A PRIMER ON THE FIBONACCI SEQUENCE: PART II

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A MATRIX WHICH GENERATES FIBONACCI IDENTITIES

The proofs of existing Fibonacci identities and the discovery of new identities can be greatly simplified if matrix algebra and a particular 2 x 2 matrix are introduced. The matrix approach to the study of recurring sequences has been used for some time [1] and the Q matrix appeared in a thesis by C. H. King [2]. We first present the basic tools of matrix algebra.

1. THE ALGEBRA OF (TWO-BY-TWO) MATRICES

The two-by-two matrix A is an array of four elements a, b, c, d:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The zero matrix Z and the identity matrix I are defined as

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix C, which is the <u>matrix sum</u> of two matrices A and B, is

$$C = A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a + e & b + f \\ c + g & d + h \end{pmatrix}$$

The matrix P, which is the <u>matrix product</u> of two matrices A and B, is

$$P = AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

The <u>determinant D(A)</u> of matrix A is

$$D(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Two matrices are equal if and only if the corresponding elements are equal. That is, for the matrices A and B above, A = B if and only if a = e, b = f, c = g, and d = h.

The proof of the following simple theorem is left as an exercise in algebra.

THEOREM: The determinant D(P) of the product P = AB of two matrices A and B is the product of the determinants D(A) and D(B). That is, D(P) = D(AB) = D(A)·D(B).

2. THE Q MATRIX

The Q matrix and the determinant of Q are

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D(Q) = -1.$$

If we designate $Q^0 = I$, the identity matrix, then $Q = Q^1 = Q^0Q = IQ = QI = QQ^0$.

Definition: $Q^{n+1} = Q^nQ^1$, an inductive definition where $Q^1 = Q$. This provides the law of exponents for matrices.

It is easily proved by mathematical induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

where F_n is the nth Fibonacci number, and the determinant of Q^n is $(-1)^n$.

3. MORE PROOFS

We may now prove several of the identities very nicely. Let us prove identity (3) from Part I:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

Proof: Evaluate the determinant of Q^n in two ways. $D(Q^n) = D^n(Q) = (-1)^n$, but by definition of determinant, $D(Q^n) = F_{n+1}F_{n-1} - F_n^2$.

Now let us prove identity (7), $F_{2n+1} = F_{n+1}^2 + F_n^2$. Since $Q^{n+1}Q^n = Q^{2n+1}$,

$$Q^nQ^{n+1} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} = \begin{pmatrix} F_{n+1}F_{n+2} + F_nF_{n+1} & F_{n+1}^2 + F_n^2 \\ F_nF_{n+2} + F_{n-1}F_{n+1} & F_nF_{n+1} + F_{n-1}F_n \end{pmatrix}$$

can also be written as

$$Q^{2n+1} = \begin{pmatrix} F_{2n+2} & F_{2n+1} \\ F_{2n+1} & F_{2n} \end{pmatrix}$$

Since these two matrices are equal, we may equate corresponding elements, so that

$$F_{2n+2} = F_{n+1}F_{n+2} + F_nF_{n+1} \qquad \text{(Upper Left)}$$

(7) $$F_{2n+1} = F_{n+1}^2 + F_n^2 \qquad \text{(Upper Right)}$$

$$F_{2n+1} = F_n F_{n+2} + F_{n-1}F_{n+1} \qquad \text{(Lower Left)}$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n \qquad \text{(Lower Right)}$$

establishing identity (7) as well as two others with some simple algebra. If we accept identity (5), $L_n = F_{n+1} + F_{n-1}$, then

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n = F_n(F_{n+1} + F_{n-1}) = F_nL_n,$$

which gives identity (9). From $F_{k+2} = F_{k+1} + F_k$, for $k = n - 1$, one can write $F_n = F_{n+1} - F_{n-1}$, so that we also have identity (8):

(8) $$F_{2n} = F_n(F_{n+1} + F_{n-1}) = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) = F_{n+1}^2 - F_{n-1}^2$$

