$L_n^2 = 4F_n^2 + 4F_{n-1}^2 - 4F_{n-2}^2 + F_{n-3}^2$$

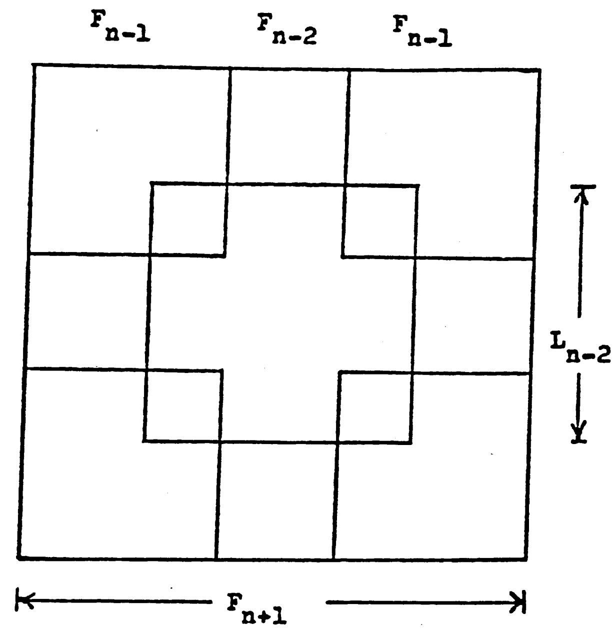
Figure 14

has a side $L_n - 2F_n = F_{n+1} + F_{n-1} - 2F_n = 2F_{n-1} - F_n = F_{n-1} - F_{n-2} = F_{n-3}$. Taking the overlapping areas into account gives the relation:

$$L_n^2 = 4F_n^2 + 4F_{n-1}^2 - 4F_{n-2}^2 + F_{n-3}^2 \quad (13)$$

FOUR CORNER SQUARES AND A CENTRAL SQUARE

A square of side $F_{n+1} = 2F_{n-1} + F_{n-2}$ has its sides divided into segments F_{n-1}, F_{n-2}, F_{n-1} in that order (Figure 15). In each corner, a square of side F_{n-1} is constructed. Then a centrally located square of side L_{n-2} is constructed. It may be wondered where the idea for doing this came from. Since $L_{n-2} = F_{n-1} + F_{n-3} = F_{n-2} + 2F_{n-3}$, it follows that such a square would



$$F_{n+1}^2 = 4F_{n-1}^2 + 4F_{n-2}^2 + L_{n-2}^2 - 4F_{n-3}^2$$

Figure 15

project into the corner squares in the amount of F_{n-3}, thus giving three squares of this dimension. Taking overlap into account leads to the formula:

(14) $$F_{n+1}^2 = 4F_{n-1}^2 + 4F_{n-2}^2 + L_{n-2}^2 - 4F_{n-3}^2 .$$

CONCLUSION

In this all too brief session we have explored some of the relations of Fibonacci numbers and geometry. It is clear that there is a field for developing geometrical ingenuity and thereby arriving simply and intuitively at algebraic relations involving Fibonacci numbers, Lucas numbers, and general Fibonacci numbers. It appears that there is a considerable wealth of enrichment and discovery material in the general area of Fibonacci numbers as related to geometry.

A PRIMER FOR THE FIBONACCI NUMBERS: PART XI

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MULTISECTION GENERATING FUNCTIONS FOR THE COLUMNS OF PASCAL'S TRIANGLE

1. INTRODUCTION

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for the sequence $\{a_n\}$. Often one desires generating functions which multisect the sequence $\{a_n\}$,

$$G_i(x) = \sum_{j=0}^{\infty} a_{i+mj} x^j, \qquad (i = 0, 1, 2, \ldots, m-1).$$

For the bisection generating functions the task is easy. Let

$$H_1(x^2) = \frac{f(x) + f(-x)}{2},$$

$$H_2(x^2) = \frac{f(x) - f(-x)}{2x};$$

then clearly $H_1(x^2)$ and $H_2(x^2)$ contain only even powers of x so that

$$H_1(x) = \sum_{n=0}^{\infty} a_{2n} x^n \qquad \text{and} \qquad H_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n$$

are what we are looking for.

