6 • Shortcuts to Large F_n and L_n

Since we shall be dealing extensively with inequalities in this section, we shall recall the following propositions.

For real numbers a, b, c, and d, we have:

(a) If $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

(b) If $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

(c) If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

(d) If $0 < \frac{a}{b} < \frac{c}{d}$, then $\frac{b}{a} > \frac{d}{c}$, and conversely.

(e) If $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$, then $\frac{a}{b} < \frac{e}{f}$.

(f) If $|b| < 1$, then $|b|^n < 1 \quad \text{for} \quad n = 1, 2, 3, \ldots.$

(g) If $a = b + c$ and $c > 0$, then $b < a$.

We shall also use an idea suggested by the following problem. Suppose that we have s pounds of sugar (s not necessarily an integer), and we ask how many one-pound sacks of sugar can be made from this quantity. Then we are interested in finding the greatest integer not greater than s. We denote this integer by $[s]$. Thus,

$$[7.2] = 7 \qquad \text{and} \qquad [7.9] = 7.$$

Similarly, we have

$$[-5.4] = -6 \qquad \text{and} \qquad [\tfrac{1}{2}] = 0.$$

We are now able to devise methods for finding values of F_n and L_n (or at least estimates of them) without doing all the additions from the beginning. The first theorem that we shall prove is the following.

THEOREM 1

$$F_n = \left[\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right] \quad \text{for} \quad n = 1, 2, 3, \ldots .$$

Proof. We have the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \qquad n = 1, 2, 3, \ldots,$$

where $\alpha = \dfrac{1+\sqrt{5}}{2} \doteq \dfrac{1 + 2.236}{2} \doteq 1.618$ and $\beta = \dfrac{1-\sqrt{5}}{2} \doteq -.618.$
We can write

$$F_n = \frac{\alpha^n}{\sqrt{5}} - \frac{\beta^n}{\sqrt{5}} = \left(\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{\beta^n}{\sqrt{5}}\right).$$

Since $0 < |\beta| < 1$, we find by (f) on page 30 that

$$0 < |\beta|^n < 1.$$

Since $1 < \dfrac{\sqrt{5}}{2}$, we find by (e) that

$$0 < |\beta|^n < \frac{\sqrt{5}}{2}.$$

Since $\sqrt{5} > 0$, we find by (a) that

(A) $$0 < \frac{|\beta|^n}{\sqrt{5}} < \frac{1}{2}.$$

Then by (c) we find (by adding $\frac{1}{2}$) that

$$\frac{1}{2} < \frac{1}{2} + \frac{|\beta|^n}{\sqrt{5}} < 1.$$

Although $\beta < 0$, we have $|\beta|^n = \beta^n$ when n is even, and so

(B) $$\text{if } n \text{ is even,} \quad \frac{1}{2} < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < 1.$$

On the other hand, we have $|\beta|^n = -\beta^n$ when n is odd, and so from (A),

$$-\frac{1}{2} < \frac{\beta^n}{\sqrt{5}} < 0.$$

Then by (c) we find (by adding $\frac{1}{2}$) that

(C) $$\text{if } n \text{ is odd,} \quad 0 < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < \frac{1}{2}.$$

Thus, from (B) and (C) we have, in general,

$$0 < \frac{1}{2} + \frac{\beta^n}{\sqrt{5}} < 1.$$

But we saw on page 31 that F_n can be expressed as

$$F_n = \left(\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{\beta^n}{\sqrt{5}}\right);$$

and so, since we have shown (page 31) that $\left(\frac{1}{2} + \frac{\beta^n}{\sqrt{5}}\right)$ is positive and less than 1, we have

$$F_n < \frac{\alpha^n}{\sqrt{5}} + \frac{1}{2} < F_n + 1,$$

or

$$F_n = \left[\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right] \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

and the theorem is proved.

Similarly, the following theorem can be proved, but we shall not give the proof here.

