8 • *Periodicity of the Fibonacci and Lucas Numbers*

What else can we discover from the list of Fibonacci numbers?

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	$F_{15}\ldots$
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610 . . .

Let us consider the sequence of primes. We observe that

2 divides F_3, F_6, and so on,
3 divides F_4, F_8, and so on,
5 divides F_5, F_{10}, and so on,
7 divides F_8, and so divides F_{16}, and so on
11 divides F_{10}, and so divides F_{20}, and so on
13 divides F_7, F_{14}, and so on.

We can speculate as to whether *every* prime number divides some Fibonacci number (and hence divides infinitely many of them). Later in this section we shall prove that *every integer* divides some Fibonacci number.

To do this, we shall use a kind of periodicity property of the Fibonacci numbers. But first we consider the following.

You are, no doubt, familiar with periodic decimals, such as the decimal representation of

$$\tfrac{1}{7}.$$

When 1 is divided by 7, the only possible remainders are

$$0, 1, 2, 3, 4, 5, 6.$$

If we had a zero remainder, our division would be exact. However, when we get any other of those seven digits, the division continues as shown, leading to the periodic decimal

$$\tfrac{1}{7} = .142857142857142857\ldots.$$

```
     .1 4 2 8 5 7 1
7 ) 1.0 0 0 0 0 0 0
     7
     3 0
     2 8
       2 0
       1 4
         6 0
         5 6
           4 0
           3 5
             5 0
             4 9
               1 0
                 7
                 3
```

Every rational number has a periodic decimal expansion and every periodic decimal expansion represents a rational number. (Even $\frac{1}{2} = .5000\ldots = .4999\ldots$.)

In our present discussion we shall use a similar argument, but this time we shall be discussing *repeating ordered pairs of remainders.*

When F_0 through F_{31} are each divided by 7, we obtain the following sequence displaying the quotients and remainders.

$F_0 =$	$0 =$	$0 \cdot 7 + 0$	$F_{16} =$	$987 =$	$141 \cdot 7 + 0$
$F_1 =$	$1 =$	$0 \cdot 7 + 1$	$F_{17} =$	$1597 =$	$228 \cdot 7 + 1$
$F_2 =$	$1 =$	$0 \cdot 7 + 1$	$F_{18} =$	$2584 =$	$369 \cdot 7 + 1$
$F_3 =$	$2 =$	$0 \cdot 7 + 2$	$F_{19} =$	$4181 =$	$597 \cdot 7 + 2$
$F_4 =$	$3 =$	$0 \cdot 7 + 3$	$F_{20} =$	$6765 =$	$966 \cdot 7 + 3$
$F_5 =$	$5 =$	$0 \cdot 7 + 5$	$F_{21} =$	$10946 =$	$1563 \cdot 7 + 5$
$F_6 =$	$8 =$	$1 \cdot 7 + 1$	$F_{22} =$	$17711 =$	$2530 \cdot 7 + 1$
$F_7 =$	$13 =$	$1 \cdot 7 + 6$	$F_{23} =$	$28657 =$	$4093 \cdot 7 + 6$
$F_8 =$	$21 =$	$3 \cdot 7 + 0$	$F_{24} =$	$46368 =$	$6624 \cdot 7 + 0$
$F_9 =$	$34 =$	$4 \cdot 7 + 6$	$F_{25} =$	$75025 =$	$10717 \cdot 7 + 6$
$F_{10} =$	$55 =$	$7 \cdot 7 + 6$	$F_{26} =$	$121393 =$	$17341 \cdot 7 + 6$
$F_{11} =$	$89 =$	$12 \cdot 7 + 5$	$F_{27} =$	$196418 =$	$28059 \cdot 7 + 5$
$F_{12} =$	$144 =$	$20 \cdot 7 + 4$	$F_{28} =$	$317811 =$	$45401 \cdot 7 + 4$
$F_{13} =$	$233 =$	$33 \cdot 7 + 2$	$F_{29} =$	$514229 =$	$73461 \cdot 7 + 2$
$F_{14} =$	$377 =$	$53 \cdot 7 + 6$	$F_{30} =$	$832040 =$	$118862 \cdot 7 + 6$
$F_{15} =$	$610 =$	$87 \cdot 7 + 1$	$F_{31} =$	$1346269 =$	$192324 \cdot 7 + 1$

You can see by the zero remainders that 7 divides F_0, F_8, F_{16}, and F_{24} in this list.