Let us illustrate this for the Fibonacci sequence. Here

$$f(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n \quad ;$$

then

$$H_1(x) = \frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n} x^n$$

and

$$H_2(x) = \frac{1 - x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n \ .$$

<u>Exercise:</u> Find the bisection generating functions for the Lucas sequence.

Let us find the general multisecting generating functions for the Fibonacci sequence, using the method of H. W. Gould (See [1].). The Fibonacci sequence enjoys the Binet Form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \ , \qquad \alpha = \frac{1 + \sqrt{5}}{2} \ , \qquad \beta = \frac{1 - \sqrt{5}}{2} \ .$$

Let $f(x) = 1/(1 - x)$; then

$$\sum_{n=0}^{\infty} F_{mn+j} x^n = \frac{\alpha^j f(\alpha^m x) - \beta^j f(\beta^m x)}{\alpha - \beta}$$

$$= \frac{1}{\alpha - \beta}\left(\frac{\alpha^j}{1 - \alpha^m x} - \frac{\beta^j}{1 - \beta^m x}\right)$$

$$= \frac{\dfrac{\alpha^j - \beta^j}{\alpha - \beta} + (\alpha\beta)^j \dfrac{\alpha^{m-j} - \beta^{m-j}}{\alpha - \beta} x}{1 - (\alpha^m + \beta^m)x + (\alpha\beta)^m x^2}$$

$$= \frac{F_j + (-1)^j F_{m-j}\, x}{1 - L_m x + (-1)^m x^2} \ , \qquad (j = 0, 1, 2, \ldots, m-1),$$

since $\alpha\beta = -1$, $\alpha^m + \beta^m = L_m$, and $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$.

Exercise: Find the general multisecting generating function for the Lucas sequence.

The same technique can be used on any sequence having a Binet Form. The general problem of multisecting a general sequence rapidly becomes very complicated according to Riordan [2], even in the classical case.

2. COLUMN GENERATORS OF PASCAL'S TRIANGLE

The column generators of Pascal's left-justified triangle [3], [4], [5], are

$$G_k(x) = \frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n, \quad k = 0, 1, 2, \ldots \quad .$$

We now seek generating functions which will m-sect these,

$$G_i(m, k; x) = \sum_{n=0}^{\infty} \binom{i+k+mn}{k} x^{n+k+1}, \quad (i = 0,1,\ldots, m-1).$$

We first cite an obvious little lemma.

Lemma 1:
$$\binom{n}{k} = \sum_{j=1}^{m} \binom{n-j}{k-1} + \binom{n-m}{k}.$$

Definition: Let $G_{i,k}(x)$, $i = 0, 1, 2, \ldots, m-1$, be the m generating functions

$$G_{i,k}(x) = \sum_{n=0}^{\infty} \binom{i+k+mn}{k} x^{i+mn+k} .$$

Lemma 2:

$$G_{i,k+1}(x) = \frac{xG_{i,k}(x) + x^2 G_{i-1,k}(x) + \ldots + x^m G_{i-m+1,k}(x)}{1 - x^m} .$$

The proof follows easily from Lemma 1.

