10 • *Selected Identities Involving the Fibonacci and Lucas Numbers*

A great many identities have been discovered that reveal interesting patterns of relationship between the Fibonacci and Lucas numbers.

In Section 2 we developed our first Fibonacci identity:

$$(\mathrm{I}_1) \qquad F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1, \quad n \geq 1$$

We shall now give several other proofs of this.

Derivation of (I_1) *from the definition.* Since for all positive integers n the Fibonacci recurrence formula

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 1,$$

holds, we may write the sequence of equations:

$$\begin{aligned} F_1 &= F_3 - F_2 \\ F_2 &= F_4 - F_3 \\ F_3 &= F_5 - F_4 \\ &\cdots\cdots\cdots \\ F_{n-1} &= F_{n+1} - F_n \\ F_n &= F_{n+2} - F_{n+1} \end{aligned}$$

By adding these term by term and noting the cancellations on the right, we find

$$\begin{aligned} F_1 + F_2 + F_3 + \cdots + F_n &= F_{n+2} - F_2 \\ &= F_{n+2} - 1. \end{aligned}$$

Derivation of (I_1) *using the Binet form.* Since

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

we have

$$F_1 + F_2 + \cdots + F_n = \frac{\alpha - \beta}{\alpha - \beta} + \frac{\alpha^2 - \beta^2}{\alpha - \beta} + \cdots + \frac{\alpha^n - \beta^n}{\alpha - \beta}$$
$$= \frac{(\alpha + \alpha^2 + \cdots + \alpha^n) - (\beta + \beta^2 + \cdots + \beta^n)}{\alpha - \beta},$$

or

$$F_1 + F_2 + \cdots + F_n$$
$$= \frac{1}{\alpha - \beta}[(\alpha + \alpha^2 + \cdots + \alpha^n) - (\beta + \beta^2 + \cdots + \beta^n)].$$

But

$$\alpha + \alpha^2 + \cdots + \alpha^n = (1 + \alpha + \alpha^2 + \cdots + \alpha^n) - 1,$$
$$\beta + \beta^2 + \cdots + \beta^n = (1 + \beta + \beta^2 + \cdots + \beta^n) - 1.$$

From the formula for the sum of a geometric progression (recalled on page 49), we see that

$$1 + \alpha + \alpha^2 + \cdots + \alpha^n = \frac{1 - \alpha^{n+1}}{1 - \alpha},$$
$$1 + \beta + \beta^2 + \cdots + \beta^n = \frac{1 - \beta^{n+1}}{1 - \beta}.$$

Therefore,

$$F_1 + F_2 + \cdots + F_n$$
$$= \frac{1}{\alpha - \beta}\left[\left(\frac{1 - \alpha^{n+1}}{1 - \alpha} - 1\right) - \left(\frac{1 - \beta^{n+1}}{1 - \beta} - 1\right)\right],$$

and since $1 - \alpha = \beta$ and $1 - \beta = \alpha$, we have

$$F_1 + F_2 + \cdots + F_n = \frac{\alpha(1 - \alpha^{n+1}) - \beta(1 - \beta^{n+1})}{\alpha\beta(\alpha - \beta)}.$$

Since $\alpha\beta = -1$, we have

$$F_1 + F_2 + \cdots + F_n = \frac{\alpha - \alpha^{n+2} - \beta + \beta^{n+2}}{(-1)(\alpha - \beta)}$$
$$= \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} - \frac{\alpha - \beta}{\alpha - \beta}$$
$$= F_{n+2} - 1.$$

Proof of (I_1) *by mathematical induction.* We wish to prove

$$P(n):\ F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$

There are two parts to the proof:

1. The statement $P(1)$ is found true by trial.
   Here $P(1)$ is $F_1 = F_3 - 1$, which is true since $1 = 2 - 1$.
2. One proves: If $P(k)$ is true for some integer k ($k \geq 1$), then $P(k + 1)$ must also be true.
   In this case, we assume

   $$P(k):\ F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$$

   and must prove

   $$P(k + 1):\ F_1 + F_2 + \cdots + F_{k+1} = F_{k+3} - 1.$$

   By adding F_{k+1} to both sides of

   $$F_1 + F_2 + \cdots + F_k = F_{k+2} - 1,$$

   we have

   $$\begin{aligned} F_1 + F_2 + \cdots + F_k + F_{k+1} &= F_{k+2} + F_{k+1} - 1 \\ &= F_{k+3} - 1, \end{aligned}$$

   since $F_{k+3} = F_{k+2} + F_{k+1}$. Therefore $P(k + 1)$ is true, and the proof is complete by mathematical induction.