Notice the pattern of the remainders (since each F_n is the sum of the two preceding Fibonacci numbers):

$$0 + 1 = 1,\ 1 + 1 = 2,\ \ldots,\ 3 + 5 = 8 = 7 + 1,\ 1 + 6 = 7 + 0,\ \ldots$$

That is, each remainder after the second is the sum of the two preceding remainders, decreased as necessary by 7.

Consider the sequences of remainders

$$1,\ 6,\ 0,\ 6 \qquad \text{(beginning with } F_6\text{)}$$

and

$$6,\ 1,\ 0,\ 1 \qquad \text{(beginning with } F_{14}\text{)}.$$

Notice that the pair (1, 6) gives a different sequence from that following the pair (6, 1). Since there are seven possible remainders (0, 1, 2, 3, 4, 5, 6) when numbers are divided by 7, there can be at most 7×7, or 49, different *ordered pairs* of remainders. Therefore, in a set of $49 + 1$, or 50, ordered pairs, at least two of the pairs must be the same. In our example, not all the possible ordered pairs of remainders appear, and there are many repetitions between the first ordered pair of remainders, (0, 1) for F_0 and F_1, and the fiftieth, (1, 1) for F_{49} and F_{50}.

We shall now show that the *first* repeated pair of remainders is (0, 1) when we consider Fibonacci numbers with positive or zero subscripts, as we did in the table above for $m = 7$. In general, let the pairs of remainders (r_k, r_{k+1}) be obtained by dividing the Fibonacci numbers F_k and F_{k+1} by some integer m. Consider the sequence of pairs

$$(r_0, r_1), (r_1, r_2), (r_2, r_3), \ldots, (r_n, r_{n+1}).$$

We shall say that pairs (a_1, b_1) and (a_2, b_2) are equal if and only if $a_1 = a_2$ and $b_1 = b_2$. Clearly, in the first $m^2 + 1$ of these pairs obtained from the division of consecutive Fibonacci numbers by m, there must be at least two which are equal. (Conceivably, there could be more than two equal pairs as in the example above.)

Let us assume that the first pair to be repeated is (r_k, r_{k+1}) for $k \geq 0$. Then in our sequence of pairs there is a later pair (r_n, r_{n+1}) equal to (r_k, r_{k+1}) with $m^2 + 1 \geq n + 1 > k + 1$. But since the pairs are equal, $r_k = r_n$ and $r_{k+1} = r_{n+1}$. Since

$$F_{k-1} = F_{k+1} - F_k \quad \text{and} \quad F_{n-1} = F_{n+1} - F_n,$$

upon dividing by m, we get

$$r_{k-1} = r_{k+1} - r_k \quad \text{and} \quad r_{n-1} = r_{n+1} - r_n.$$

Since $r_{k+1} = r_{n+1}$ and $r_k = r_n$,

$$r_{k-1} = r_{n-1},$$

although it may be necessary to add m to r_{k-1} and r_{n-1} to make them positive. Thus, the pairs (r_{k-1}, r_k) and (r_{n-1}, r_n) are also equal. Therefore, every repeated pair has a predecessor that is also repeated except the first (0, 1), which has no predecessor. This first (0, 1) is the first pair in our list; see the table made for divisor $m = 7$ on page 43.

Let $(r_n, r_{n+1}) = (0, 1)$, $n > 0$, be the second appearance of (0, 1) in the sequence of ordered pairs of remainders. Then $r_n = 0$, so that F_n is divisible by m where $1 < n + 1 \leq m^2 + 1$, or $0 < n \leq m^2$. Therefore, we have proved the following theorem.

THEOREM I

Every integer m divides some Fibonacci number ($> F_0$) whose subscript does not exceed m^2.

As we progress up the sequence of remainders from the second (0, 1), we simply repeat the pairs in the same order as before so that F_{2n} is also divisible by m. We define n as the *period* of m in the Fibonacci sequence and denote it by K_m. For example,

$$K_7 = 16,$$

as you can see by looking back at the remainder pairs obtained by dividing the Fibonacci numbers by 7 (page 43). The pair (0, 1) occurred first with F_0 and F_1. The next occurrence of these remainders was for F_{16} and F_{17}, and F_{16} was divisible by 7. The pairs of remainders now repeat in cycles of 16; hence, $K_7 = 16$.

However, we noticed that F_8 also has a zero remainder and is divisible by 7. The subscript of the first positive Fibonacci number divisible by m is called the *rank of apparition* or *entry point* of the number m in the Fibonacci numbers. Thus, the entry point of 7 in the Fibonacci numbers is 8.

