9 • *Pascal's Triangle and the Fibonacci Numbers*

If we consider the expansions of the binomial $(x + y)^n$ for $n = 0, 1, 2, 3, 4, 5, \ldots,$ we can write them in the form:

$$\begin{aligned}
(x+y)^0 &= x^0y^0 \\
(x+y)^1 &= x^1y^0 + x^0y^1 \\
(x+y)^2 &= x^2y^0 + 2x^1y^1 + x^0y^2 \\
(x+y)^3 &= x^3y^0 + 3x^2y^1 + 3x^1y^2 + x^0y^3 \\
(x+y)^4 &= x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + x^0y^4 \\
(x+y)^5 &= x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + x^0y^5 \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Recalling that $x^0 = y^0 = 1$ (x and y nonzero), we can write the array of coefficients, which is called *Pascal's triangle*, as follows:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Before we proceed further, let us introduce some special symbolism. If we write

$$1 + 2 + 3 + 4 + \cdots + n,$$

we mean that starting with one, we add the consecutive integers until we come to n, which is the last number added. By using a notation based on

$\sum$, the Greek letter sigma for s, the first letter in *summation*, we can write

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n,$$

where the *index* is i and the *limits of summation* are 1 and n. The index progresses from $i = 1$ to $i = n$ in unit increases. A *summand* is a quantity obtained by putting a certain value of i into a formula that yields the quantities to be added. For example, in

$$\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6 = 21$$

the summands are 1, 2, 3, 4, 5, and 6.

You may recall from your study of arithmetic and geometric progressions the general formulas

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

$$\sum_{i=0}^{n-1} r^i = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}, \qquad r \neq 1.$$

Returning now to Pascal's triangle, let $\binom{n}{m}$ represent the mth term in the nth row ($n \geq m \geq 0$). Thus, for $n = 4$, we have

$$(x + y)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$
$$= (1)x^4 + (4)x^3 y + (6)x^2 y^2 + (4)x y^3 + (1)y^4.$$

The numbers represented by the symbols $\binom{n}{m}$ are called the *binomial coefficients*. In the expansion of $(x + y)^4$ the numbers are the binomial coefficients for the fourth-power expansion.

In general, from a study of Pascal's triangle, we can define

$$\binom{0}{0} = \binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} \quad \text{for } n \geq m \geq 1.$$

We see that these interesting numbers are thus defined by a recurrence formula for each positive integer n. For fixed n, the binomial coefficients are the entries across the nth level of Pascal's triangle. We can now write

$$(x + y)^4 = \sum_{i=0}^{4} \binom{4}{i} x^{4-i} y^i \quad (\text{since } y^0 = x^0 = 1)$$

and, in general (as can be proved by mathematical induction),

$$\text{(A)} \qquad (x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Although we can generate any number of the binomial coefficients by writing down successive levels of Pascal's triangle, it is possible to calculate them directly. Let $1 \cdot 2 \cdot 3 \cdot \cdots \cdot n = n!$ (called "n factorial") and $0! = 1$. Then it can be shown that

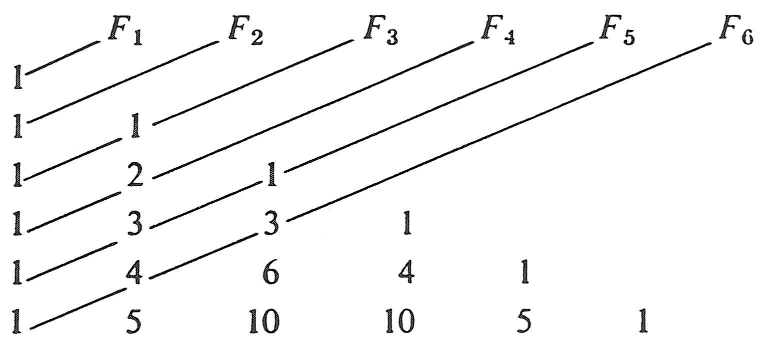
$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

For example,

$$\binom{5}{3} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1 \cdot 2 \cdot 3)(1 \cdot 2)} = 10.$$

Compare this with the coefficient for $m = 3$ in the expansion of $(x + y)^5$

How are the Fibonacci numbers related to Pascal's triangle? Let us write Pascal's triangle in the following form:



We can now see that the sums along the rising diagonals are the Fibonacci numbers:

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 1 + 1, \quad F_4 = 1 + 2,$$
$$F_5 = 1 + 3 + 1, \quad F_6 = 1 + 4 + 3, \quad F_7 = 1 + 5 + 6 + 1, \quad \text{etc.}$$

It can be proved that in general,

$$F_{n+1} = \sum_{i=0}^{[n/2]} \binom{n-i}{i},$$

where $[x]$ denotes the greatest integer not greater than x (Section 6). For example,

$$F_5 = \sum_{i=0}^{2} \binom{4-i}{i} = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 1 + 3 + 1 = 5,$$

$$F_8 = \sum_{i=0}^{3} \binom{7-i}{i} = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3}$$
$$= 1 + \frac{6!}{(1!)(5!)} + \frac{5!}{(2!)(3!)} + \frac{4!}{(3!)(1!)}$$
$$= 1 + 6 + 10 + 4 = 21.$$

Now suppose that we look at $\sum_{i=0}^{n} \binom{n}{i} F_i$, where $F_i = \dfrac{\alpha^i - \beta^i}{\alpha - \beta}\cdot$ Thus,

$$\sum_{i=0}^{n} \binom{n}{i} F_i = \frac{1}{\alpha - \beta}\left[\sum_{i=0}^{n} \binom{n}{i} \alpha^i - \sum_{i=0}^{n} \binom{n}{i} \beta^i\right]$$

Since from formula (A) on page 49 we have $\sum_{i=0}^{n} \binom{n}{i} x^i = (1 + x)^n$ (see Exercise 1 below), we can write

$$\sum_{i=0}^{n} \binom{n}{i} F_i = \frac{1}{\alpha - \beta}[(1 + \alpha)^n - (1 + \beta)^n]$$

But, $1 + \alpha = \alpha^2$ and $1 + \beta = \beta^2$; therefore,

$$\sum_{i=0}^{n} \binom{n}{i} F_i = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = F_{2n}.$$

For example,

$$\sum_{i=0}^{5} \binom{5}{i} F_i = \binom{5}{0} F_0 + \binom{5}{1} F_1 + \binom{5}{2} F_2 + \binom{5}{3} F_3 + \binom{5}{4} F_4 + \binom{5}{5} F_5$$

$$= (1)(0) + (5)(1) + (10)(1) + (10)(2) + (5)(3) + (1)(5)$$

$$= 0 + 5 + 10 + 20 + 15 + 5 = 55 = F_{10}.$$

EXERCISES

Demonstrate the following:

1. $\sum_{i=0}^{n} \binom{n}{i} x^i = (1 + x)^n$ **2.** $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ **3.** $\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0,$ $n \geq 1$

4. Verify that the two formulas for binomial coefficients agree by using the factorial notation to demonstrate that

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}.$$

5. Verify that $\sum_{i=0}^{4} \binom{4}{i} F_i = F_8$.

Using the Binet form show that

6. $\sum_{i=0}^{n} \binom{n}{i} F_{i+j} = F_{2n+j}$.

7. $\sum_{i=0}^{n} \binom{n}{i} L_{i+j} = L_{2n+j}$.

8. $\sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{i+j} = (-1)^{j+1} F_{n-j}$

9. $\sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{2i+j} = (-1)^n F_{n+j}$