A PRIMER FOR THE FIBONACCI NUMBERS: PART VII

Marjorie Bicknell
A. C. Wilcox High School, Santa Clara, California

AN INTRODUCTION TO FIBONACCI POLYNOMIALS AND THEIR DIVISIBILITY PROPERTIES

An elementary study of the Fibonacci polynomials yields some general divisibility theorems, not only for the Fibonacci polynomials, but also for Fibonacci numbers and generalized Fibonacci numbers. This paper is intended also to be an introduction to the Fibonacci polynomials.

Fibonacci and Lucas polynomials are special cases of Chebyshev polynomials, and have been studied on a more advanced level by many mathematicians. For our purposes, we define only Fibonacci and Lucas polynomials.

1. THE FIBONACCI POLYNOMIALS

The Fibonacci polynomials $\{F_n(x)\}$ are defined by

(1.1) $$F_1(x) = 1, \quad F_2(x) = x, \quad \text{and} \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x) .$$

Notice that, when x = 1, $F_n(1) = F_n$, the nth Fibonacci number. It is easy to verify that the relation

(1.2) $$F_{-n}(x) = (-1)^{n+1}F_n(x)$$

extends the definition of Fibonacci polynomials to all integral subscripts. The first ten Fibonacci polynomials are given below:

$$F_1(x) = 1$$

$$F_2(x) = x$$

$$F_3(x) = x^2 + 1$$

$$F_4(x) = x^3 + 2x$$

$$F_5(x) = x^4 + 3x^2 + 1$$

$$F_6(x) = x^5 + 4x^3 + 3x$$

$$F_7(x) = x^6 + 5x^4 + 6x^2 + 1$$

$$F_8(x) = x^7 + 6x^5 + 10x^3 + 4x$$

$$F_9(x) = x^8 + 7x^6 + 15x^4 + 10x^2 + 1$$

$$F_{10}(x) = x^9 + 8x^7 + 21x^5 + 20x^3 + 5x$$

It is important for Section 4, to notice that the degree of $F_n(x)$ is $|n| - 1$ for $n \neq 0$. Also, $F_0(x) = 0$.

In Table 1, the coefficients of the Fibonacci polynomials are arranged in ascending order. The sum of the nth row is F_n, and the sum of the nth diagonal of slope one, formed by beginning on the nth row, left-most column, and going one up and one right to get the next term, is given by

$$2^{(n-1)/2} = 2 \cdot 2^{(n-3)/2}$$

when n is odd.

Table 1

Fibonacci Polynomial Coefficients Arranged in Ascending Order

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
1	1								
2	0	1							
3	1	0	1						
4	0	2	0	1					
5	1	0	3	0	1				
6	0	3	0	4	0	1			
7	1	0	6	0	5	0	1		
8	0	4	0	10	0	6	0	1	
9	1	0	10	0	15	0	7	0	1

. . . .

To compare with Pascal's triangle, the sum of the nth row there is 2^n, and the sum of the nth diagonal of slope one is F_n. In fact, the (alternate) diagonals of slope one in Table 1 produce Pascal's triangle.

If the successive binomial expansions of $(x + 1)^n$ are written in descending order,

$n = 0$:	1								
$n = 1$:	x	$+$	1						
$n = 2$:	x^2	$+$	$2x$	$+$	1				
$n = 3$:	x^3	$+$	$3x^2$	$+$	$3x$	$+$	1		
$n = 4$:	x^4	$+$	$4x^3$	$+$	$6x^2$	$+$	$4x$	$+$	1
. . .				. . .					

the sum of the 4th diagonal of slope one is $F_4(x) = x^4 + 3x^2 + 1$, and the sum of the nth diagonal of slope one is $F_n(x)$, or,

$$(1.3) \qquad F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1}$$

for $[x]$ the greatest integer contained in x, and binomial coefficient $\binom{n}{j}$, as given by Swamy [1] and others.