We can assert that 7 divides a Fibonacci number if and only if *its subscript* is divisible by 8. We note that the *only if* follows from the *periodicity* and not from Theorem III in Section 7, since 7 is not a Fibonacci number. (We also note that 7 does *not* divide F_7.) If a prime divides some Fibonacci number F_d, then it divides every Fibonacci number F_{nd}, since F_d divides F_{nd}. (Note that, since the number 7, which we chose to illustrate periodicity, happens to be a Lucas number, for this particular divisor we could have applied Theorem IV in Section 7.)

Now let us turn to the Lucas numbers. It can easily be shown that 5 does not divide any Lucas number:

$$\begin{aligned} L_0 &= 2 = 0 \cdot 5 + 2 & \quad L_4 &= 7 = 1 \cdot 5 + 2 \\ L_1 &= 1 = 0 \cdot 5 + 1 & \quad L_5 &= 11 = 2 \cdot 5 + 1 \\ L_2 &= 3 = 0 \cdot 5 + 3 & \quad L_6 &= 18 = 3 \cdot 5 + 3 \\ L_3 &= 4 = 0 \cdot 5 + 4 & \quad L_7 &= 29 = 5 \cdot 5 + 4 \end{aligned}$$

Thus, in considering the remainder pairs when the Lucas numbers are divided by 5, we come to a repeated remainder pair (2, 1) again at L_4 and L_5. We note that in the period of length 4 there were no zeros. (Zero is not a Lucas number, and so the remainder 0 may not occur at all.) Thus, we see that *no Lucas number* is divisible by 5.

Of course, some Lucas numbers are divisible by other integers, for example 6 (see Exercise 4, page 47).

In the booklet *An Introduction to Fibonacci Discovery* by Brother U. Alfred (available from the author, St. Mary's College, California 94575) there is a nice table (Table 3, page 55) of entry points and periods of primes 2 to 269 in both the Fibonacci numbers and the Lucas numbers.

The Fibonacci and Lucas numbers have interesting remainder properties. We state the following theorems without proof.

THEOREM II

If F_n is divided by F_m $(n > m)$, then either the remainder R is a Fibonacci number or $F_m - R$ is a Fibonacci number.

For example,

$$89 = 2 \cdot 34 + 21, \quad \text{or} \quad F_{11} = 2 \cdot F_9 + F_8;$$
$$144 = 6 \cdot 21 + 18, \quad \text{or} \quad F_{12} = 6 \cdot F_8 + 18.$$

In the second case, although 18 is not a Fibonacci number, the difference

$$F_8 - 18 = 21 - 18 = 3 = F_4$$

is a Fibonacci number. We can write

$$F_{12} = 6 \cdot F_8 + (F_8 - F_4).$$

Of course, $F_0 = 0$ is a Fibonacci number, and thus when F_{10} is divided by F_5, the zero remainder is a Fibonacci number.

With one exception, the property given in Theorem II is shared by the Lucas numbers. We have this theorem.

THEOREM III

If L_n is divided by L_m, $n > m$, then the remainder R is zero, or R is a Lucas number, or $L_m - R$ is a Lucas number.

For example,

$$76 = 19 \cdot 4 + 0, \quad \text{or} \quad L_9 = 19 \cdot L_3 + 0,$$

where $R = 0$, which is not a Lucas number but is the first possibility given in the theorem. Also,

$$18 = 1 \cdot 11 + 7, \quad \text{or} \quad L_6 = 1 \cdot L_5 + L_4;$$
$$47 = 6 \cdot 7 + 5, \quad \text{or} \quad L_8 = 6 \cdot L_4 + (L_4 - L_0).$$

A proof of the "Remainder Property" occurs in a short article by John H. Halton in *The Fibonacci Quarterly*, Vol. 2, No. 3 (October, 1964), pages 217–218, titled "Fibonacci Residues." The Lucas property is discussed in a paper by Laurence Taylor in *The Fibonacci Quarterly*, Vol. 5, No. 3 (October, 1967), pages 298–304, titled "Residues of Fibonacci-Like Sequences"; in the same paper, we find that the Fibonacci and Lucas sequences are the only sequences which satisfy the recurrence relation $u_{n+2} = u_{n+1} + u_n$ and have the properties given in Theorems II and III.

EXERCISES

1. Show that $K_{13} = 28$ for the Fibonacci sequence, and find the entry point of 13 in that sequence.
2. Show that 13 does not divide any Lucas number, but $K_{13} = 28$ also for the Lucas sequence.
3. Show that $K_{10} = 60$ for the Fibonacci sequence, and find the entry point of 10 in that sequence.
4. Show that $K_6 = 24$ for the Lucas sequence, and state the entry point of 6 in that sequence.
5. Show that $K_{10} = 12$ for the Lucas sequence, and state why 10 has no entry point in that sequence.