Let

$$(1 + x + x^2 + \dots + x^{m-1})^n = \sum_{j=0}^{n(m-1)} \binom{n}{j}_m x^j$$

define the row elements of the m-nomial triangle. Further, let

$$f_i(m,k;\ x) = \sum_{j=0} \binom{k}{i + jm}_m x^j , \quad i = 0, 1, \dots, m-1,$$

where j is such that $i + jm \le k(m - 1)$. These are multisecting polynomials for the rows of the m-nomial triangle. Now, we can state an interesting theorem:

<u>Theorem:</u> For $i = 0, 1, 2, \dots, m - 1$,

$$G_i(m,k;\ x) = \frac{x^{k+i} f_i(m,k;\ x)}{(1 - x)^{k+1}} .$$

<u>Proof:</u> Recall first that the m-nomial coefficients obey

$$\binom{n}{r}_m = \binom{n-1}{r}_m + \binom{n-1}{r-1}_m + \dots + \binom{n-1}{r-m+1}_m$$

where the lower arguments are non-negative and less than or equal to $n(m - 1)$.

Clearly, for $k = 0$, from the definition just before Lemma 2,

$$G_{i,0}(x) = \frac{x^i}{1 - x^m} , \quad i = 0, 1, 2, \dots, m-1 .$$

Assume now that

$$G_{i,k}(x) = \frac{x^{k+i} f_i(m,k;\ x^m)}{(1 - x^m)^{k+1}}$$

for $i = 0, 1, 2, 3, \dots, (m - 1)$. From Lemma 2,

$$G_{i,k+1}(x) = \frac{xG_{i-1,k}(x) + \dots + x^m G_{i-m+1,k}(x)}{1 - x^m} .$$

Thus,

$$G_{i,k+1}(x) = \frac{\sum_{s=0}^{m-1}\left(\sum_{j=0}\binom{k}{i-s+jm}_m\right)x^{k+(i-s)+s+jm+1}}{(1-x^m)^{k+2}}$$

$$= \frac{\sum_{j=0}\left(\sum_{s=0}^{m-1}\binom{k}{i-s+jm}_m\right)x^{k+1+i+jm}}{(1-x^m)^{k+2}}$$

$$= \frac{x^{k+1+i}\sum_{j=0}\binom{k+1}{i+jm}_m x^{jm}}{(1-x^m)^{k+2}}$$

$$= \frac{x^{k+1+i}\, f_i(m,k;\, x^m)}{(1-x^m)^{k+2}} \; .$$

This completes the induction.

The x^{k+1+i} merely position the column generators. Here the non-zero entries are separated by $m-1$ zeros. To get rid of the zeros, let

$$G_i(m,k;\, x) = \frac{x^{k+i}\, f_i(m,k;\, x)}{(1-x)^{k+1}}$$

for $i = 0, 1, 2, \ldots, m-1$. This concludes the proof of the theorem.

If we write this in the form

$$G_i(m,k;\, x) = \sum_{j=0}^{\infty}\binom{i+jm+k}{k}x^{j+k+1} = \frac{\sum_{j=0}\binom{k}{i+jm}_m x^{k+i+j}}{(1-x)^{k+1}}$$

it emphasizes the relation of the multisection of the kth column of Pascal's triangle and the multisection of the kth row of the m-nomial triangle.

3. A NEAT GENERATING FUNCTION

<u>Lemma 3:</u>

$$\binom{n}{k} = \sum_{j=0}^{r} \binom{r}{j}\binom{n-r}{k-j}$$

This is easy to prove by starting with

$$\text{(A)} \qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k-2}$$

$$= 1\cdot\binom{n-2}{k} + 2\cdot\binom{n-2}{k-1} + 1\cdot\binom{n-2}{k-2}.$$

Apply (A) to each term on the right repeatedly.

Now let $H_i(m,k;\ x)$ m-sect the kth column of Pascal's triangle ($i = 0, 1, 2, \ldots, m-1$); then, using Lemma 3, it follows that

<u>Lemma 4:</u>

$$H_i(m,k;\ x) = \frac{x}{1-x} \sum_{j=1}^{m} \binom{m}{j} H_i(m,k-j;\ x) .$$

The results using the method of Polya for small m and i seem to indicate the following (See [3].).