2. LUCAS POLYNOMIALS AND GENERAL FIBONACCI POLYNOMIALS

The Lucas polynomials $\{L_n(x)\}$ are defined by

$$(2.1) \qquad L_0(x) = 2, \quad L_1(x) = x, \qquad L_{n+1}(x) = xL_n(x) + L_{n-1}(x) .$$

Again, when $x = 1$, $L_n(1) = L_n$, the nth Lucas number. Lucas polynomials have the properties

$$(2.2) \qquad L_n(x) = F_{n+1}(x) + F_{n-1}(x) = xF_n(x) + 2F_{n-1}(x)$$

$$xL_n(x) = F_{n+2}(x) - F_{n-2}(x)$$

and can be extended to negative subscripts by

$$(2.3) \qquad L_{-n}(x) = (-1)^n L_n(x) .$$

If the Lucas polynomial coefficients are arranged in ascending order in a left-justified triangle similar to that of Table 1, the sum of the nth row is L_n, and the sum of the nth diagonal of slope one is given by $3 \cdot 2^{(n-2)/2}$ for even n, $n \geq 2$. The degree of $L_n(x)$ is $|n|$, as can be observed in the following list of the first ten Lucas polynomials:

$$L_1(x) = x$$

$$L_2(x) = x^2 + 2$$

$$L_3(x) = x^3 + 3x$$

$$L_4(x) = x^4 + 4x^2 + 2$$

$$L_5(x) = x^5 + 5x^3 + 5x$$

$$L_6(x) = x^6 + 6x^4 + 9x^2 + 2$$

$$L_7(x) = x^7 + 7x^5 + 14x^3 + 7x$$

$$L_8(x) = x^8 + 8x^6 + 20x^4 + 16x^2 + 2$$

$$L_9(x) = x^9 + 9x^7 + 27x^5 + 30x^3 + 9x$$

$$L_{10}(x) = x^{10} + 10x^8 + 35x^6 + 50x^4 + 25x^2 + 2 \quad .$$

When general Fibonacci polynomials are defined by

(2.4) $$H_1(x) = a, \quad H_2(x) = bx, \quad H_n(x) = xH_{n-1}(x) + H_{n-2}(x) ,$$

then

(2.5) $$H_n(x) = bxF_{n-1}(x) + aF_{n-2}(x) .$$

If the coefficients of the $\{H_n(x)\}$, written in ascending order, are placed in a left-justified triangle such as Table 1, then the sum of the nth diagonal of slope one is

$$(a + b)\cdot 2^{(n-3)/2} = (a + b)\cdot 2^{[(n-2)/2]}$$

for odd n, $n \geq 3$. (Notice that, if a = 2, b = 1, then $H_{n+1}(x) = L_n(x)$, and if a = b = 1, $H_n(x) = F_n(x)$.)

3. A MATRIX GENERATOR FOR FIBONACCI POLYNOMIALS

Since Fibonacci polynomials appear as the elements of the matrix defined below, many identities can be derived for Fibonacci polynomials using matrix theory, as done by Hayes [2] and others, and as done for Fibonacci numbers by Basin and Hoggatt [3].

It is easily established by mathematical induction that the matrix

$$Q = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},$$

when raised to the kth power, is given by

$$Q^k = \begin{pmatrix} F_{k+1}(x) & F_k(x) \\ F_k(x) & F_{k-1}(x) \end{pmatrix}$$

for any integer k, where Q^0 is the identity matrix and Q^{-k} is the matrix inverse of Q^k. Since det Q = -1, det Q^k = (det Q)^k = $(-1)^k$ gives us

$$(3.2) \qquad F_{k+1}(x)F_{k-1}(x) - F_k^2(x) = (-1)^k .$$

Since $Q^mQ^n = Q^{m+n}$ for all integers m and n, matrix multiplication of Q^m and Q^n gives

$$Q^mQ^n = \begin{pmatrix} F_{m+1}(x)F_{n+1}(x) + F_m(x)F_n(x) & F_{m+1}(x)F_n(x) + F_m(x)F_{n-1}(x) \\ F_m(x)F_{n+1}(x) + F_{m-1}(x)F_n(x) & F_m(x)F_n(x) + F_{m-1}(x)F_{n-1}(x) \end{pmatrix}$$

while

$$Q^{m+n} = \begin{pmatrix} F_{m+n+1}(x) & F_{m+n}(x) \\ F_{m+n}(x) & F_{m+n-1}(x) \end{pmatrix} .$$

