LINEAR RECURSION RELATIONS

ANSWERS TO PROBLEMS

LESSON ONE

1. $a_n = n(n+1)$; $T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$
2. $a_n = 3n - 2$; $T_{n+2} = 2T_{n+1} - T_n$
3. $a_n = n^3$; $T_{n+4} = 4T_{n+3} - 6T_{n+2} + 4T_{n+1} - T_n$
4. $T_{6n+k} = 1, 3, 3, 1, 1/3, 1/3$, for $k = 1, 2, 3, 4, 5, 6$, respectively
5. $T_{n+1} = \sqrt{1 + T_n^2}$
6. $T_{n+4} = 4T_{n+3} - 6T_{n+2} + 4T_{n+1} - T_n$
7. $T_{n+1} = aT_n$
8. $T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$
9. $T_{2n-1} = a$, $T_{2n} = 1/a$
10. $T_{n+1} = 1/(2 - T_n)$

LESSON TWO

1.
$$T_n = -4 + (7/2)\,2^n$$

First ten terms: 3, 10, 24, 52, 108, 220, 444, 892, 1788, 3580.

2.
$$T_n = (13/6)\,3^n + (-3/10)\,5^n$$

3.
$$T_n = 17/5 + (4/15)\,6^n$$
$$T_{n+1} = 7T_n - 6T_{n-1}$$

4.
$$T_n = -2 + 3 \cdot 2^n + (-1/3)\,3^n$$

5.
$$T_{n+1} = 3T_n + T_{n-1} - 3T_{n-2}$$
$$T_n = 1/4 + (7/8)(-1)^n + (13/24)\,3^n \,.$$

6.
$$T_n = 5/3 + (1/3)(-1)^n/2^{n-2}$$

7.
$$T_n = 5/2 + (9/2)(-1/3)^n$$

8.
$$T_n = 2^{n/2}\left[\frac{5 + 3\sqrt{2}}{4} + (-1)^n\,\frac{5 - 3\sqrt{2}}{4}\right]$$

9. $$T_n = 3 + (-1)^n$$

10. $$T_n = \frac{-2+\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)^n + \frac{-2-\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right)^n.$$

LESSON THREE

2. $$2(-1)^n$$

3. $$L_{2n} + (-1)^n$$

4. $$L_{4n} + (-1)^n L_{2n} + 1$$

5. $$L_{2n} + (-1)^{n+1}$$

6. $$L_{4n} + (-1)^{n+1} L_{2n} + 1$$

7. $$T_n = \frac{10+\sqrt{5}}{5} r^n + \frac{10-\sqrt{5}}{5} s^n$$

8. $$F_n = 2^{-n+1}\left[n + 5\binom{n}{3} + 5^2\binom{n}{5} + 5^3\binom{n}{7} \cdots\right]$$

9 $$L_n = 2^{-n+1}\left[1 + 5\binom{n}{2} + 5^2\binom{n}{4} + 5^3\binom{n}{6} \cdots\right]$$

10. $$F_{2n+1}.$$

LESSON FOUR

1. For any modulus m, there are m possible residues $(0, 1, 2, \cdots, m-1)$. Successive pairs may come in m^2 ways. Two successive residues determine all residues thereafter. Now in an infinite sequence of residues there is bound to be repetition and hence periodicity.

Since m divides T_0, it must by reason of periodicity divide an infinity of members of the sequence.

2. $n = mk$, where m and k are odd. V_n can be written

$$V_n = (r^m)^k + (s^m)^k ,$$

which is divisible by $V_m = r^m + s^m$.

3. $r = 2 + 2i\sqrt{2}$, $s = 2 - 2i\sqrt{2}$.

$$T_n = \left(\frac{2 - 3i\sqrt{2}}{16}\right) r^n + \left(\frac{2 + 3i\sqrt{2}}{16}\right) s^n .$$

4. The auxiliary equation is $(x - 1)^2 = 0$, so that T_n has the form

$$T_n = An \times 1^n + B \times 1^n = An + B .$$

5.
$$T_n = 2^n \left[\left(\frac{b - 2a}{4}\right) n + \frac{4a - b}{4} \right] .$$

6.
$$T_n = -(-i)^n$$

7.
$$T_{n+1} = 5T_n - 6T_{n-1}$$
$$T_n = 2^n + 3^{n-1}$$

8.
$$r = \frac{5 + \sqrt{29}}{2}, \qquad s = \frac{5 - \sqrt{29}}{2}$$

$$T_n = \frac{r^n - s^n}{\sqrt{29}} \quad \text{with terms } 1, 5, 26, 135, \cdots$$

$$V_n = r^n + s^n \quad \text{with terms } 5, 27, 140, \cdots$$

9.
$$r = \frac{3 + i\sqrt{11}}{2}, \qquad s = \frac{3 - i\sqrt{11}}{2}$$

$$T_n = \left(\frac{33 - 16i\sqrt{11}}{55}\right) r^n + \left(\frac{33 + 16i\sqrt{11}}{55}\right) s^n$$