<u>Theorem:</u> The generating functions for the rising diagonal sums of the rows of Pascal's triangle $i + jm$ (all other rows are deleted) are given by

$$H_i(x) = \frac{(1+x)^i}{1 - x(1+x)^m}, \quad i = 0, 1, \ldots, m-1.$$

<u>Exercise:</u> Show that

$$\sum_{i=0}^{m-1} x^i H_i(x^m) = \frac{1}{1 - x(1+x^m)} .$$

This is a necessary condition which now makes the theorem plausible. These are the generalized Fibonacci numbers obtained as rising diagonal sums from Pascal's triangle, beginning in the left-most column and going over 1 and up m. (See [3]) The theorem is proved by careful examination of its meaning with regards to Pascal's triangle as follows:

$$\frac{(1+x)^i}{1-x(1+x)^m} = \sum_{n=0}^{\infty} x^n(1+x)^{mn+i} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{m(n-j)+i}{j} x^n ,$$

$i = 0, 1, 2, \ldots, m-1$. Recall that $\binom{n}{k} = 0$ if $0 \le n < k$.

ILLUSTRATION

$$\begin{array}{lll}
n = 0 & x^0(1+x)^{0+1} = & 1 + x \\
n = 1 & x^1(1+x)^{2+1} = & x + 3x^2 + 3x^3 + x^4 \\
n = 2 & x^2(1+x)^{4+1} = & x^2 + 5x^3 + 10x^4 + 10x^5 + 5x^6 + x^7 \\
n = 3 & x^3(1+x)^{6+1} = & x^3 + 7x^4 + 21x^5 + \ldots \\
\ldots & \ldots & \ldots\ldots\ldots\ldots \\
\text{Sum:} & & 1 + 2x + 4x^2 + 9x^3 + 19x^4 + \ldots
\end{array}$$

Here, m = 2 and i = 1. Now, write a left-justified Pascal's triangle. Form the sequence of sums of elements found by beginning in the left-most column and proceeding right one and up 2 throughout the array: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, ... Notice that the coefficients of successive powers of x give every other term in that sequence.

The general problem of finding generating functions which multisect the column generators of Pascal's triangle has been solved by Nilson [6], although interpretation of the numerator polynomial coefficients has not been achieved as in our last few theorems.

REFERENCES

1. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 1-16.
2. John Riordan, Combinatorial Identities, Wiley, 1968, Section 4.3.
3. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Diagonal Sums of Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 7, No. 4, Nov., 1969, pp. 341-358.
4. Marjorie Bicknell, this publication pp. 98-103.
5. V. E. Hoggatt, Jr, "A New Slant on Pascal's Triangle," Fibonacci Quarterly, Vol. 6, No. 4, October, 1968, pp. 221-234.
6. Paul Nilson, "Column Generating Functions in Recurrence Triangles," San Jose State University Master's Thesis, August, 1972.

SOLUTIONS TO PROBLEMS

Solutions to problems posed previously are given here. Where a problem solution appeared in the Fibonacci Quarterly, date and page numbers are given.

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The solutions to "Problems For Exploration" were given by Ken Siler in "Fibonacci Summations," Fibonacci Quarterly, Vol. 1, No. 3, October, 1963, pp. 67-69, as follows:

$$\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1 \qquad \sum_{k=1}^{n} F_{4k-2} = F_{2n}^2$$

$$2\sum_{k=1}^{n} F_{3k-1} = F_{3n+1} - 1 \qquad \sum_{k=1}^{n} F_{4k} = F_{2n+1}^2 - 1$$

$$2\sum_{k=1}^{n} F_{3k-2} = F_{3n} \qquad \sum_{k=1}^{n} F_{4k-3} = F_{2n-1}F_{2n}$$

$$2\sum_{k=1}^{n} F_{3k} = F_{3n+2} - 1 \qquad \sum_{k=1}^{n} F_{4k-1} = F_{2n}F_{2n+1}$$

In that paper is derived the general formula,

$$\sum_{k=1}^{n} F_{ak-b} = \frac{(-1)^a F_{an-b} - F_{a(n+1)-b} + (-1)^{a-b} F_b + F_{a-b}}{(-1)^a + 1 - L_a}$$

for the a-th Lucas number L_a and the Fibonacci numbers with subscript ak-b .

