A PRIMER FOR THE FIBONACCI NUMBERS: PART V

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INFINITE SERIES AND FIBONACCI ARCTANGENTS

1. INTRODUCTION

In Section 8 of Part IV, we discussed an alternating series. This time we shall lay down some brief foundations of sequences and infinite series. This leads to some very interesting results and to the broad topics of generating functions and continued fractions. Many Fibonacci numbers shall appear.

2. SEQUENCES

Definition: An ordered set of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called an infinite sequence of numbers. If there are but a finite number of the a's, $a_1, a_2, \ldots, a_n$, then it is a finite sequence of numbers.

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to have a real number a as a limit, written $\lim_{n\to\infty} a_n = a$, if for every positive real number ϵ, $|a_n - a| < \epsilon$ for all but a finite number of the members of the sequence $\{a_n\}$. If the sequence $\{a_n\}$ has a limit, this limit is unique and the sequence is said to converge to this limit. If the sequence $\{a_n\}$ fails to approach a limit, then the sequence is said to diverge. We now give examples of each kind.

If $a_n = 1$, $\{a_n\} = 1, 1, 1, \ldots$ converges since $\lim_{n\to\infty} a_n = 1$.

If $a_n = 1/n$, $\{a_n\} = 1, 1/2, 1/3, \ldots, 1/n, \ldots$ converges to zero.

If $a_n = (-1)^n$, $\{a_n\} = 1, -1, +1, -1, +1, \ldots$ diverges by oscillation. That is, it does not approach any limit.

If $a_n = n$, $\{a_n\} = 1, 2, 3, \ldots$ diverges to positive infinity.

Finally, if $a_n = n/(n+1)$, then $\{a_n\} = 1/2, 2/3, \ldots$ converges to one.

Some limit theorems for sequences are the following:

If $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers with limits a and b respectively, then

$$\lim_{n\to\infty} (a_n + b_n) = a + b$$

$$\lim_{n\to\infty} (a_n - b_n) = a - b$$

$$\lim_{n\to\infty} (ca_n) = ca, \text{ any real } c$$

$$\lim_{n\to\infty} (a_n b_n) = ab$$

$$\lim_{n\to\infty} (a_n/b_n) = a/b \,, \quad b \neq 0 \,.$$

3. BOUNDED MONOTONE SEQUENCES

The sequence $\{a_n\}$ is said to be bounded if there exists a positive number K such that $|a_n| < K$ for all $n \geq 1$. If $a_{n+1} \geq a_n$ for $n \geq 1$, the sequence $\{a_n\}$ is said to be a monotone increasing sequence; if $a_n \geq a_{n+1}$ for $n \geq 1$, the sequence is monotone decreasing. If a sequence is such that it is either monotone increasing or monotone decreasing, it will be called a monotone sequence.

The following useful and important theorem is stated without proof:

Theorem 1: A bounded monotone sequence converges.

As an example, consider the sequence $\{(1 + 1/n)^n\}$, which is monotone increasing and bounded above by 3. The limit of this sequence is well known. We will use Theorem 1 in the material to come.

4. ANOTHER IMPORTANT THEOREM

The following sufficient conditions for the convergence of an alternating series are given below.

Theorem 2: If, for the sequence $\{S_n\}$,

1. $S_1 > 0$,

2. $(S_{n-1} - S_n)(-1)^n > (S_n - S_{n+1})(-1)^{n+1} > 0$, for $n \geq 2$,

3. $\lim_{n\to\infty} (S_n - S_{n+1}) = 0$,

then the sequence $\{S_n\}$ converges to a limit S such that $0 < S < S_1$.

5. AN EXAMPLE OF AN APPLICATION OF THEOREM 2

For the following example a limit is known to exist by the application of Theorem 2 of Section 4.

Let $S_n = F_n/F_{n+1}$, where $\{F_n\}$ is the Fibonacci sequence. Then $S_{n-1} - S_n = (-1)^n/(F_nF_{n+1})$. By Theorem 2 above, $\lim_{n\to\infty} S_n$ exists.

To find the limit, consider

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} ,$$

which in terms of $\{S_n\}$ is $1/S_n = 1 + S_{n-1}$. Let the limit of S_n as n tends to infinity be S. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = S > 0$. Applying the limit theorems of Section 2, it follows that S satisfies

$$S = \frac{1}{1 + S} \qquad \text{or} \qquad S^2 + S - 1 = 0 .$$

Thus $S > 0$ is given by $S = (\sqrt{5} - 1)/2$, the positive root of the quadratic equation $S^2 + S - 1 = 0$.