The first part of the proof, where $P(1)$ is verified by direct trial, is often called the *basis for induction.* Clearly, if no statements can be found to be true by trial, we have no basis for our attempted proof. The second part is often called the *inductive transition* or *inductive step.* The assumption that $P(k)$ is true for some integer k ($k \geq 1$) is the *inductive hypothesis.* If we can show that the inductive hypothesis is sufficient to prove that $P(k + 1)$ is true, then we have completed the inductive transition.

It is important that *both* part 1 and part 2 be satisfied.

The declaration, "The proof is complete by mathematical induction," following such a proof is a simple statement telling the reader what method has been used.

There is a corresponding identity for Lucas numbers:

(I_2) $$L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3, \quad n \geq 1$$

This is to be proved in Exercise 2, page 60.

As our next identity, let us consider:

$$(I_3) \qquad F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}, \quad n \geq 1$$

We shall prove this by mathematical induction. Here we have

$$P(n): F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

Then

$$P(1): F_1^2 = F_1 F_2$$

is easily seen to be true, since $(1)^2 = (1)(1)$. Thus, we have completed the *basis for induction*.

Now we *suppose* that

$$P(k): F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1}$$

is true (the *inductive hypothesis*), and we undertake to prove:

$$P(k+1): F_1^2 + F_2^2 + \cdots + F_{k+1}^2 = F_{k+1} F_{k+2}$$

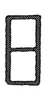
To do so, add F_{k+1}^2 to both sides of equality $P(k)$, obtaining

$$(F_1^2 + F_2^2 + \cdots + F_k^2) + F_{k+1}^2 = F_k F_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1})$$
$$= F_{k+1} F_{k+2}.$$

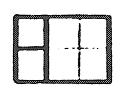
Therefore, we have shown that if $P(k)$ is true, then $P(k+1)$ is true, and we have completed the *inductive transition*. The proof is complete by mathematical induction.

Formula (I_3) may be pictured geometrically as follows:

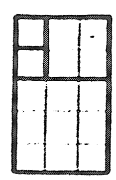
$$F_1^2 + F_2^2 = F_2 \times F_3$$
$$(1)^2 + (1)^2 = 1 \times 2$$



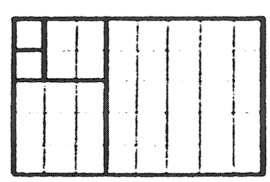
$$F_1^2 + F_2^2 + F_3^2 = F_3 \times F_4$$
$$(1)^2 + (1)^2 + (2)^2 = 2 \times 3$$



$$F_1^2 + F_2^2 + F_3^2 + F_4^2 = F_4 \times F_5$$
$$(1)^2 + (1)^2 + (2)^2 + (3)^2 = 3 \times 5$$



$$F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 = F_5 \times F_6$$
$$(1)^2 + (1)^2 + (2)^2 + (3)^2 + (5)^2 = 5 \times 8$$



There is a corresponding identity for Lucas numbers:

$$(I_4) \qquad L_1^2 + L_2^2 + L_3^2 + \cdots + L_n^2 = L_n L_{n+1} - 2, \quad n \geq 1$$

This is to be proved in Exercise 5, page 60.

Now consider the following pair of identities (to be proved in Exercises 6 and 8, page 60):

(I$_5$) $$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}, \quad n \geq 1$$

(I$_6$) $$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1, \quad n \geq 1$$

Verify by adding term by term that the sum of (I$_5$) and (I$_6$) is the same as (I$_1$) with n replaced by $2n$.

In Section 5 we found two interesting identities connecting the Fibonacci and Lucas numbers:

(I$_7$) $$F_{2n} = F_n L_n, \quad n \geq 1$$

(I$_8$) $$L_n = F_{n-1} + F_{n+1}, \quad n \geq 1$$

These were to be proved in Exercises 1 and 2, page 29. We also have:

(I$_9$) $$F_n = \tfrac{1}{5}(L_{n-1} + L_{n+1}), \quad n \geq 1$$

This can be proved by applying (I$_8$) (see Exercise 12, page 61).

From (I$_7$) and (I$_8$) we can derive:

(I$_{10}$) $$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \quad n \geq 1$$

See Exercise 13, page 61.