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B-1, B-2, B-3 are each proved by mathematical induction in Vol. 1, No. 3, October, 1963, pp. 76-78.

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Use the formulas for the lambda number developed in the article <u>or</u> use elementary row and column operations to simplify the resulting determinant.

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B-4 (Solution by Joseph Erbacher and J. L. Brown, Jr., FQ, 2:1, February, 1964, p. 80) Using the Binet formula,

$$F_{2n+j} = \frac{(a^2)^n a^j - (b^2)^n b^j}{a - b} = \frac{(1 + a)^n a^j - (1 + b)^n b^j}{a - b}$$

Since

$$a^2 = a + 1, \quad b^2 = b + 1 \quad \text{when} \quad a = (1 + \sqrt{5})/2, \quad b = (1 - \sqrt{5})/2,$$

we have

$$F_{2n+j} = \frac{1}{a - b}\left[\sum_{i=0}^{n} \binom{n}{i} a^{i+j} - \sum_{i=0}^{n} \binom{n}{i} b^{i+j}\right] = \sum_{i=0}^{n} \binom{n}{i} \frac{a^{i+j} - b^{i+j}}{a - b}$$

$$= \sum_{i=0}^{n} \binom{n}{i} F_{i+j} \quad .$$

If $j = 0$, we have the original problem. The identity also holds, with arbitrary j, for Lucas numbers $L_n = F_{n+1} + F_{n-1}$.

B-5 (Solution by J. L. Brown, Jr., FQ, 1:3, October, 1963, p. 79.)

Let a_n for $n \geq 1$ be the number of different ways of being paid n dollars in one and two dollar bills, taking order into account. Consider the case where $n \geq 2$. Since a one-dollar bill is received as the last bill if and only if $n - 1$ dollars have been received previously and a two-dollar bill is received as the last bill if and only if $n - 2$ dollars have been received previously, the two possibilities being mutually exclusive, we have $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. But $a_1 = 1$, $a_2 = 2$; therefore, $a_n = F_{n+1}$ for $n \geq 1$.

B-9 (Solution by Francis D. Parker, FQ, 1:4, Dec., 1963, p. 76)

Since

$$\frac{1}{F_{n-1}F_{n+1}} = \frac{F_n}{F_{n-1}F_nF_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_nF_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}} \quad ,$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}\right) = \left(\frac{1}{1\cdot 1} - \frac{1}{1\cdot 2}\right) + \left(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right)$$

$$+ \left(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 5}\right) + \dots = 1$$

Similarly,

$$\frac{F_n}{F_{n-1}F_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}} - \frac{1}{F_{n+1}} \qquad \text{and}$$

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{8}\right) + \ldots = 2$$

B-10 (Solution by Charles Wall, FQ, 1:4, Dec., 1963, p. 77)
Since

$$\frac{L_n + \sqrt{5}\,F_n}{2} = \frac{a^n + b^n + a^n - b^n}{2} = a^n$$

where $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$, we have

$$\left(\frac{L_n + \sqrt{5}\,F_n}{2}\right)^p = a^{np} = \frac{a^{np} + b^{np} + a^{np} - b^{np}}{2} = \frac{L_{np} + \sqrt{5}\,F_{np}}{2} \quad .$$