Equating elements in the upper right corner gives

$$(3.3) \qquad F_{m+n}(x) = F_{m+1}(x)F_n(x) + F_m(x)F_{n-1}(x) .$$

Replacing n by (-n) and using the identity (1.2) gives

$$F_{m-n}(x) = (-1)^n[-F_{m+1}(x)F_n(x) + F_m(x)F_{n+1}(x)] .$$

Then,

$$F_{m+n}(x) + (-1)^nF_{m-n}(x) = F_m(x)F_{n-1}(x) + F_m(x)F_{n+1}(x) = F_m(x)L_n(x) .$$

If we replace n by k and m by m - k above, we can obtain finally

(3.4) $$F_m(x) = L_k(x)F_{m-k}(x) + (-1)^{k+1}F_{m-2k}(x)$$

which results in the divisibility theorems of the next section.

4. DIVISIBILITY PROPERTIES OF FIBONACCI AND LUCAS POLYNOMIALS

Lemma. The Fibonacci polynomials $F_m(x)$ satisfy

$$F_m(x) = F_{m-k}(x)\left(\sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{(2i+1)k-2im}(x)\right) + (-1)^{p(m-k)+m+1}F_{(2p-1)m-2pk}(x)$$

for all integers m and k, and for $p \geq 1$.

Proof: If $p = 1$, the Lemma is just Equation (3.4). For convenience, call $Q_p(x)$ the sum of Lucas polynomials in the Lemma. Then, assume that the Lemma holds when $p = j$, or that

(A) $$F_m(x) = F_{m-k}(x)Q_j(x) + (-1)^{j(m-k)+m+1}F_{(2j-1)m-2jk}(x) .$$

Substitute $[2jk - (2j - 1)m]$ for m in Equation (3.4), giving

$$F_{2jk-(2j-1)m}(x) = L_{k'}(x)F_{2jk-(2j-1)m-k'}(x) + (-1)^{k'+1}F_{2jk-(2j-1)m-2k'}(x) .$$

Since we want to express $F_{2jk-(2j-1)m}(x)$ in terms of $F_{m-k}(x)$, set

$$2jk - (2j - 1)m - k' = m - k$$

and solve for k', yielding $k' = (2j + 1)k - 2jm$, so that

$$F_{2jk-(2j-1)m}(x) = L_{(2j+1)k-2jm}(x)F_{m-k}(x) + (-1)^{k+1}F_{(2j+1)m-(2j+2)k}(x) .$$

Substituting into (A) and using Equation (1.2) to simplify gives

$$F_m(x) = [Q_j(x) + (-1)^{j(m-k)}L_{(2j+1)k-2jm}(x)]F_{m-k}(x)$$
$$+ (-1)^{(j+1)(m-k)+m+1}F_{(2j+1)m-(2j+2)k}(x) ,$$

which is the Lemma when $p = j + 1$, completing a proof by mathematical induction.

Notice that the Lemma yields an interesting identity for Fibonacci numbers, given below:

$$(4.1)\quad F_m = F_{m-k}\left(\sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{m-(2i+1)(m-k)}\right) + (-1)^{p(m-k)}F_{m-2p(m-k)},$$

where $p \geq 1$. To establish (4.1), use algebra on the subscripts of the Lemma and then take $x = 1$.

