FIBONACCI MATRICES AND LAMBDA FUNCTIONS

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When we speak of a Fibonacci matrix, we shall have in mind matrices which contain members of the Fibonacci sequence as elements. An example of a Fibonacci matrix is the Q matrix as defined by King in [1], where

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The determinant of Q is -1, written det Q = -1. From a theorem in matrix theory, $\det Q^n = (\det Q)^n = (-1)^n$. By mathematical induction, it can be shown that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

so that we have the familiar Fibonacci identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ by finding $\det Q^n$.

The lambda function of a matrix was studied extensively in [2] by Fenton S. Stancliff, who was a professional musician. Stancliff defined the lambda function $\lambda(M)$ of a matrix M as the change in the value of the determinant of M when the number one is added to each element of M. If we define (M + k) to be that matrix formed from M by adding any given number k to each element of M, we have the identity

$$(1) \qquad \det (M + k) = \det M + k\lambda(M).$$

For an example, the determinant $\lambda(Q^n)$ is given by

$$\lambda(Q^n) = \begin{vmatrix} F_{n+1} + 1 & F_n + 1 \\ F_n + 1 & F_{n-1} + 1 \end{vmatrix} - \det Q^n$$

$$= (F_{n+1}F_{n-1} - F_n^2) + (F_{n-1} + F_{n+1} - 2F_n) - \det Q^n$$

$$= F_{n-3}$$

which follows by use of Fibonacci identities. Now if we add k to each element of Q^n, the resulting determinant is

$$\begin{vmatrix} F_{n+1} + k & F_n + k \\ F_n + k & F_{n-1} + k \end{vmatrix} = \det Q^n + k\, F_{n-3}.$$

However, there are more convenient ways to evaluate the lambda function. For simplicity, we consider only 3 x 3 matrices.

THEOREM. For the given general 3 x 3 matrix M, $\lambda(M)$ is expressed by either of the expressions (2) or (3). For

$$(2) \qquad M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}, \qquad \lambda(M) = \begin{vmatrix} a + e - b - d & b + f - c - e \\ d + h - g - e & e + j - h - f \end{vmatrix},$$

$$(3) \qquad \lambda(M) = \begin{vmatrix} 1 & b & c \\ 1 & e & f \\ 1 & h & j \end{vmatrix} + \begin{vmatrix} a & 1 & c \\ d & 1 & f \\ g & 1 & j \end{vmatrix} + \begin{vmatrix} a & b & 1 \\ d & e & 1 \\ g & h & 1 \end{vmatrix}$$

Proof: This is made by direct evaluation and a simple exercise in algebra.

An application of the lambda function is in the evaluation of determinants. Whenever there is an obvious value of k such that det (M + k) is easy to find, we can use equation (1) advantageously. To illustrate this fact, consider

$$M = \begin{pmatrix} 1000 & 998 & 554 \\ 990 & 988 & 554 \\ 675 & 553 & 554 \end{pmatrix}.$$

We notice that, if we add k = -554 to each element of M, then det (M + k) = 0 since every element in the third column will be zero. From (2) we compute

$$\lambda(M) = \begin{vmatrix} 0 & 10 \\ -120 & 435 \end{vmatrix} = 1200;$$

and from (1) we find that 0 = det M + (-554)(1200), so that det M = (554)(1200).

Readers who enjoy mathematical curiosities can create determinants which are not changed in value when any given number k is added to each element, by writing any matrix D such that $\lambda(D) = 0$.

LEMMA: If two rows (or columns) of a matrix D have a constant difference between corresponding elements, then $\lambda(D) = 0$.

Proof: Evaluate $\lambda(D)$ directly, by (2) or (3).