6. INFINITE SERIES

If we add together the members of a sequence $\{a_n\}$, we get the <u>infinite series</u> $a_1 + a_2 + \dots + a_n + \dots$. We now get another sequence from this infinite series.

Define a sequence $\{S_n\}$ in the following way. Let $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, ... , or in general, $S_n = a_1 + a_2 + a_3 + \dots + a_n$. This is called the <u>sequence of partial sums</u> of the infinite series. The infinite series is said to converge to the limit S if the sequence $\{S_n\}$ converges to the limit S; otherwise, the series is said to diverge.

7. SPECIAL RESULTS CONCERNING SERIES

1. If an infinite series $a_1 + a_2 + a_3 + \dots + a_n + \dots$ converges, then $\lim_{n\to\infty} a_n = 0$. This is immediate since $a_n = S_n - S_{n-1}$.

2. From section 3 above, an infinite series of positive terms converges if the partial sums are bounded above since the partial sums form a monotone increasing sequence.

3. For the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ such that $a_n > 0$, $n \geq 1$; $a_{n+1} \leq a_n$, $n \geq 1$; and $\lim_{n\to\infty} a_n = 0$, by Section 4 above, the infinite series converges. In the theorem, $S_n = \sum_{j=1}^{n} (-1)^j a_j$. An example of an alternati series was seen in Part IV, Section 8, of this Primer.

8. FIBONACCI NUMBERS, LUCAS NUMBERS, AND PI

It is well known and easily verified that

$$\frac{\pi}{4} = \text{Tan}^{-1}\frac{1}{1} = \text{Tan}^{-1}\frac{1}{2} + \text{Tan}^{-1}\frac{1}{3} .$$

Also one can verify

$$\frac{\pi}{4} = \text{Tan}^{-1}\frac{1}{1} = \text{Tan}^{-1}\frac{1}{2} + \text{Tan}^{-1}\frac{1}{5} + \text{Tan}^{-1}\frac{1}{8} ,$$

$$\frac{\pi}{4} = \text{Tan}^{-1}\frac{1}{3} + \text{Tan}^{-1}\frac{1}{5} + \text{Tan}^{-1}\frac{1}{7} + \text{Tan}^{-1}\frac{1}{8} .$$

We note Fibonacci and Lucas numbers here, surely. We shall here easily extend these results in several ways.

In this section we shall use several new identities which are left as exercises for the reader and will be marked with an asterisk.

*Lemma 1: $L_{2n}L_{2n+2} - 1 = 5F_{2n+1}^2$

Lemma 2: $L_n^2 = L_{2n} + 2(-1)^n$

Lemma 3: $L_n^2 - 5F_n^2 = 4(-1)^n$

*Lemma 4: $L_nL_{n+1} = L_{2n+1} + (-1)^n$

We now discuss

Theorem 3: If $\tan\theta_n = 1/L_n$, then $\tan(\theta_{2n} + \theta_{2n+2}) = 1/F_{2n+1}$, or,

$$\text{Tan}^{-1}\frac{1}{F_{2n+1}} = \text{Tan}^{-1}\frac{1}{L_{2n}} + \text{Tan}^{-1}\frac{1}{L_{2n+2}} .$$

Proof:

$$\text{Tan}(\theta_{2n} + \theta_{2n+2}) = \frac{L_{2n} + L_{2n+2}}{L_{2n}L_{2n+2} - 1} = \frac{1}{F_{2n+1}}$$

using the trigonometric identity $\tan(x + y) = (\tan x + \tan y)/(1 - \tan x \tan y)$ with Lemma 1 above and the identity $L_{2n+2} + L_{2n} = 5F_{2n+1}$.

Theorem 4: If $\tan\theta_n = 1/F_n$, then $\tan(\theta_{2n} - \theta_{2n+2}) = 1/F_{2n+1}$, or

$$\mathrm{Tan}^{-1}\frac{1}{F_{2n+1}} = \mathrm{Tan}^{-1}\frac{1}{F_{2n}} - \mathrm{Tan}^{-1}\frac{1}{F_{2n+2}} .$$

Proof:

$$\mathrm{Tan}\,(\theta_{2n} - \theta_{2n+2}) = \frac{F_{2n+2} - F_{2n}}{F_{2n}F_{2n+2} + 1} = \frac{1}{F_{2n+1}}$$

since $F_{2n+2} - F_{2n} = F_{2n+1}$ and $F_{2n}F_{2n+2} - F_{2n+1}^2 = (-1)^{2n+1} = -1$.