10.
$$T_{n+1} = 5T_n + 2T_{n-1}; \quad T_1 = 3, \quad T_2 = 7 .$$

LESSON FIVE

1. $$T_{n+1} = 8T_n - 18T_{n-1} + 16T_{n-2} - 5T_{n-3}$$

2. $$T_n = -5/2 + 7 \times 2^n - (7/6)3^n$$

3. $$T_{n+1} = 4T_n - 2T_{n-1} - 3T_{n-2}$$

4. $$T_{n+1} = 2T_n + T_{n-1} - 3T_{n-2} + T_{n-4}$$

5. $$T_n = 12 + \frac{1}{\sqrt{13}}\left(\frac{3+\sqrt{13}}{2}\right)^n - \frac{1}{\sqrt{13}}\left(\frac{3-\sqrt{13}}{2}\right)^n$$

6. $$T_n = (-135/20)(-1)^n + (19/10)(-2)^n + (41/60)3^n$$

7. $$T_{n+1} = 3T_{n-1} + 2T_{n-2}$$

8. $$T_n = -1/3 + 4n - (-2)^n/6$$

9. $T_{n+1} = 3T_n - 3T_{n-1} + T_{n-2}$ and $T_n = 2 + n/2 + 3n^2/2$

10. $T_{n+1} = -T_{n-1}$ and $T_n = \frac{-3-i}{2} i^n + \frac{-3+i}{2}(-i)^n$

LESSON SIX

1. $$T_{n+1} = 5T_n + 2T_{n-1} - 9T_{n-2} - 5T_{n-3}$$

2. $$T_{n+1} = 5T_n - 4T_{n-1} - 9T_{n-2} + 7T_{n-3} + 6T_{n-4}$$

3. $$T_{n+1} = 5T_n - 7T_{n-1} + 3T_{n-2}$$

4. $$T_{n+4} = 4T_{n+3} - 2T_{n+2} - 5T_{n+1} + 2T_n$$

5. $$T_{n+6} = 2T_{n+5} + 4T_{n+4} - 4T_{n+3} - 6T_{n+2} + T_n$$

6. $$T_{n+1} = 7T_n - 17T_{n-1} + 17T_{n-2} - 6T_{n-3}$$

7. $$T_{n+4} = T_n \quad \text{and} \quad T_n = (-1)^n/2 + \frac{-3-5i}{4} i^n + \frac{-3+5i}{4}(-i)^n$$

8. $$T_{n+1} = 4T_n - 5T_{n-1} + T_{n-2} + 2T_{n-3} - T_{n-4}$$

9. $$T_{n+1} = 6T_n - 11T_{n-1} + 5T_{n-2} + 4T_{n-3} - 3T_{n-4}$$

10. $$T_{n+1} = 9T_n - 27T_{n-1} + 25T_{n-2} + 13T_{n-3} - 19T_{n-4} - 6T_{n-5} .$$

LESSON SEVEN

1. $$5n^3 - 4n^2 + 3n - 8$$

2. $3n^2 - 8n + 4$ and the Fibonacci sequence: 1, 4, 5, 9, 14, ···

3. $$7n^3 + 3n^2 - 5n + 2 + 3 \times 2^n$$

4. $$4n + 3 + 3(-1)^n$$

5. $2n^3 - 3n^2 - n + 5$ and the Fibonacci sequence $4L_n$

6. $$5 \times 4^{n-1} + 17n + 19$$

7. The Fibonacci sequence 1, 4, 5, 9, 14, 23, ··· and the arithmetic progression $6n + 1$

8. $$7 \times 3^{n-1} + n^2/2 + n/2 + 2$$

9. The Fibonacci sequence 3, 7, 10, 17, ··· and the polynomial
$$(7n^2 - 27n + 28)/2$$

10. The Fibonacci sequence 5, 11, 16, 27, ··· and $6 \times 2^{n-1}$.

LINEAR RECURSION RELATIONS

LESSON EIGHT

1. 11.2556550
2. The roots are 3, and

$$\frac{-3 \pm \sqrt{5}}{2} .$$

Limiting ratio is 3.

3. The roots are -2, -2, r and s. Limiting ratio is -2.
4. The roots of the combined recursion relation will be 1, r, s. Limiting ratio is r.
5. The roots of the combined recursion relation are +2, +2, +2,

$$\frac{3 \pm \sqrt{13}}{2} .$$

The limiting ratio is

$$\frac{3 + \sqrt{13}}{2} = 3.3027756 .$$

6. The roots of the auxiliary equation are 2,

$$\frac{1 \pm \sqrt{19}\, i}{2} .$$

The absolute value of the complex roots is greater than 2. Thus the sequences will not have a limiting ratio.

7. The limiting ratio is 1.
8. The limiting ratio is 3.
9. The roots of the auxiliary equation are 1, 1 $\pm$ i. Since the absolute value of the complex root is greater than 1, there is no limiting ratio.
10. The recursion relation is $T_{n+1} = -T_{n-1}$ with roots $\pm i$ for the auxiliary equation. Hence, there is no limiting ratio.