5 • *Some Fibonacci Algebra*

Recall from Section 3 that $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of

(F) $$x^2 - x - 1 = 0,$$

and so $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. Also, $\alpha + \beta = 1$ and $\alpha - \beta = \sqrt{5}$. Moreover,

(A) $$\alpha^{n+2} = \alpha^{n+1} + \alpha^n$$

and

(B) $$\beta^{n+2} = \beta^{n+1} + \beta^n$$

and by using these equations, we found that the Fibonacci numbers can be expressed in the so-called Binet form:

(C) $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \qquad n = 1, 2, 3, \ldots$$

Now suppose that we add the members of equation (B) to the members of equation (A), giving

$$(\alpha^{n+2} + \beta^{n+2}) = (\alpha^{n+1} + \beta^{n+1}) + (\alpha^n + \beta^n).$$

If we let $u_n = \alpha^n + \beta^n$, then we have

$$u_{n+2} = u_{n+1} + u_n$$

and

$$u_1 = \alpha + \beta = 1,$$
$$u_2 = \alpha^2 + \beta^2 = \alpha + 1 + \beta + 1 = (\alpha + \beta) + 2 = 1 + 2 = 3.$$

Thus, this sequence u_n is the sequence of Lucas numbers defined in Section 2, and so we have a Binet form for the Lucas numbers:

(D) $$L_n = \alpha^n + \beta^n, \qquad n = 1, 2, 3, \ldots$$

Now look at the following comparison of the Fibonacci numbers and the Lucas numbers:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	...
1	1	2	3	5	8	13	21	34	55	...
1	3	4	7	11	18	29	47	76	123	...
L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	...

Notice that

$$F_1 + F_3 = L_2, \quad F_2 + F_4 = L_3, \quad \text{and so on.}$$

It can be proved (Exercise 2, page 29) that in general

$$L_n = F_{n-1} + F_{n+1},$$

from which, since $F_{n+1} = F_n + F_{n-1}$, it follows that

(E) $$L_n = F_n + 2F_{n-1}.$$

You can verify this latter statement for specific examples; that is, you can show that $L_6 = F_6 + 2F_5$, and so on.

We now have F_n and L_n expressed in terms of α^n and β^n. We can also find α^n and β^n in terms of F_n and L_n. If we note that $\alpha - \beta = \sqrt{5}$, then, from the Binet forms, we have

$$\sqrt{5}\, F_n = \alpha^n - \beta^n,$$
$$L_n = \alpha^n + \beta^n.$$

Adding, we find

$$2\alpha^n = L_n + \sqrt{5}\, F_n$$

or

$$\alpha^n = \frac{L_n + \sqrt{5}\, F_n}{2};$$

subtracting, we find

$$\beta^n = \frac{L_n - \sqrt{5}\, F_n}{2}.$$

Recall that in Section 2 we had occasion to define F_0 as $F_2 - F_1$ (page 6). Similarly, we can define L_0 as $L_2 - L_1 = 3 - 1 = 2$. (Notice that this agrees with the definition by the Binet form, since $\alpha^0 + \beta^0 = 1 + 1 = 2$.) Since $L_1 = \alpha + \beta = 1$, we can now write expression (E) for L_n (above) as

(E′) $$L_n = L_1 F_n + L_0 F_{n-1}.$$

The method of defining F_0 and L_0 suggests that we can also define F_{-1}, L_{-1}, and so on, by applying the formulas

$$F_{n-1} = F_{n+1} - F_n$$

and

$$L_{n-1} = L_{n+1} - L_n$$

repeatedly. Thus, we have:

...	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	...
...	−3	2	−1	1	0	1	1	2	3	...
...	7	−4	3	−1	2	1	3	4	7	...
...	L_{-4}	L_{-3}	L_{-2}	L_{-1}	L_0	L_1	L_2	L_3	L_4	...