THEOREM II

$$L_n = [\alpha^n + \tfrac{1}{2}] \qquad \text{for } n = 2, 3, 4, \ldots.$$

It is shown in *The Fibonacci Numbers* by N. N. Vorobyov* that F_n is the integer nearest to $\frac{\alpha^n}{\sqrt{5}}$, that is,

$$\left|F_n - \frac{\alpha^n}{\sqrt{5}}\right| < \frac{1}{2}.$$

Using this result, F_n can be computed if logarithms to a sufficient number of places are used. A similar development can be given to show that

$$|L_n - \alpha^n| < \tfrac{1}{2}.$$

PROBLEM 1

Find $F_{16} \doteq \frac{\alpha^{16}}{\sqrt{5}}$.

Solution.
$$\log \frac{\alpha^{16}}{\sqrt{5}} \doteq 16 \log \alpha - \log 2.236$$

Since we are going to multiply $\log \alpha$ by 16, we should find $\log \alpha$ to more

* For one translation see the reference on page 23.

decimal places than we plan to use for the remainder of the computation.

$$\alpha = \frac{1+\sqrt{5}}{2}$$

$$\doteq \frac{1+2.23607}{2} \doteq 1.6180$$

From a five-place table of common, or base 10, logarithms we find

$$\log \alpha \doteq 0.20898.$$

Thus, we have:

$$\log \frac{\alpha^{16}}{\sqrt{5}} \doteq 16(0.20898) - 0.3494$$

$$\doteq 3.3437 - 0.3494 \doteq 2.9943$$

$$\frac{\alpha^{16}}{\sqrt{5}} \doteq 987.0$$

Therefore, $F_{16} = 987$. You can check this answer in the list on page 83.

For larger indices we can find only the first three digits accurately if we are using four-place logarithms.

PROBLEM 2

Estimate F_{30}.

Solution. $$\log \frac{\alpha^{30}}{\sqrt{5}} \doteq 30(0.20898) - 0.3494 \doteq 5.9200$$

$$\frac{\alpha^{30}}{\sqrt{5}} \doteq 831800$$

Therefore, $F_{30} \doteq 832{,}000$. You can find the exact value in the list on page 83.

PROBLEM 3

Find $L_{14} \doteq \alpha^{14}$.

Solution. $$\log \alpha^{14} \doteq 14(0.20898) \doteq 2.9257$$

$$\alpha^{14} \doteq 842.8$$

Therefore, $L_{14} = 843$. You can check this value in the list on page 83.

We shall now develop methods for finding F_{n+1} from F_n and L_{n+1} from L_n *even if we do not know the value of* n. To achieve this, we shall prove the following new theorem.

THEOREM III

$$F_{n+1} = [\alpha F_n + \tfrac{1}{2}], \qquad n = 2, 3, 4, \ldots.$$

Proof. Since

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

we have

$$\alpha F_n = \frac{\alpha^{n+1} - \alpha\beta^n}{\sqrt{5}} = \frac{\alpha^{n+1} - \alpha\beta^n - \beta^{n+1} + \beta^{n+1}}{\sqrt{5}}$$

$$= \frac{\alpha^{n+1} - (\alpha\beta)\beta^{n-1} - \beta^{n+1} + \beta^{n+1}}{\sqrt{5}}.$$

Since $\alpha\beta = -1$, we have

$$\alpha F_n = \frac{\alpha^{n+1} - \beta^{n+1} + \beta^{n+1} + \beta^{n-1}}{\sqrt{5}}$$

$$= F_{n+1} + \beta^{n-1}\left(\frac{\beta^2 + 1}{\sqrt{5}}\right)$$

But

$$\beta^2 + 1 = \beta + 2 = \frac{1 - \sqrt{5} + 4}{2} = \frac{5 - \sqrt{5}}{2} = \sqrt{5}\left(\frac{\sqrt{5} - 1}{2}\right) = -\sqrt{5}\,\beta.$$

Therefore,

$$\alpha F_n = F_{n+1} + \beta^{n-1}(-\beta) = F_{n+1} - \beta^n,$$

and

(A) $$\alpha F_n + \tfrac{1}{2} = F_{n+1} + (\tfrac{1}{2} - \beta^n).$$

Now, since $|\beta| < .62$, we have $|\beta|^2 < \frac{1}{2}$, and so

$$|\beta|^n < \tfrac{1}{2} \quad \text{for} \quad n \geq 2.$$

Also, $|\beta^n| = |\beta|^n$, and so

$$|\beta^n| < \tfrac{1}{2} \qquad \text{or} \qquad -\tfrac{1}{2} < \beta^n < \tfrac{1}{2}.$$