<u>Theorem 1:</u> Whenever a Fibonacci polynomial $F_m(x)$ is divided by a Fibonacci polynomial $F_{m-k}(x)$, $m \neq k$, of lesser or equal degree, the remainder is always a Fibonacci polynomial or the negative of a Fibonacci polynomial, and the quotient is a sum of Lucas polynomials whenever the division is not exact. Explicitly, for $p \geq 1$,

(i) the remainder is

$$\pm F_{(2p-1)m-2kp}(x) \quad \text{when} \quad \frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1},$$

or, equivalently, the remainder is

$$\pm F_{m-2p(m-k)}(x) \quad \text{for} \quad \frac{|m|}{2p+1} < |m-k| < \frac{|m|}{2p-1};$$

(ii) the quotient is $\pm L_k(x)$ when $|k| < 2|m|/3$;

(iii) the quotient is given by

$$Q_p(x) = \sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{(2i+1)k-2im}(x)$$

$$= \sum_{i=0}^{p-1}(-1)^{i(m-k)}L_{m-(2i+1)(m-k)}(x)$$

for m, k, and p related as in (i), and by $Q_p(x) + (-1)^{p(m-k)}$ if $k = 2pm/(2p+1)$;

(iv) the division is exact when $k = 2pm/(2p+1)$ or $k = (2p-1)m/2p$.

Proof: When

$$\frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1}$$

and the degree of $F_m(x)$ is greater than that of $F_{m-k}(x)$, we can show that

$$|m| > |m-k| > |(2p-1)m - 2pk| .$$

Since the degree of $F_n(x)$ is $|n| - 1$, we can interpret the Lemma in terms of quotients and remainders for the restrictions on m, k, and p above, establishing (i), (ii), and (iii). As for (iv), the division is exact if we have $k = \frac{(2p-1)m}{2p}$, for then

$$F_{(2p-1)m-2pk}(x) = F_0(x) = 0 .$$

When $k = \frac{2pm}{2p+1}$,

$$F_{(2p-1)m-2pk}(x) = F_{k-m}(x) = (-1)^{m-k+1}F_{m-k}(x)$$

$$= (-1)^{m+1}F_{m-k}(x)$$

because k is an even integer. Referring to the Lemma, increasing the quotient by $(-1)^{p(m-k)+m+1+m+1} = (-1)^{p(m-k)}$ will make the division exact.

Corollary 1.1: $F_q(x)$ divides $F_m(x)$ if and only if q divides m.

Proof: If q divides m, then either $m/2p = q$ or $m/(2p+1) = q$. Let $q = m - k$ and apply Theorem 1.

If $F_q(x)$ divides $F_m(x)$, then let $q = m - k$ and consider the remainder of Theorem 1. Either

$$F_{(2p-1)m-2pk}(x) = F_0(x) \quad \text{or} \quad F_{(2p-1)m-2pk}(x) = \pm F_{m-k}(x) ,$$

giving

$$k = \frac{(2p-1)m}{2p}, \quad k = \frac{(2p-2)m}{2p-1}, \quad \text{or} \quad k = \frac{2pm}{2p+1}$$

by equating subscripts. The possibilities give $q = m - k = m/2p$, $q = m/(2p-1)$, or $q = m/(2p+1)$, so that q divides m.

Corollary 1.2: If the Fibonacci number F_m is divided by F_{m-k}, $m \neq k$, then the remainder of least absolute value is always a Fibonacci number or its negative. Further,

(i) the remainder is

$$\pm F_{m-2p(m-k)} \quad \text{when} \quad \frac{|m|}{2p+1} < |m-k| < \frac{|m|}{2p-1},$$

$m - k \neq 2$, and the quotient is the sum of Lucas numbers;

(ii) the quotient is $\pm L_k$ when $|k| < 2|m|/3$, for Lucas number L_k.

Proof: Let $x = 1$ throughout Theorem 1. Since the magnitudes of Fibonacci numbers are ordered by their subscripts, $\pm F_{m-2p(m-k)}$ represents a remainder (unless $m - k = 2$ since $F_2 = F_1 = 1$).

To illustrate Corollary 1.2, divide F_{13} by F_7:

$$233 = 17 \cdot 13 + 12 = 18 \cdot 13 + (-1).$$

Now, 12 is the remainder in usual division, but we consider the positive and negative remainders with absolute value less than that of the divisor, so that $(-1) = -F_1$ is the remainder of least absolute value. Here, $m = 13$, $k = 6 < 2m/3$, $p = 1$, and the quotient is $L_6 = 18$. The remainders found upon dividing one Fibonacci number by another have been discussed by Taylor [4] and Halton [5].