We can derive a formula for F_{-n}, $n > 0$, by assuming that the Binet form also holds for negative values of the exponents (compare derivation on pages 10–11):

$$F_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{\left(\frac{1}{\alpha}\right)^n - \left(\frac{1}{\beta}\right)^n}{\alpha - \beta}$$

Since $\alpha\beta = -1$ (Exercise 3c, page 13), we have

$$\frac{1}{\alpha} = -\beta \quad \text{and} \quad \frac{1}{\beta} = -\alpha.$$

Therefore

$$F_{-n} = \frac{(-\beta)^n - (-\alpha)^n}{\alpha - \beta} = \frac{(-1)^n(\beta^n - \alpha^n)}{\alpha - \beta} = (-1)^{n+1}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right),$$

and so

$$F_{-n} = (-1)^{n+1}F_n.$$

Similarly, you can show (Exercise 3, page 29) that for $n > 0$,

$$L_{-n} = (-1)^n L_n.$$

Now suppose that we compute the first 14 successive ratios $\frac{F_{n+1}}{F_n}$ and $\frac{L_{n+1}}{L_n}$. The values of the successive ratios as shown at the top of the next page suggest that in both cases the value of the ratio becomes closer and closer to α as we take larger and larger values of n. However, we shall not undertake to prove this here. We can also observe that the first Fibonacci ratio is less than α, the second is greater than α, and so on, while the first Lucas ratio is greater than α, the second is less than α, and so on. Moreover,

$$\frac{F_2}{F_1} < \alpha < \frac{L_2}{L_1}, \qquad \frac{F_3}{F_2} > \alpha > \frac{L_3}{L_2}, \qquad \text{and so on.}$$

$\frac{F_{n+1}}{F_n}$	$\frac{L_{n+1}}{L_n}$
$\frac{1}{1} = 1.0000$	$\frac{3}{1} = 3.0000$
$\frac{2}{1} = 2.0000$	$\frac{4}{3} \doteq 1.3333$
$\frac{3}{2} = 1.5000$	$\frac{7}{4} = 1.7500$
$\frac{5}{3} \doteq 1.6667$	$\frac{11}{7} \doteq 1.5714$
$\frac{8}{5} = 1.6000$	$\frac{18}{11} \doteq 1.6363$
$\frac{13}{8} = 1.6250$	$\frac{29}{18} \doteq 1.6111$
$\frac{21}{13} \doteq 1.6154$	$\frac{47}{29} \doteq 1.6207$
$\frac{34}{21} \doteq 1.6190$	$\frac{76}{47} \doteq 1.6170$
$\frac{55}{34} \doteq 1.6176$	$\frac{123}{76} \doteq 1.6184$
$\frac{89}{55} \doteq 1.6182$	$\frac{199}{123} \doteq 1.6179$
$\frac{144}{89} \doteq 1.6180$	$\frac{322}{199} \doteq 1.6181$
$\frac{233}{144} \doteq 1.6181$	$\frac{521}{322} \doteq 1.6180$
$\frac{377}{233} \doteq 1.6180$	$\frac{843}{521} \doteq 1.6180$
$\frac{610}{377} \doteq 1.6180$	$\frac{1364}{843} \doteq 1.6180$

$$\alpha \doteq 1.61803398875\ldots$$

EXERCISES

Using the Binet form:

1. Show that $F_{2n} = F_n L_n$, $n \geq 1$.
2. Show that $L_n = F_{n-1} + F_{n+1}$, $n \geq 1$. *Hint:* Use $\alpha\beta = -1$.
3. Show that $L_{-n} = (-1)^n L_n$ for $n > 0$.
4. Show that $5F_n^2 = L_{2n} - 2(-1)^n$.
5. Show that $L_n^2 = L_{2n} + 2(-1)^n$.
6. Show that $F_{n+1}L_n - L_{n+1}F_n = 2(-1)^n$.

Using previously found results:

7. Show that $\dfrac{F_{n+1}}{F_n} - \dfrac{L_{n+1}}{L_n} = \dfrac{2(-1)^n}{F_{2n}}$.
8. Show that $F_{-n} = (-1)^{n+1}F_n$ holds also when $n < 0$.
9. Show that $L_{-n} = (-1)^n L_n$ holds also when $n < 0$.
10. Show that $5F_n^2 = L_n^2 - 4(-1)^n$.
11. Show from $L_n = F_{n+1} + F_{n-1}$ that $L_n = F_{n+2} - F_{n-2}$.