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H-8 (Solution by John Allen Fuchs and Joseph Erbacher, FQ, 1:3, October, 1963, pp. 51-52.) The squares of the Fibonacci numbers satisfy the linear homogeneous recursion relationship $F^2_{n+3} = 2F^2_{n+2} + 2F^2_{n+1} - F^2_n$. We may use this recursion formula to substitute for the last row of the given determinant, D_n, and then apply standard row operations to get

$$D_n = \begin{vmatrix} F^2_n & F^2_{n+1} & F^2_{n+2} \\ F^2_{n+1} & F^2_{n+2} & F^2_{n+3} \\ 2F^2_{n+1} + 2F^2_n - F^2_{n-1} & 2F^2_{n+2} + 2F^2_{n+1} - F^2_n & 2F^2_{n+3} + 2F^2_{n+2} - F^2_{n+1} \end{vmatrix}$$

$$= \begin{vmatrix} F^2_n & F^2_{n+1} & F^2_{n+2} \\ F^2_{n+1} & F^2_{n+2} & F^2_{n+3} \\ -F^2_{n-1} & -F^2_n & -F^2_{n+1} \end{vmatrix} = -D_{n-1} \ .$$

It follows that $D_n = (-1)^{n-1}D_1$. Since $D_1 = 2$, $D_n = 2(-1)^{n-1} = 2(-1)^{n+1}$.

B-28 (Solution by Marjorie Bicknell, FQ, 2:2 , April, 1964, p. 159)

By considering combinations of Fibonacci numbers which give minimum and maximum values to sums of the form abc + def + ghi, the following determinant seems to have the maximum value obtainable with the nine Fibonacci numbers given:

$$\begin{vmatrix} F_{10} & F_4 & F_7 \\ F_6 & F_9 & F_3 \\ F_2 & F_5 & F_8 \end{vmatrix} = F_{10}F_9F_8 + F_7F_6F_5 + F_4F_3F_2 - (F_{10}F_3F_5 + F_9F_2F_7 + F_8F_4F_6)$$

$$= 39796 - 1496 = 38300 .$$

B-13 Expand the determinant by its last row, obtaining $F_n = F_{n-1} + F_{n-2}$, making possible a proof by mathematical induction since $F_1 = 1$ and $F_2 = 2$.

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B-14 (Solution by Charles Wall, FQ, 1:4, Dec., 1963, pp. 79-80)

Since

$$\sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

let x = 0.1 in one case and (-0.1) in the other.

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Since Euler's famous formula gives $e^{i\pi} = -1$, the curious formula becomes just $\emptyset = 2 \cos \pi/5$, which is proved in the article referred to when the problem was posed.

B-18 (Solution by J. L. Brown, Jr., FQ, 2:1, Feb., 1964, pp. 74-75.)

It is well known (e. g., I. J. Schwatt, "An Introduction to the Operations with Series," Chelsea Pub. Co., p. 177) that $\cos \pi/5 = (1 + \sqrt{5})/4$ and $\sin \pi/10 = (\sqrt{5} - 1)/4$. Therefore, $a = (1 + \sqrt{5})/2 = 2 \cos \pi/5$ and $b = (1 - \sqrt{5})/2 = -2 \sin \pi/10$, and

$$F_n = \frac{a^n - b^n}{a - b} = 2^{n-1} \cdot \frac{\cos^n \frac{\pi}{5} - (-1)^n \sin^n \frac{\pi}{10}}{\cos \frac{\pi}{5} + \sin \frac{\pi}{10}}$$

$$= 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1} \frac{\pi}{5} \sin^k \frac{\pi}{10}$$

as stated. We have made use of the algebraic identity

$$\frac{x^n - y^n}{x - y} = \sum_{k=0}^{n-1} x^{n-k-1} y^k .$$

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H-19 (Solution by Michael Goldberg, FQ, 2:2, April, 1964, pp. 130-131)