By writing

$$F_{2n+1} = F_{2n+2} - F_{2n} = F_{2(n+1)} - F_{2n}$$

and applying (I$_{10}$), we can derive:

(I$_{11}$) $$F_{2n+1} = F_{n+1}^2 + F_n^2, \quad n \geq 1$$

See Exercise 14, page 61.

Also in Section 5 we found this identity:

(I$_{12}$) $$5F_n^2 = L_n^2 - 4(-1)^n$$

This was to be proved in Exercise 10 on page 29.

Let us now look again at the Fibonacci sequence:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	$F_{10}\ldots$
1	1	2	3	5	8	13	21	34	55 ...

Notice that:

$$F_1F_3 - F_2^2 = 1$$
$$F_2F_4 - F_3^2 = -1$$
$$F_3F_5 - F_4^2 = 1$$
$$F_4F_6 - F_5^2 = -1$$

and so on.

In general, we may write:

$$(I_{13}) \qquad F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \quad n \geq 1$$

We shall prove this by mathematical induction, starting with $n = 1$.

$$P(1)\colon F_0F_2 - F_1^2 = 0 \cdot 1 - 1^2 = -1$$

We have established the *inductive basis.*

Assume for some integer $k \geq 1$ that

$$P(k)\colon F_{k+1}F_{k-1} - F_k^2 = (-1)^k.$$

Let us look for $P(k+1)$:

$$\begin{aligned} F_{k+2}F_k - F_{k+1}^2 &= (F_{k+1} + F_k)F_k - F_{k+1}^2 \\ &= F_k^2 + F_{k+1}(F_k - F_{k+1}) \\ &= F_k^2 - F_{k+1}F_{k-1} \\ &= (-1)(F_{k+1}F_{k-1} - F_k^2) \\ &= (-1)(-1)^k = (-1)^{k+1} \end{aligned}$$

Thus,

$$P(k+1)\colon F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}$$

is true. The proof is complete by mathematical induction.

The mathematician Charles Lutwidge Dodgson, whose pen name was Lewis Carroll, liked to entertain his friends with puzzles. According to his nephew,* one of his favorites was this geometrical paradox. Let us cut the square on the left in Figure 18 as marked and reassemble the pieces to form the rectangle on the right.

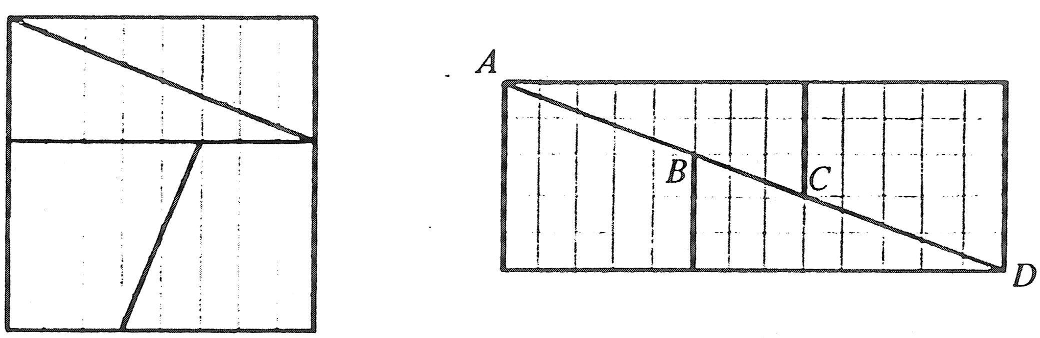


Figure 18

The area of the square is $8 \times 8 = 64$, while the area of the rectangle is $5 \times 13 = 65$. What happened?

* *Diversions and Digressions of Lewis Carroll*, edited by Stuart Dodgson Collingwood (New York: Dover Publications, Inc., 1961), pp. 316–317.

If you draw the diagrams pictured in Figure 18 making the square units rather large, you can discover what happened. The points A, B, C, and D, which in the small figure appear to lie on a straight line, actually are the vertices of a parallelogram, the area of which is the unexpectedly added unit. If you make the unit squares 1 inch on a side, the height of the added parallelogram will be nearly $\frac{1}{8}$ inch (see Exercise 15, page 61).

How is such a puzzle discovered? Have you noticed that the numbers 5, 8, and 13 are the consecutive Fibonacci numbers F_5, F_6, and F_7? Therefore from identity (I_{13}) we have

$$5 \cdot 13 - 8^2 = (-1)^6,$$

or

$$65 - 64 = 1.$$

Any triple of Fibonacci numbers such that the middle one has an even subscript (index) will produce this difference of 1. However, the larger the numbers used, the less noticeable is the added parallelogram.

