LINEAR RECURSION RELATIONS – LESSON FOUR
SECOND-ORDER LINEAR RECURSION RELATIONS

Given a second-order linear recursion relation

(1) $$T_{n+1} = a\,T_n + b\,T_{n-1} ,$$

where a and b are real numbers and the values T_i of the sequence are real as well, there is an auxiliary equation:

(2) $$x^2 - ax - b = 0 ,$$

with roots

(3) $$r = \frac{a + \sqrt{a^2 + 4b}}{2}$$
$$s = \frac{a - \sqrt{a^2 + 4b}}{2} .$$

As is usual with quadratic equations, three cases may arise depending on whether

(4)
$a^2 + 4b > 0$, roots real and distinct;
$a^2 + 4b = 0$, roots real and equal;
$a^2 + 4b < 0$, roots complex numbers.

CASE 1. $a^2 + 4b > 0$.

In previous lessons we have considered cases of this kind. It has been noted that the roots may be rational or irrational. There seems to be nothing to add for the moment to the discussion of these cases.

CASE 2. $a^2 + 4b = 0$.

The presence of multiple roots in the auxiliary equation clearly requires some modification in the previous development. If

$$x^2 - ax - b = 0$$

$$x^n - ax^{n-1} - bx^{n-2} = 0 \ .$$

Since the equation has a multiple root $(a/2)$, the derivative of this equation will have this same root. Hence

(5) $$nx^{n-1} - a(n-1)x^{n-2} - b(n-2)x^{n-3} = 0$$

is satisfied by the multiple root also.

Thus the multiple root, r, satisfies the following two relations:

(6)
$$r^n = ar^{n-1} + br^{n-2}$$

$$nr^n = a(n-1)r^{n-1} + b(n-2)r^{n-2} \ .$$

The result is that if we formulate T_n as

(7)
$$T_n = A \qquad {}^{n} + B r^n$$

$$T_{n+1} = A(n+1)r^{n+1} + Br^{n+1}$$

it follows that

(8)
$$T_{n+2} = a T_{n+1} + b T_n$$

$$= A[a(n+1)r^{n+1} + bn r^n] + B[ar^{n+1} + br^n]$$

$$= A(n+2)r^{n+2} + B r^{n+2}$$

so that the form of T is maintained.

EXAMPLE

Find the expression for T_n in terms of the roots of the auxiliary equation corresponding to the linear recursion relation

$$T_{n+1} = 6T_n - 9T_{n-1}$$

if $T_1 = 4$, $T_2 = 7$. Here the auxiliary equation is $x^2 - 6x + 9 = 0$ with a double root of 3. Hence T_n has the form

$$T_n = Anx3^n + Bx3^n .$$

Using the values of T_1 and T_2

$$4 = Ax3 + Bx3$$

$$7 = 2Ax3^2 + Bx3^2$$

with solutions $A = -5/9$, $B = 17/9$. Hence

$$T_n = \frac{-5nx3^n + 17x3^n}{9} = 3^{n-2}[(-5n + 17)].$$

It may be noted that for any non-zero multiple root r, the determinant of the coefficients in the set of equations for T_1 and T_2 is

$$\begin{vmatrix} r & r \\ 2r^2 & r^2 \end{vmatrix} = -r^3$$

which is not zero, so that these equations will always have a solution.

CASE 3. $a^2 + 4b < 0$.

The case of complex roots is quite similar to that of real and distinct roots as far as determining coefficients from initial value equations is concerned. However, since we have specified that the terms of the sequence and the coefficients in the recursion relation are real, there will have to be a special relation between A and B in the expression for T_n:

$$T_n = A r^n + B s^n .$$

Note that r and s are complex conjugates, so that r^n and s^n are of the form $P + Qi$ and $P - Qi$ respectively, where P and Q are real. If T_n is to be real, then A and B must be complex conjugates as well.