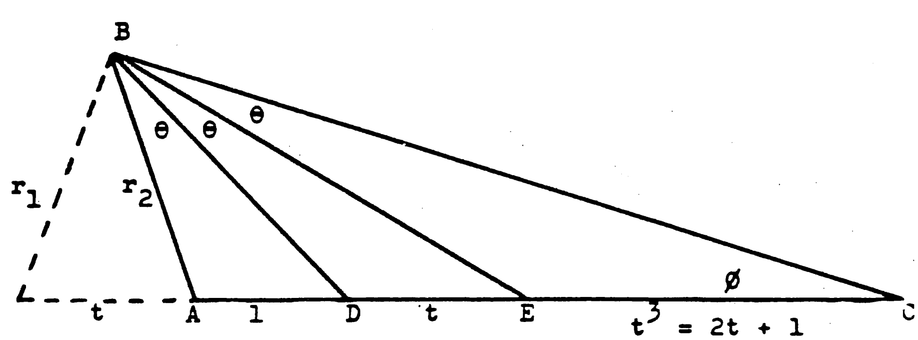
As $n \to \infty$, the ratio F_{n+1}/F_n approaches $t = (\sqrt{5} + 1)/2$, and F_{n+3}/F_n approaches $t^3 = 2t + 1$. Hence, the limiting triangle ABC can be drawn by taking points D and E on AC so that AD = 1, DE = t, and EC = 2t + 1. Since BD is a bisector of ∠ABE, the point B must lie on the circle which is the locus of points whose distances to A and E are in the ratio AD/DE = 1/t. The circle passes through D. If the diameter of the circle is $2r_1 = x + 1$, then $x/(x + 1 + t) = 1/t$ from which

$$r_1 = t/(t - 1) = t^2 = t + 1 .$$

Simarly, BE is a bisector of the angle DBC. The point B must lie on a circle which is the locus of points whose distances from D and C are in the ratio $DE/EC = t/t^3 = 1/t^2$. If the diameter of the circle is $2r_2 = y + t$, then $y/(y + t + t^2) = 1/t^2$ from which

$$r_2 = t^2 = t + 1 = r_1 .$$

Hence, $\cos \angle BAE = - t/2(t + 1) = - (\sqrt{5} - 1)/4$ and $\angle BAE = 108°$. From which $2\theta = 90° - 108°/2 = 36°$, $\theta = 18°$; $\phi = 180° - 108° - 3\theta = 18°$.



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B-39 (Solution by Brian Scott, FQ, 2:4, Dec., 1964, p. 327)

The solution is by induction on n. $F_{3+2} = F_5 = 5 < 8 = 2^3$ and $F_{4+2} = F_6 = 8 < 16 = 2^4$. Assume as the induction hypothesis that $F_{(n-2)+2} < 2^{n-2}$ and $F_{(n-1)+2} < 2^{n-1}$. Then

$$F_{n+2} = F_{(n-1)+2} + F_{(n-2)+2} < 2^{n-1} + 2^{n-2} = 2^{n-2}(2 + 1) < 2^{n-2} \cdot 2^2 = 2^n .$$

Therefore, $F_{n+2} < 2^n$ for all $n \geq 3$.

B-41(Solution by John L. Brown, Jr., FQ, 2:4, Dec., 1964, pp. 328-329.)

No. For, assume $F_i < F_j < F_h < F_k$ are in arithmetic progression, so that $F_j - F_i = d = F_k - F_h$. Then

$$d = F_j - F_i < F_j$$

while

$$d = F_k - F_h \geq F_k - F_{k-1} = F_{k-2} \geq F_j ,$$

since $k \geq j + 2$. This is a contradiction, so that four distinct positive Fibonacci numbers cannot be in arithmetic progression.

B-42 (Solution by H. H. Ferns)

The following three identities are readily proved by applying Binet's formula.

(1) $$2F_{n+1} = F_n + L_n$$

(2) $$L_n^2 - 5F_n^2 = 4(-1)^n$$

(3) $$2L_{n+1} = 5F_n + L_n$$

Eliminating L_n from (1) and (2) gives F_{n+1}, while eliminating F_n from (2) and (3) gives L_{n+1}:

$$F_{n+1} = \frac{F_n + \sqrt{5F_n^2 + 4(-1)^n}}{2} , \qquad L_{n+1} = \frac{L_n + \sqrt{5}\sqrt{L_n^2 - 4(-1)^n}}{2} .$$