Therefore, by (b) on page 30, we have

$$\tfrac{1}{2} > -\beta^n > -\tfrac{1}{2}, \qquad \text{or} \qquad -\tfrac{1}{2} < -\beta^n < \tfrac{1}{2},$$

and by (c) we have

$$0 < \tfrac{1}{2} - \beta^n < 1.$$

Now since $\frac{1}{2} - \beta^n > 0$, we find from equation (A) and (g) on page 30 that

$$F_{n+1} < \alpha F_n + \tfrac{1}{2}.$$

Moreover, since $\frac{1}{2} - \beta^n < 1$, we have

$$F_{n+1} + (\tfrac{1}{2} - \beta^n) < F_{n+1} + 1.$$

Applying these to equation (A), we have

$$F_{n+1} < \alpha F_n + \tfrac{1}{2} < F_{n+1} + 1,$$

or

$$F_{n+1} = [\alpha F_n + \tfrac{1}{2}], \qquad n = 2, 3, 4, \ldots,$$

and the theorem is proved.

COROLLARY*

$$F_{n+1} = \left[\frac{F_n + 1 + \sqrt{5F_n^2}}{2}\right], \qquad n \geq 2.$$

Proof. From Theorem III, we have

$$F_{n+1} = \left[\alpha F_n + \tfrac{1}{2}\right] = \left[F_n\left(\frac{1+\sqrt{5}}{2}\right) + \frac{1}{2}\right]$$
$$= \left[\frac{F_n + \sqrt{5}\,F_n + 1}{2}\right] = \left[\frac{F_n + 1 + \sqrt{5F_n^2}}{2}\right].$$

This corollary shows that we can compute F_{n+1} from F_n without using either n or α.

We could prove in a similar way the corresponding theorem and corollary for Lucas numbers.

THEOREM IV

$$L_{n+1} = [\alpha L_n + \tfrac{1}{2}], \qquad n \geq 4.$$

COROLLARY*

$$L_{n+1} = \left[\frac{L_n + 1 + \sqrt{5L_n^2}}{2}\right], \qquad n \geq 4.$$

* A slightly different form of these corollaries appears as Theorem 4 in V. E. Hoggatt, Jr., and D. A. Lind, "The Heights of the Fibonacci Polynomials and an Associated Function," *The Fibonacci Quarterly*, Vol. 5, No. 2 (April, 1967), page 144.

PROBLEM 4

Given that 610 is a Fibonacci number, use the Corollary to Theorem III to find the next one.

Solution.

$$F_{n+1} = \left[\frac{610 + 1 + \sqrt{5(610)^2}}{2}\right]$$
$$= \left[\frac{611 + \sqrt{1860500}}{2}\right]$$
$$= \left[\frac{611 + 1364.0}{2}\right]$$
$$= \left[\frac{1975.0}{2}\right] = 987$$

You can check this value in the list on page 83.

Alternatively, you may compute as follows:

$$F_{n+1} = \left[\frac{610 + 1 + \sqrt{5}\,(610)}{2}\right]$$
$$= \left[\frac{611 + (2.236)(610)}{2}\right]$$
$$= \left[\frac{611 + 1363.96}{2}\right]$$
$$= \left[\frac{1974.96}{2}\right] = 987$$

For larger Fibonacci numbers you may need to find $\sqrt{5}$ to more decimal places.

EXERCISES

In Exercises 1–4, use four-place tables of logarithms except for $\log \alpha = 0.20898$.

1. Find F_{12}, using the method of Problem 1.
2. Find L_{12}, using the method of Problem 3.
3. Find the first three digits of F_{34}, using the method of Problem 2.
4. Find the first three digits of L_{33}.
5. Given that 1597 is a Fibonacci number, find the next one.
6. Given that 2207 is a Lucas number, find the next one.