LINEAR RECURSION RELATIONS — LESSON SIX
COMBINING LINEAR RECURSION RELATIONS

Suppose we have two sequences $P_i(1, 5, 25, 125, 625, 3125, \cdots)$ with a recursion relation

(1) $$P_{n+1} = 5P_n$$

and $Q_i(3, 10, 13, 23, 36, 59, \cdots)$, a Fibonacci sequence with recursion relation:

(2) $$Q_{n+1} = Q_n + Q_{n-1} .$$

Let

(3) $$T_n = P_n + Q_n .$$

What is the recursion relation of T_n and how can it be conveniently obtained from the recursion relations of P_n and Q_n?

Proceeding in a straightforward manner, we could first eliminate P_n as follows:

$$T_{n+1} = P_{n+1} = Q_{n+1}$$

$$5T_n = 5P_n + 5Q_n .$$

Subtracting and using relation (1),

$$T_{n+1} - 5T_n = Q_{n+1} - 5Q_n .$$

We can proceed likewise for Q. Thus

$$T_{n+1} - 5T_n = Q_{n+1} - 5Q_n$$
$$T_n - 5T_{n-1} = Q_n - 5Q_{n-1}$$
$$T_{n-1} - 5T_{n-2} = Q_{n-1} - 5Q_{n-2} \,.$$

Now subtract the sum of the last two equations from the first and use relation (2). The result is:

$$T_{n+1} - 6T_n + 4T_{n-1} + 5T_{n-2} = 0 \,,$$

a recursion relation involving only T_i.

A much simpler approach is by means of an operator E, such that

(3) $$(E)\,T_n = T_{n+1} \,.$$

The effect of E is to increase the subscript by 1. A relation

$$P_{n+1} - 5P_n = 0 \,,$$

can be written

$$(E - 5)P_n = 0 \,,$$

and a relation

$$Q_{n+1} - Q_n - Q_{n-1} = 0 \,,$$

can be written

$$(E^2 - E - 1)Q_{n-1} = 0 \,.$$

It is not difficult to convince onself that these operators obey the usual algebraic laws. As a result, if

$$T_n = P_n + Q_n \,,$$

$(E - 5)(E^2 - E - 1)T_n = (E - 5)(E^2 - E - 1)P_n + (E - 5)(E^2 - E - 1)Q_n$. But $(E - 5)P_n = 0$ and $(E^2 - E - 1)Q_n = 0$, so that

$$(E - 5)(E^2 - E - 1)T_n = 0$$

or

$$(E^3 - 6E^2 + 4E + 5)T_n = 0 ,$$

which is equivalent to the recursion relation

$$T_{n+3} = 6T_{n+2} - 4T_{n+1} - 5T_n .$$

In general, if we have linear operators such that:

$$f(E)P_n = 0 \quad \text{and} \quad g(E)Q_n = 0 \quad \text{and} \quad T_n = AP_n + BQ_n,$$

where A and B are constants, then

$$f(E)g(E)T_n = Af(E)g(E)P_n + Bf(E)g(E)Q_n = 0 ,$$

since $f(E)P_n = 0$, and $g(E)Q_n = 0$. Thus when T_n is the sum of terms of two sequences with different recursion relations, the recursion relation for T_n is found by multiplying T_n by the two recursion operators for the two sequences.

Example. What is the recursion relation for $T_n = 2 \times 5^n + n^2 - n + 4$? The recursion relation for 2×5^n is $(E - 5)P_n = 0$, and that for $n^2 - 4 + 4$ is $(E^3 - 3E^2 + 3E - 1)Q_n = 0$. Thus the recursion relation for the given sequence is

$$(E - 5)(E^3 - 3E^2 + 3E - 1)T_n = 0,$$

which is equivalent to:

$$T_{n+4} = 8T_{n+3} - 18T_{n+2} + 16T_{n+1} - 5T_n .$$

Example. Find the recursion relation corresponding to T_n if

$P_{n+1} = P_n + P_{n-1} + P_{n-2}$ and $Q_n = 3n^2 - 4n + 5$ and $T_n = P_n + Q_n$.