B-44 (Solution by Douglas Lind, FQ, 3:1, February, 1965, p. 75)
Assume the maximum,

(1) $$n^k < F_{r+1}, F_{r+2}, \ldots, F_{r+n} < n^{k+1} .$$

Now

$$\sum_{j=1}^{n-1} F_{r+j} = \sum_{j=1}^{r+n-1} F_j - \sum_{j=1}^{r} F_j = F_{r+n+1} - F_{r+2} .$$

But by (1),

$$\sum_{j=1}^{n-1} F_{r+j} + F_{r+2} > n \cdot n^k$$

and hence

$$F_{r+n+1} > n^{k+1}$$

thus proving the proposition.

B-47 (Solution by Sidney Kravitz, FQ, 3:1, February, 1965, p. 77)
Let F_n be the n-th Fibonacci number. We note that $F_n > 1$ for $n > 2$, that F_j divides F_{mj}, and that j is a divisor of $(k + 2)! + j$ for $3 \le j \le k + 2$. Thus the k consecutive Fibonacci numbers $F_{(k+2)!+3}$, $F_{(k+2)!+4}$, $\ldots$, $F_{(k+2)!+k+2}$ are divisible by F_3, F_4, $\ldots$, F_{k+2} respectively.

B-58 (Solution by Douglas Lind, FQ, 3:3, October, 1965, pp. 236-237)
Since $L_k = F_{k-1} + F_{k+1}$, the assertion is equivalent to

(1) $$F_n = F_{k-1} + F_{k+1} .$$

If $k \ge 3$, then $n > k + 1$, and (1) is clearly impossible since

$$F_{k-1} + F_{k+1} < F_k + F_{k+1} = F_{k+2} \le F_n .$$

Impossibility for $k \ge 3$ implies impossibility for $k \le -3$ since only signs are different. For $-3 < k < 3$ we find $F_{-2} = L_{-1} = 1$, $F_3 = L_0 = 2$, $F_1 = L_1 = 1$, and $F_4 = L_2 = 3$, corresponding to $k = -1, 0, 1,$ and 2 respectively. Hence these are the only solutions.

B-62 (Solution by J. L. Brown, Jr., FQ, 3:3, October, 1965, p. 239)

From the identity $F_{2n+1} = F_n^2 + F_{n+1}^2$, $(n \geq 1)$ it follows that $F_{2n+1} < (F_n + F_{n+1})^2 = F_{n+2}^2$. Therefore, any representation $F_{2n+1} = F_k^2 + F_m^2$ $(k \leq m)$ must have both k and $m \leq n + 1$. Then $k \geq n$, for otherwise $F_k^2 + F_m^2 < F_n^2 + F_{n+1}^2 = F_{2n+1}$ for $k > 2$.

B-95 (Solution by Charles W. Trigg, FQ, 5:2, April, 1967, p. 204)

For $n \geq 3$, F_k is divisible by 2^n if k is of the form $2^{n-2} \cdot 3(1 + 2m)$. F_k is divisible by 2^n but by no higher power of 2. Hence, the highest power of 2 that exactly divides $F_1F_2F_3 \cdots F_{100}$ is

$$[103/6] + 3[106/12] + 4[112/24] + 5[124/48] + 6[148/96] + 7[196/192]$$

or 80. As usual, $[x]$ indicates the largest integer in x.

(Editorial note: The results in the above solution indicate that the answer may also be expressed as

$$[100/3] + 2[100/6] + [100/12] + [100/24] + [100/48] + [100/96]$$
$$= 33 + 32 + 8 + 4 + 2 + 1 = 80.\ )$$

H-2 This was a world famous problem. J. H. E. Cohn proved the truth of the conjecture in "Square Fibonacci Numbers, Etc.," Fibonacci Quarterly, Vol. 2, No. 2, April, 1964, pp. 109-113. Also, it was established that $L_1 = 1$ and $L_3 = 4$ are the only Lucas numbers which are perfect squares.