It is a simple task to verify that $Q^2 = Q + I$, leading to

$$Q^{n+2} = Q^{n+1} + Q^n \qquad \text{and} \qquad Q^n = Q\cdot F_n + I\cdot F_{n-1},$$

where F_n is the nth Fibonacci number and the multiplication of matrix A, by a number q, is defined by

$$qA = q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aq & bq \\ cq & dq \end{pmatrix}$$

4. GENERATION OF FIBONACCI NUMBERS BY LONG DIVISION

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^2 + \ldots + F_nx^{n-1} + \ldots$$

In the process of long division below

$$1 - x - x^2 \overline{)\quad 1 \qquad\qquad}$$

there is no ending. As far as you care to go the process will yield Fibonacci numbers as the coefficients.

5. F_n AS A FUNCTION OF ITS SUBSCRIPT

It is not difficult to show by mathematical induction that

$$P(n): \qquad F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

This can be derived in many ways. P(1) and P(2) are clearly true. From $F_k = F_{k-1} + F_{k-2}$ and the inductive assumption that P(k-2) and P(k-1) are true,

$$P(k-2): \qquad F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right\}$$

$$P(k-1): \qquad F_{k-1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right\}$$

Adding, after a simple algebra step, we get

$$F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\left(\frac{1+\sqrt{5}}{2} + 1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\left(\frac{1-\sqrt{5}}{2} + 1\right)\right\}$$

Observing that

$$\frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 \text{ and } \frac{1-\sqrt{5}}{2} + 1 = \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2$$

it follows simply that if P(k-2) and P(k-1) are true, then for n = k,

$$P(k): \qquad F_k = F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right\}$$

making the proof complete by mathematical induction.

Similarly, it may be shown that

$$L_n = F_{n+1} + F_{n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The formulas given ior F_n and L_n in terms of the subscripts are called the <u>Binét forms</u> for F_n and L_n.

Now let us use the Binét forms for F_n and L_n to prove identity (9), $F_{2n} = F_nL_n$, a second way:

$$F_{2n} = \frac{1}{\sqrt{5}}\left\{\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\right]^2 - \left[\left(\frac{1-\sqrt{5}}{2}\right)^n\right]^2\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} = F_nL_n.$$

6. MORE IDENTITIES

(15) $$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

(16) $$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

(17) $$F_1^3 + F_2^3 + F_3^3 + \dots + F_n^3 = \frac{F_{3n+2} + (-1)^{n+1}6\cdot F_{n-1} + 5}{10}$$

(18) $$1\cdot F_1 + 2\cdot F_2 + 3\cdot F_3 + \dots + n\cdot F_n = (n+1)F_{n+2} - F_{n+4} + 2$$

(19) $$F_2 + F_4 + \dots + F_{2n} = F_{2n+1} - 1$$

(20) $$F_1F_2 + F_2F_3 + F_3F_4 + \dots + F_{n-1}F_n = \frac{1}{2}(F_{2n-1} + F_nF_{n-1} - 1)$$

(21) $$\sum_{i=0}^{n}\binom{n}{i}F_{n-i} = F_{2n}, \text{ where } \binom{n}{i} = \frac{n!}{(n-i)!i!}, \quad m! = 1\cdot 2\cdot 3\cdot \dots \cdot m.$$

(22) $$F_{3n+3} = F_{n+1}^3 + F_{n+2}^3 - F_n^3$$

(23) $$F_nF_m - F_{n-k}F_{m+k} = (-1)^{n-k}F_kF_{m+k-n}$$

(24) $$F_{n+4}^3 = 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3$$

REFERENCES

1. J. S. Frame, "Continued Fractions and Matrices", American Mathematical Monthly, February, 1949, p. 98.
2. Charles H. King, "Some Properties of the Fibonacci Numbers", Unpublished Master's Thesis, San Jose State College, June, 1960.