LINEAR RECURSION RELATIONS
LESSON THREE—THE BINET FORMULAS

In the previous lesson, the technique of relating the terms of a linear recursion relation to the roots of an auxiliary equation was studied and illustrated. The Fibonacci sequences are characterized by the recursion relation:

(1) $$T_{n+1} = T_n + T_{n-1} ,$$

which is a linear recursion relation of the second order having an auxiliary equation:

(2) $$x^2 = x + 1$$

or

(3) $$x^2 - x - 1 = 0 .$$

The roots of this equation are:

(4) $$r = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad s = \frac{1 - \sqrt{5}}{2}$$

From the theory of the relation of roots to coefficients or by direct calculation it can be ascertained that:

(5) $$r + s = 1 \quad \text{and} \quad rs = -1 .$$

It follows from what has been developed in the previous lesson that the terms of any Fibonacci sequence can be written in the form:

(5) $$T_n = ar^n + bs^n ,$$

where a and b are suitable constants. For example, let

$$T_1 = 2, \; T_2 = 5 \; .$$

The relations that must be satisfied are:

$$2 = ar + bs$$

$$5 = ar^2 + bs^2 \; .$$

These give solutions:

$$a = \frac{15 + \sqrt{5}}{10} \quad \text{and} \quad b = \frac{15 - \sqrt{5}}{10} \; ,$$

so that

$$T_n = \frac{15 + \sqrt{5}}{10} r^n + \frac{15 - \sqrt{5}}{10} s^n \; .$$

Let us apply this technique to what is commonly known as the Fibonacci sequence whose initial terms are $F_1 = 1$ and $F_2 = 1$. Then

$$1 = ar + bs$$

$$1 = ar^2 + bs^2 \; ,$$

with solutions

$$a = \frac{1}{\sqrt{5}}$$

and

$$b = \frac{-1}{\sqrt{5}}$$

so that

(6) $$F_n = \frac{r^n - s^n}{\sqrt{5}} ,$$

the BINET FORMULA for the Fibonacci sequence.

Similarly, for the Lucas sequence with $L_1 = 1$ and $L_2 = 3$,

$$1 = ar + bs$$

$$3 = ar^2 + bs^2 ,$$

one obtains $a = 1$, $b = 1$, so that:

(7) $$L_n = r^n + s^n ,$$

the BINET FORMULA for the Lucas sequence.

THE GOLDEN SECTION RATIO

With this formulation it is easy to see the connection between the Fibonacci sequences and the Golden Section Ratio. To divide a line segment in what is known as "extreme and mean ratio" or to make a Golden Section of the line segment, one finds a point on the line such that the length of the entire line is to the larger segment as the larger segment is to the smaller segment. To produce an exact parallel with the Fibonacci sequence auxiliary equation, let x be the length of the line and 1 the length of the larger segment. Then:

$$x : 1 = 1 : 1 - x ,$$

which leads to the equation

$$x^2 - x - 1 = 0 .$$

Clearly, we are interested in the positive root

$$r = \frac{1 + \sqrt{5}}{2}$$

The other root $s = -1/r$ is the negative reciprocal of r, the Golden Section Ratio. (It may be noted that

$$\frac{1}{r} = \frac{\sqrt{5} - 1}{2}$$

is also considered the Golden Section Ratio by some authors. This is a matter of point of view: whether one is taking the ratio of the larger segment to the smaller segment or vice-versa.)

USING THE BINET FORMULAS

The Binet formulas for the Fibonacci and Lucas sequences are certainly not the practical means of calculating the terms of these sequences. Algebraically, however, they provide a powerful tool for creating or verifying Fibonacci-Lucas relations. Let us consider a few examples.

Example 1

If we study the terms of the Fibonacci sequence and the Lucas sequence in the following table:

n	F_n	L_n
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76
10	55	123

it is a matter of observation that:

$$F_4L_4 = 3 \times 7 = 21 = F_8$$

$$F_5L_5 = 5 \times 11 = 55 = F_{10}$$

and in general it appears that:

$$F_n L_n = F_{2n} .$$

Why is this so? Using the Binet formula for F_{2n},

$$F_{2n} = \frac{r^{2n} - s^{2n}}{\sqrt{5}} = \frac{(r^n - s^n)}{\sqrt{5}} (r^n + s^n) = F_n L_n$$

Example 2

$$F_{kn} = \frac{r^{kn} - s^{kn}}{\sqrt{5}} = \frac{(r^k)^n - (s^k)^n}{\sqrt{5}}$$

has a factor

$$\frac{r^k - s^k}{\sqrt{5}} = F_k ,$$

which proves that if k is a divisor of the subscript of a Fibonacci number F_m, then F_k divides F_m.

Example 3

By taking successive values of k, one can intuitively surmise the formula:

$$F_{n+k} F_{n-k} - F_n^2 = (-1)^{n+k+1} F_k^2$$

To prove this relation, use the Binet formula for F. This gives:

$$F_{n+k}F_{n-k} - F_n^2 = \frac{r^{n+k} - s^{n+k}}{\sqrt{5}} \cdot \frac{r^{n-k} - s^{n-k}}{\sqrt{5}} - \frac{(r^n - s^n)^2}{5}$$

$$= \frac{r^{2n} + s^{2n} - r^{n+k}s^{n-k} - r^{n-k}s^{n+k} - r^{2n} + 2r^n s^n - s^{2n}}{5}$$

$$= -\frac{r^{n-k}s^{n-k}(r^{2k} - 2r^k s^k + s^{2k})}{5} = (-1)^{n+k+1} F_k^2 .$$

LINEAR RECURSION RELATIONS

PROBLEMS

1. Prove that

$$L_{2n} = L_n^2 + 2(-1)^{n+1} .$$

2. Using the Binet formulas, find the value of:

$$L_n F_{n-1} - F_n L_{n-1} .$$

3. $F_{3n} = F_n(\qquad)$. Determine the expression for the cofactor of F_n.
4. $F_{5n} = F_n(\qquad)$. Determine the expression for the cofactor of F_n.
5. $L_{3n} = L_n(\qquad)$. Find the expression for the cofactor of L_n.
6. $L_{5n} = L_n(\qquad)$. Find the expression for the cofactor of L_n.
7. For the Fibonacci relation with $T_1 = 3$, $T_2 = 7$, find the expression for T_n in terms of powers of r and s.
8. Using the binomial expansion, find an expression for F_n in terms of powers of 5 and binomial coefficients.
9. Do likewise for L_n.
10. Assuming the relation

$$L_n + L_{n+2} = 5F_{n+1} ,$$

determine an equivalent single Fibonacci number for $F_n^2 + F_{n+1}^2$ using the Binet formula.

(See page 56 for answers to problems.)