EXAMPLE

Find the expression for T_n in terms of the roots of the auxiliary equation for the linear recursion relation

$$T_{n+1} = 3 T_n - 4 T_{n-1} ,$$

with $T_1 = 5$, $T_2 = 9$. Here the auxiliary equation is:

$$x^2 - 3x + 4 = 0$$

with roots

$$r = \frac{3 + i\sqrt{7}}{2} , \quad s = \frac{3 - i\sqrt{7}}{2} .$$

Then

$$5 = Ar + Bs$$

$$9 = A r^2 + B s^2$$

from which one finds that

$$A = \frac{21 - 11i\sqrt{7}}{28} , \quad B = \frac{21 + 11i\sqrt{7}}{28} .$$

Accordingly,

$$T_n = \left(\frac{21 - 11i\sqrt{7}}{28}\right) r^n + \left(\frac{21 + 11i\sqrt{7}}{28}\right) s^n .$$

AN ANALOGUE

Because of the similarities among second-order linear recursion relations it is possible to find close analogues among them to the Fibonacci and Lucas sequences. Let us consider as an example the second-order linear recursion relation

$$T_{n+1} = 3\,T_n + T_{n-1} .$$

The auxiliary equation is

$$x^2 - 3x - 1 = 0$$

with roots

$$r = \frac{1 + \sqrt{13}}{2}, \quad s = \frac{1 - \sqrt{13}}{2} .$$

If the initial terms are taken as $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, then

$$1 = Ar + Bs$$

$$3 = Ar^2 + Bs^2 ,$$

with resulting values $A = 1/\sqrt{13}$ and $B = -1/\sqrt{13}$ so that

$$T_n = \frac{r^n - s^n}{\sqrt{13}} = \frac{r^n - s^n}{r - s}$$

has precisely the same form as the expression for F_n with 13 replacing 5 under the square root sign.

If the relation $V_n = T_{n+1} + T_{n-1}$ is used to define the corresponding "Lucas" sequence, the terms of this sequence are:

$$V_0 = 2, \; V_1 = 3, \; V_2 = 11, \; V_3 = 36, \cdots .$$

Solving for A and B from

$$3 = Ar + Bs$$

$$11 = Ar^2 + Bs^2$$

gives values of $A = 1$, $B = 1$, so that

$$V_n = r^n + s^n$$

in perfect correspondence to the expression for the Lucas sequence. As a result of this similarity, many relations in the Fibonacci-Lucas complex can be taken over (sometimes with the slight modification of replacing 5 by 13) to this pair of sequences. Thus:

$$T_{2n} = T_n V_n$$

$$T_{2n+1} = T_n^2 + T_{n+1}^2$$

$$T_{n+1}T_{n-1} - T_n^2 = (-1)^{n-1}$$

$$V_{2n} = V_n^2 + 2(-1)^{n+1}$$

$$V_n + V_{n+2} = 13T_{n+1}$$

$$V_n^2 + V_{n+1}^2 = 13(T_n^2 + T_{n+1}^2) .$$

PROBLEMS

1. For the sequence $T_1 = 1$, $T_2 = 3$, obeying the linear recursion relation

$$T_{n+1} = 3T_n + T_{n-1}$$

show that every integer divides an infinity of members of the sequence.

2. For the corresponding "Lucas" sequence, prove that if m divides n, where n is odd, then V_m divides V_n.

3. Find the expression for the sequence $T_1 = 2$, $T_2 = 5$ in terms of the roots of the auxiliary equation corresponding to the linear recursion relation

$$T_{n+1} = 4T_n + 4T_{n-1}.$$

4. Prove that the second-order linear recursion relation

$$T_{n+1} = 2T_n - T_{n-1}$$

defines an arithmetic progression.

5. If $T_1 = a$, $T_2 = b$, find the expression for T_n in terms of the roots of the auxiliary equation corresponding to $T_{n+1} = 4T_n - 4T_{n-1}$.

6. If $T_1 = i$, $T_2 = 1$ and $T_{n+1} = -T_{n-1}$, find the general expression for T_n in terms of the roots of the auxiliary equation.

7. $T_1 = 3$, $T_2 = 7$, $T_3 = 17$, $T_4 = 43$, $T_5 = 113, \cdots$ are terms of a second-order linear recursion relation. Find this relation and express T_n in terms of the roots of the auxiliary equation.

8. For the second-order linear recursion relation $T_{n+1} = 5T_n + T_{n-1}$ find the particular sequences analogous to the Fibonacci and Lucas sequences and express their terms as functions of the roots of the auxiliary equation.

9. For $T_1 = 5$, $T_2 = 9$, $T_{n+1} = 3T_n - 5T_{n-1}$, find T_n in terms of the roots of the auxiliary equation.

10. If

$$T_n = \left(\frac{-66 + 13\sqrt{33}}{33}\right)\left(\frac{5 + \sqrt{33}}{2}\right)^n + \left(\frac{-66 - 13\sqrt{33}}{33}\right)\left(\frac{5 - \sqrt{33}}{2}\right)^n$$

determine the recursion relation obeyed by T_n and find T_1 and T_2.

(See page 56 for solutions to problems.)