SOME NEW FIBONACCI IDENTITIES

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In this paper, some new Fibonacci and Lucas identities are generated by matrix methods.

The matrix

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

satisfies the matrix equation $R^3 - 2R^2 - 2R + I = 0$. Multiplying by R^n yields

(1) $$R^{n+3} - 2R^{n+2} - 2R^{n+1} + R^n = 0 \quad .$$

It has been shown by Brennan [1] and appears in an earlier article [2] that

(2) $$R^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_nF_{n-1} & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix} ,$$

where F_n is the nth Fibonacci number.

By the definition of matrix addition, corresponding elements of R^{n+3}, R^{n+2}, R^{n+1}, and R^n must satisfy the recursion formula given in Equation (1). That is, for example,

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0,$$

$$F_{n+3}F_{n+4} - 2F_{n+2}F_{n+3} - 2F_{n+1}F_{n+2} + F_nF_{n+1} = 0.$$

Returning again to $R^3 - 2R^2 - 2R + I = 0$, this equation can be rewritten as

$$(R + I)^3 = R^3 + 3R^2 + 3R + I = 5R(R + I).$$

In general, by induction, it can be shown that

(3) $$R^p(R + I)^{2n+1} = 5^nR^{n+p}(R + I).$$

Equating the elements in the first row and third column of the above matrices, by means of Equation (2), we obtain

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i+p}^2 = 5^n F_{2(n+p)+1} \quad . \tag{4}$$

It is not difficult to show that the Lucas numbers and members of the Fibonacci sequence have the relationship

$$L_n^2 - 5F_n^2 = (-1)^n 4 \quad .$$

Since also

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} (-1)^{i+p} = 0,$$

we can derive the following sum of squares of Lucas numbers,

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L_{i+p}^2 = 5^{n+1} F_{2(n+p)+1} \quad ,$$

by substitution of the preceding two identities in Equation (4).

Upon multiplying Equation (3) on the right by (R + I), we obtain

$$R^p(R+I)^{2n+2} = 5^n R^{n+p}(R+I)^2 \quad . \tag{5}$$

Then, using the expression for R^n given in Equation (2) and the identity $L_k = F_{k-1} + F_{k+1}$, we find that

$$(R^{n+1} + R^n)(R+I) = \begin{pmatrix} F_{2n-1} & F_{2n} & F_{2n+1} \\ 2F_{2n} & 2F_{2n+1} & 2F_{2n+2} \\ F_{2n+1} & F_{2n+2} & F_{2n+3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} L_{2n} & L_{2n+1} & L_{2n+2} \\ 2L_{2n+1} & 2L_{2n+2} & 2L_{2n+3} \\ L_{2n+2} & L_{2n+3} & L_{2n+4} \end{pmatrix} \quad .$$

Finally, by equating the elements in the first row and third column of the matrices of Equation (5), we derive the two identities

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} F_{i+p}^2 = 5^n L_{2(n+p)+2},$$

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} L_{i+p}^2 = 5^{n+1} L_{2(n+p)+2}.$$

By similar steps, by equating the elements appearing in the first row and second column of the matrices of Equations (3) and (5), we can write the additional identities,

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i-1+p} F_{i+p} = 5^n F_{2(n+p)} \quad ,$$

$$\sum_{i=0}^{2n+2} \binom{2n+2}{i} F_{i-1+p} F_{i+p} = 5^n L_{2(n+p)+1} \quad .$$

REFERENCES

1. From the unpublished notes of Terry Brennan.
2. Marjorie Bicknell and Verner E. Hoggatt, Jr., "Fibonacci Matrices and Lambda Functions," The Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 47-52.

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Editorial Comment

Form the (n + 1) X (n + 1) matrix P_n with Pascal's triangle appearing on and below its secondary diagonal, e. g.,

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Surely the reader will see $R = P_2$ and matrix P_1 very like Q in the lower left. The element occurring in the lower left corner of P_n^k is always F_k^n, and the characteristic equation of P_n has the Fibonomial coefficients appearing, leading to identities such as described in the next article.