Corollary 1.3: The Fibonacci number F_q divides F_m if and only if q divides m, $|q| \neq 2$.

Proof: If q divides m, let $x = 1$ in Corollary 1.1. If F_q divides F_m, let $q = m - k$. The remainder of Corollary 1.2 becomes $F_{m-2p(m-k)} = F_0 = 0$ or $F_{m-2p(m-k)} = \pm F_{m-k}$. The algebra on the subscripts follows the proof of Corollary 1.1, which will prove that q divides m, provided that there are no cases of mistaken identity, such as $F_s = F_q$, $|s| \neq |q|$, and such that s does not divide m. Thus, the restriction $|q| \neq 2$ since $F_2 = F_1 = 1$.

Unfortunately, as pointed out by E. A. Parberry, Corollary 1.3 cannot be proved immediately from Corollary 1.1 by simply taking $x = 1$. That F_q

divides F_m does not imply that $F_q(x)$ divides $F_m(x)$, just as that f(1) divides g(1) does not imply that f(x) divides g(x) for arbitrary polynomials f(x) and g(x). Also, Webb and Parberry [8] have proved that a Fibonacci polynomial $F_m(x)$ is irreducible over the integers if and only if m is prime. But, if m is prime, while F_m is not divisible by any other Fibonacci number F_q, $q \geq 3$, F_m is not necessarily a prime. How to determine all values of m for which F_m is prime when m is prime, is an unsolved problem.

Corollary 1.4: There exist an infinite number of sequences $\{S_n\}$ having the division property that, when S_m is divided by S_{m-k}, $m \neq k$, the remainder of least absolute value is always a member of the sequence or the negative of a member of the sequence.

Proof: We can let x be any integer in the Lemma and throughout Theorem 1. If x = 2, one such sequence is ..., 0, 1, 2, 5, 12, 29, 70, 169, ...

Theorem 2: Whenever a Lucas polynomial $L_m(x)$ is divided by a Lucas polynomial $L_{m-k}(x)$, $m \neq k$, of lesser degree, a non-zero remainder is always a Lucas polynomial or the negative of a Lucas polynomial. Explicitly,

(i) non-zero remainders have the form

$$\pm L_{(2p-1)m-2pk}(x) \qquad \text{when} \qquad \frac{2p|m|}{2p+1} > |k| > \frac{(2p-2)|m|}{2p-1},$$

or, equivalently,

$$\pm L_{2p(m-k)-m}(x) \qquad \text{for} \qquad \frac{|m|}{2p+1} < |m-k| < \frac{|m|}{2p-1} \quad ;$$

(ii) if $|k| < 2|m|/3$, the quotient is $\pm L_k(x)$;

(iii) the division is exact when $k = 2pm/(2p+1)$, $p \neq 0$.

Proof: Since the proof parallels that of the Lemma and Theorem 1, details are omitted. Identity (3.4) is used to establish

$$(4.2) \qquad L_m(x) = L_k(x)L_{m-k}(x) + (-1)^{k+1}L_{m-2k}(x) .$$

Since $L_{-n}(x) = (-1)^n L_n(x)$, it can be proved that

$$L_m(x) = Q_p(x)L_{m-k}(x) \pm L_{(2p-1)m-2pk}(x) ,$$

for $|m| \geq |m - k| \geq |(2p - 1)m - 2pk|$. Since the degree of $L_n(x)$ is $|n|$, the rest of the proof is similar to that of Theorem 1. However, notice that it is necessary to both proofs that $F_{-n}(x) = \pm F_n(x)$ and $L_{-n}(x) = \pm L_n(x)$.

Corollary 2.1: The Lucas polynomial $L_q(x)$ divides $L_m(x)$ if and only if m is an odd multiple of q .

Proof: If $m = (2p + 1)q$, let $q = m - k$ and Theorem 2 guarantees that $L_q(x)$ divides $L_m(x)$.