In Figure 19 the situation is shown schematically with the parallelogram exaggerated:

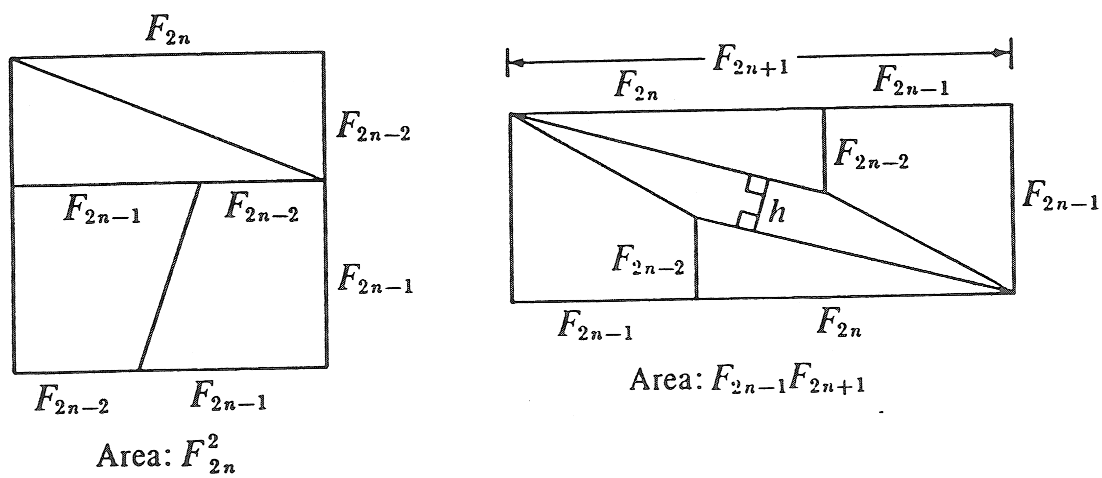


Figure 19

Since

$$F_{2n-1}F_{2n+1} - F_{2n}^2 = (-1)^{2n} = 1,$$

the area of the parallelogram is one square unit. The greatest width, the height h, of the parallelogram is easily found. Since the area equals the product of the height and the length of the base, we have

$$1 = h\sqrt{F_{2n}^2 + F_{2n-2}^2} \quad \text{(by the Pythagorean theorem)},$$

or

$$h = \frac{1}{\sqrt{F_{2n}^2 + F_{2n-2}^2}}.$$

A great many more identities have been discovered. The following additional list will indicate the various types.