The operator expressions for these recursion relations are:

$$(E^3 - E^2 - E - 1)P_{n-2} = 0 \quad \text{and} \quad (E^3 - 3E^2 + 3E - 1)Q_{n-2} = 0 .$$

Thus the recursion relation for T_n is:

$$(E^3 - E^2 - E - 1)(E^3 - 3E^2 + 3E - 1)T_n = 0 ,$$

which is equivalent to

$$T_{n+6} = 4T_{n+5} - 5T_{n+4} + 2T_{n+3} - T_{n+2} + 2T_{n+1} - T_n .$$

It may be noted that two apparently different recursion relations may conceal the fact that they embody partly identical recursion relations. For example, if

$$\begin{aligned} P_n &= 4P_{n-1} - 3P_{n-2} - 2P_{n-3} + P_{n-4} \\ Q_n &= 3Q_{n-1} - 2Q_{n-2} - Q_{n-3} + Q_{n-4} \end{aligned} ,$$

and we proceed directly to find the recursion operator and corresponding recursion relation for $T_n = P_n + Q_n$, we arrive at a recursion relation of order eight. However, in factored form, we have:

$$(E^2 - E - 1)(E^2 - 3E + 1)P_{n-4} = 0 ,$$

and

$$(E^2 - E - 1)(E^2 - 2E + 1)Q_{n-4} = 0 .$$

The recursion relation for T_n in simpler form would thus be:

$$(E^2 - E - 1)(E^2 - 3E + 1)(E^2 - 2E + 1)T_n = 0,$$

which is only of order six.

If the terms of the two sequences are given explicitly, a slightly different but equivalent procedure using the auxiliary equation is possible. Thus if

$$P_n = 5n + 2 + 2 \times 3^n + F_n$$
$$Q_n = n^2 - 3n + 5 - 6 \times 2^n + L_n,$$

the roots of the auxiliary equation for P_n are 1, 1, 3, r, and s, while those of the auxiliary equation for Q_n are 1, 1, 1, 2, r, s. Hence the roots for the auxiliary equation of T_n would be 1, 1, 1, 2, 3, r, s, where r and s are the roots of the equation $x^2 - x - 1 = 0$. Thus the auxiliary equation for T_n would be:

$$(x - 1)^3(x - 2)(x^2 - x - 1) = 0$$

which leads equivalently to the recursion relation

$$T_{n+7} = 9T_{n+6} - 31T_{n+5} + 50T_{n+4} - 33T_{n+3} - 5T_{n+2} + 17T_{n+1} - 6T_n .$$

PROBLEMS

1. If P_n is the geometric progression 3, 15, 75, 375, 1875, $\cdots$ and

$$Q_n = 5F_n + 2(-1)^n,$$

what is the recursion relation for $T_n = P_n + Q_n$?

2. Given recursion relations

$$P_{n+1} = 4P_n - P_{n-1} - 6P_{n-2} \qquad Q_{n+1} = 6Q_n = 10Q_{n-1} + Q_{n-2} + 6Q_{n-3},$$

with $T_{n+1} = P_{n+1} + Q_{n+1}$, determine the recursion relation of lowest order satisfied by T_{n+1}.

3. Determine the recursion relation for $T_n = P_n + Q_n$ where P_n is the arithmetic progression 3, 7, 11, 15, 19, ⋯ and Q_n is the geometric progression 2, 6, 18, 54, ⋯ .

4. Determine the recursion relation for $T_n = 2^n + F_n^2$ given that the recursion relation for F_n^2 is

$$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \; .$$

5. Determine the recursion relation for

$$T_n = 5L_n^2 + (-1)^{n-1} + 4F_n \; .$$

6. If $P_n = 3^n + 2n - 4$ and $Q_n = 2^n - 3n + 2$, find the linear recursion relation for $4P_n + 5Q_n$.

7. Given the sequences 1, -1, -2, 2, 1, -1, -2, 2, ⋯, and 1, 3, -1, -3, 1, 3, -1, -3, ⋯ find the linear recursion relation for the sum of the sequences and an explicit expression for the n^{th} term in terms of the roots of the auxiliary equation.

8. If $P_n = L_n + 2n - 3$ and $Q_n = F_n + n^2$, find the linear recursion relation for the sum $P_n + Q_n$.

9. $P_n = 2 \times 3^n + 5n + 4$ and $Q_n = F_n + 2n - 3$. Find the linear recursion relation for the sum $P_n + Q_n$.

10. If $P_n = 2^n + F_n$ and $Q_n = 3^n + V_n$, determine the linear recursion relation for the sum $P_n + Q_n$.

$$V_1 = 1, \quad V_2 = 3, \quad V_{m+1} = 3V_m + V_{m-1} \; .$$

(See page 58 for solutions to problems.)