For example, we write the matrix D, where corresponding elements in the first and second rows differ by 4, such that

$$\det D = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 4 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 1 + k & 2 + k & 3 + k \\ 5 + k & 6 + k & 7 + k \\ 4 + k & 9 + k & 8 + k \end{vmatrix} = 24.$$

Now, we consider other Fibonacci matrices. Suppose that we want to write a Fibonacci matrix U such that det U = F_n. We can write $F_n = F_1F_2F_n$ for any n, and for some n we will also have other Fibonacci factorizations. Hence, for

$$U = \begin{pmatrix} F_1 & F_0 & F_0 \\ F_m & F_2 & F_0 \\ F_k & F_p & F_n \end{pmatrix},$$

det U = F_n where $F_0 = 0$. If we choose m = k = 3 and p = 2, we find that $\lambda(U) = 0$. If we choose m = 1 or 2, k = 1 or 2, and let p be an arbitrary integer, then $\lambda(U) = F_n$.

A more elegant way to write such a matrix was suggested by Ginsburg in [3], who wrote a matrix with the same first two columns as U below but with all elements in the third column equal to n and thus with determinant value n. We can write F_m = det U, where

$$U = \begin{pmatrix} F_{2p} & F_{2p+1} & F_m \\ F_{2p+1} & F_{2p+2} & F_m \\ F_{2p+2} & F_{2p+3} & F_m \end{pmatrix}$$

We have, using equation (3),

$$\lambda(U) = \begin{vmatrix} 1 & F_{2p+1} & F_m \\ 1 & F_{2p+2} & F_m \\ 1 & F_{2p+3} & F_m \end{vmatrix} + \begin{vmatrix} F_{2p} & 1 & F_m \\ F_{2p+1} & 1 & F_m \\ F_{2p+2} & 1 & F_m \end{vmatrix} + \begin{vmatrix} F_{2p} & F_{2p+1} & 1 \\ F_{2p+1} & F_{2p+2} & 1 \\ F_{2p+2} & F_{2p+3} & 1 \end{vmatrix}$$

$$= 0 + 0 + 1 = 1$$

If we let k = F_{m-1}, from (1) we see that det $(U + F_{m-1}) = F_m + (F_{m-1})(1) = F_{m+1}$.

Notice the possibilities for finding Fibonacci identities using the lambda function and evaluation of determinants. As a brief example, we let k = F_n and consider det $(Q^n + F_n)$, which gives us

$$\begin{vmatrix} F_{n+1} + F_n & F_n + F_n \\ F_n + F_n & F_{n-1} + F_n \end{vmatrix} = \det Q^n + F_n\,\lambda(Q^n)$$

or

$$\begin{vmatrix} F_{n+2} & 2F_n \\ 2F_n & F_{n+1} \end{vmatrix} = (-1)^n + F_n F_{n-3}$$

so that

$$4F_n^2 = F_{n+2}F_{n+1} - F_nF_{n-3} + (-1)^{n+1}.$$

As a final example of a Fibonacci matrix, we take the matrix R, given by

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

which has been considered by Brennan [4]. It can be shown that

$$R^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix}$$

by mathematical induction. The reader may verify that by equation (2) and by Fibonacci identities,

$$\lambda(R^n) = (-1)^n(F_{n-1}^2 - F_{n-3}F_{n-2}),$$

the center element of R^{n-2} multiplied by $(-1)^n$.

REFERENCES

1. Charles H. King, "Some Properties of the Fibonacci Numbers", (Master's Thesis), San Jose State College, June, 1960, pp. 11-27.
2. From the unpublished notes of Fenton S. Stancliff.
3. Jekuthiel Ginsburg, "Determinants of a Given Value", Scripta Mathematica, Vol. 18, Issues 3-4, Sept.-Dec., 1952, p. 219.
4. From the unpublished notes of Terry Brennan.

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Problem B-24 (Proposed by Brother Alfred Brousseau): It is evident that the determinant below has a value of zero. Prove that if the same quantity k is added to each element, the value becomes $(-1)^{n-1}k$.

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+2} & F_{n+3} & F_{n+4} \end{vmatrix}$$