If $L_q(x)$ divides $L_m(x)$, then let $q = m - k$. For the division to be exact, the term $\pm L_{(2p-1)m-2pk}(x)$ must equal $L_{m-k}(x)$ since it cannot be the zero polynomial. Then, either $k = 2pm/(2p + 1)$ or $k = 2pm/(2p - 1)$, so $q = m - k = m/(2p + 1)$ or $q = m/(2p - 1)$. In either case, m is an odd multiple of q .

Corollary 2.2: If a Lucas number L_m is divided by L_{m-k}, then the non-zero remainder of least absolute value is always a Lucas number or its negative with the form

$$\pm L_{2p(m-k)-m} \qquad \text{for} \qquad \frac{|m|}{2p + 1} < |m - k| < \frac{|m|}{2p - 1} \ ,$$

and the quotient is $\pm L_k$ when $|k| < 2|m|/3$.

Proof: Let $x = 1$ throughout the development of Theorem 2.

Corollary 2.3: The Lucas number L_q divides L_m if and only if $m = (2s + 1)q$ for some integer s. (This result is due to Carlitz [6]).

Proof: If $m = (2s + 1)q$, let $x = 1$ in Corollary 2.1. If L_q divides L_m, take $q = m - k$ and examine the remainder $L_{2p(m-k)-m}$ of Corollary 2.2 which must equal L_{m-k} or L_{k-m} since it cannot be zero. The algebra follows that given in Corollary 2.1. Since there are no Lucas numbers such that $L_q = L_s$ where $|q| \neq |s|$, and since $L_q \neq 0$ for any q , there are no restrictions.

Since the generalized Fibonacci polynomials $H_m(x)$ satisfy Equation (2.5), $H_m(x) = bxF_{m-1}(x) + aF_{m-2}(x)$, we can show that

$$H_m(x) = L_k(x)H_{m-k}(x) + (-1)^{k+1}H_{m-2k}(x) \ , \tag{4.3}$$

but since $H_m(x) \neq \pm H_{-m}(x)$, we have a more limited theorem.

Theorem 3: Whenever a generalized Fibonacci polynomial $H_m(x)$ is divided by $H_{m-k}(x)$, $2m/3 > k > 0$, any non-zero remainder is always another generalized Fibonacci polynomial or its negative, and the quotient is $L_k(x)$.

As a consequence of Theorem 3, when a generalized Fibonacci number H_m is divided by H_q, a non-zero remainder of least absolute value is guaranteed to be another generalized Fibonacci number only when $|m - q| < 2m/3$. Taylor [4] has proved that, of all generalized Fibonacci sequences $\{H_m\}$ satisfying the recurrence $H_m = H_{m-1} + H_{m-2}$, the only sequences with the division property that the non-zero remainders of least absolute value are always a member of the sequence or the negative of a member of the sequence, are the Fibonacci and Lucas sequences. For your further reading, Hoggatt [7] gives a lucid description of divisibility properties of Fibonacci and Lucas numbers.

REFERENCES

1. M. N. S. Swamy, "Further Properties of Morgan-Voyce Polynomials," The Fibonacci Quarterly, Vol. 6, No. 2, April, 1968, pp. 167-175.
2. Richard A. Hayes, Fibonacci and Lucas Polynomials, Unpublished Master's Thesis, San Jose State College, January, 1965, pp. 36-39.
3. S. L. Basin and Verner E. Hoggatt, Jr., "A Primer on the Fibonacci Sequence: Part II," Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 61-68.
4. Laurence Taylor, "Residues of Fibonacci-Like Sequences," Fibonacci Quarterly, Vol. 5, No. 3, October, 1967, pp. 298-304.
5. J. H. Halton, "On Fibonacci Residues," Fibonacci Quarterly, Vol. 2, No. 3, October, 1964, pp. 217-218.
6. Leonard Carlitz, "A Note on Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 1, February, 1964, pp. 15-28.
7. Verner E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton-Mifflin Mathematics Enrichment Series, Houghton-Mifflin, Boston, 1969, Chapters 7 and 8.
8. W. A. Webb and E. A. Parberry, "Divisibility Properties of Fibonacci Polynomials," Fibonacci Quarterly, Vol. 7, No. 5, December, 1969, pp. 457-463.