(I_{14}) $$L_n = F_{n+2} - F_{n-2}$$

(I_{15}) $$L_{4n} + 2 = L_{2n}^2$$

(I_{16}) $$L_{4n} - 2 = 5F_{2n}^2$$

(I_{17}) $$L_{4n+2} + 2 = 5F_{2n+1}^2$$

(I_{18}) $$L_{4n+2} - 2 = L_{2n+1}^2$$

(I_{19}) $$F_{n-k}F_{n+k} - F_n^2 = (-1)^{n+k+1}F_k^2$$

(I_{20}) $$L_{n-k}L_{n+k} - L_n^2 = 5(-1)^{n+k}F_k^2$$

(I_{21}) $$F_{n+p} + F_{n-p} = F_nL_p, \quad p \text{ even}$$

(I_{22}) $$F_{n+p} + F_{n-p} = L_nF_p, \quad p \text{ odd}$$

(I_{23}) $$F_{n+p} - F_{n-p} = F_nL_p, \quad p \text{ odd}$$

(I_{24}) $$F_{n+p} - F_{n-p} = L_nF_p, \quad p \text{ even}$$

(I_{25}) $$F_{n+p}^2 - F_{n-p}^2 = F_{2n}F_{2p}$$

(I_{26}) $$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$$

(I_{27}) $$L_{m+n+1} = F_{m+1}L_{n+1} + F_mL_n$$

(I_{28}) $$F_nL_{n+k} - F_{n+k}L_n = 2(-1)^{n+1}F_k$$

(I_{29}) $$F_kF_{k+1}F_{k+3}F_{k+4} = F_{k+2}^4 - 1$$

(I_{30}) $$L_{2n}L_{2n+2} - 1 = 5F_{2n+1}^2$$

(I_{31}) $$L_nL_{n+1} - L_{2n+1} = (-1)^n$$

(I_{32}) $$5F_n = L_{n+2} - L_{n-2}$$

(I_{33}) $$L_n^2 + 4L_{n-1}L_{n+1} = 25F_n^2$$

(I_{34}) $$L_{n-1}L_{n+1} + F_{n-1}F_{n+1} = 6F_n^2$$

(I_{35}) $$F_n^2 + 4F_{n-1}F_{n+1} = L_n^2$$

(I_{36}) $$F_{n+1}^2 - 4F_nF_{n-1} = F_{n-2}^2$$

(I_{37}) $$F_{m-1}F_{m+1} - F_{m+2}F_{m-2} = 2(-1)^m$$

(I_{38}) $$L_nF_{m-n} + F_nL_{m-n} = 2F_m$$

(I_{39}) $$F_{2m}^2 = 5F_m^4 + 4(-1)^mF_m^2$$

(I_{40}) $$\sum_{k=1}^{n} kF_k = (n+1)F_{n+2} - F_{n+4} + 2$$

(I41) $$\sum_{k=0}^{2n} \binom{2n}{k} F_{2k+p} = 5^n F_{2n+p}$$

(I42) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_{2k+p} = 5^n L_{2n+p+1}$$

(I43) $$L_m L_n + L_{m+1} L_{n+1} = 5F_{m+n+1}$$

(I44) $$\left(\frac{L_p + \sqrt{5}\, F_p}{2}\right)^n = \frac{L_{np} + \sqrt{5}\, F_{np}}{2}$$

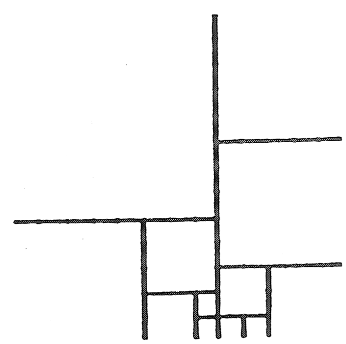
(I45) $$\sum_{k=0}^{2n+2} \binom{2n+2}{k} F_k^2 = 5^n L_{2n+2}$$

(I46) $$\sum_{k=0}^{2n+2} \binom{2n+2}{k} L_k^2 = 5^{n+1} L_{2n+2}$$

(I47) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k} F_k^2 = 5^n F_{2n+1}$$

EXERCISES

1. Write (I_1) using summation notation for the left-hand member.
2. Prove (I_2) by mathematical induction.
3. Write (I_3) using summation notation for the left-hand member.
4. The diagram at the right illustrates the statement
$$F_7^2 = F_6^2 + 3F_5^2 + 2(F_4^2 + F_3^2 + F_2^2 + F_1^2).$$
Use (I_3) to prove that, in general,
$$F_{n+1}^2 = F_n^2 + 3F_{n-1}^2 + 2(F_{n-2}^2 + \cdots + F_1^2).$$
Show that the number of squares is $2n$.



5. Prove (I_4) by mathematical induction.
6. Derive (I_5) by using the sequence of equations $F_1 = F_2$, $F_3 = F_4 - F_2$, $F_5 = F_6 - F_4$, and so on.
7. Write (I_5) using summation notation for the left-hand member.
8. Prove (I_6) by mathematical induction.
9. Write (I_6) using summation notation for the left-hand member.

10. Find and prove an identity for Lucas numbers corresponding to (I_5).

11. Find and prove an identity for Lucas numbers corresponding to (I_6).

12. Use (I_8) to prove (I_9).

13. Use (I_7) and (I_8) to prove (I_{10}).

14. Use (I_{10}) to prove (I_{11}).

15. Verify that if the small squares in Figure 18 are drawn 1 inch square on a side, the height h of the parallelogram is less than $\frac{1}{8}$ inch.

16. On graph paper, carefully draw a square with each side $F_8 = 21$ units, cut it as indicated in Figure 19, and assemble the pieces as a rectangle as shown there. (You will be able to surprise your friends with the result.)

17. Recall the definition of a generalized Fibonacci sequence on page 7. Show by mathematical induction that if $H_1 = p$, $H_2 = q$, and $H_{n+2} = H_{n+1} + H_n$, $n \geq 1$, then $H_{n+2} = qF_{n+1} + pF_n$.

18. Using (I_{26}) on page 59, show by mathematical induction that F_n divides F_{nk}, $k > 0$.
Hint: Let $m + 1 = nk$. Then (I_{26}) becomes $F_{(k+1)n} = F_{kn}F_{n+1} + F_{kn-1}F_n$.