From Theorem 4,

$$\sum_{n=1}^{M} \mathrm{Tan}^{-1}\frac{1}{F_{2n+1}} = \sum_{n=1}^{M}\left(\mathrm{Tan}^{-1}\frac{1}{F_{2n}} - \mathrm{Tan}^{-1}\frac{1}{F_{2n+2}}\right) = \mathrm{Tan}^{-1}\frac{1}{F_2} - \mathrm{Tan}^{-1}\frac{1}{F_{2M+2}}$$

and since $\lim_{M\to\infty} \mathrm{Tan}^{-1}\frac{1}{F_{2M+2}} = 0$ by continuity of $\mathrm{Tan}^{-1} x$ at $x = 0$, we may write

<u>Theorem 5</u>:
$$\frac{\pi}{4} = \mathrm{Tan}^{-1}\, 1 = \sum_{n=1}^{\infty} \mathrm{Tan}^{-1}\frac{1}{F_{2n+1}} .$$

This is the celebrated result of D. H. Lehmer, Nov., 1936, <u>American Mathematical Monthly</u> , p. 632, Problem 3801.

We note in passing that the partial sums

$$S_M = \sum_{n=1}^{M} \mathrm{Tan}^{-1}\frac{1}{F_{2n+1}} = \mathrm{Tan}^{-1}\frac{1}{F_2} - \mathrm{Tan}^{-1}\frac{1}{F_{2M+2}}$$

are all bounded above by $\mathrm{Tan}^{-1}\, 1 = \pi/4$ and S_M is monotone. Thus Theorem 1 can be applied. From Theorem 3,

$$\sum_{n=1}^{M} \mathrm{Tan}^{-1}\frac{1}{F_{2n+1}} = \sum_{n=1}^{M}\left(\mathrm{Tan}^{-1}\frac{1}{L_{2n}} + \mathrm{Tan}^{-1}\frac{1}{L_{2n+2}}\right)$$

so that

$$\sum_{n=1}^{M} \mathrm{Tan}^{-1}\frac{1}{F_{2n+1}} + \mathrm{Tan}^{-1}\frac{1}{3} = 2\sum_{n=1}^{M} \mathrm{Tan}^{-1}\frac{1}{L_{2n}} + \mathrm{Tan}^{-1}\frac{1}{L_{2M+2}}$$

The limit on the left tends to $\text{Tan}^{-1} 1 + \text{Tan}^{-1} 1/3 = \text{Tan}^{-1} 2$, and the right-hand side tends to this same limit. Since $\lim_{M\to\infty} \text{Tan}^{-1} \frac{1}{L_{2M+2}} = 0$, then

Theorem 6: $$\sum_{n=1}^{\infty} \text{Tan}^{-1} \frac{1}{L_{2n}} = \text{Tan}^{-1} \frac{\sqrt{5}-1}{2} = \frac{1}{2} \text{Tan}^{-1} 2 .$$

Compare with Theorem 6 in Part IV.

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FIBONACCI DETERMINANTS

Below are reprinted a selection of problems which appeared in early issues of the Fibonacci Quarterly.

H-8 (Proposed by Brother Alfred Brousseau) Prove that

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = 2(-1)^{n+1} ,$$

where F_n is the nth Fibonacci number.

B-28 (Proposed by Brother Alfred Brousseau) Using the nine Fibonacci numbers F_2 through F_{10} (1, 2, 3, 5, 8, 13, 21, 34, 55), determine a third-order determinant having each of these numbers as elements so that the value of the determinant is a maximum.

B-13 (Proposed by S. L. Basin) Prove the (n - 1)st order determinant below has value F_n. (This is a special case of B-13)

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n-1}$$

Such determinants are called continuants.

A problem which predates B-28 is to determine the third-order determinant of maximum value which has each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 as elements, and to determine the complete set of determinant values possible. (See Bicknell and Hoggatt, "An Investigation of Nine-Digit Determinants," Mathematics Magazine, May-June, 1963, pp. 147-152.)