Pattern Avoidance on k ary Heaps
(Work in Progress aren t they all)

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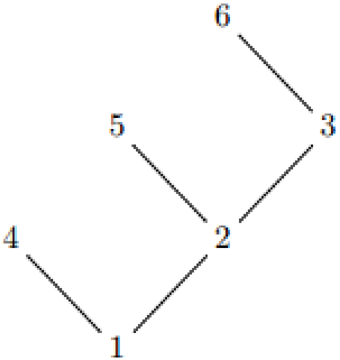
Permutation Patterns - July 10, 2014

Once upon a time. . .

Motivation

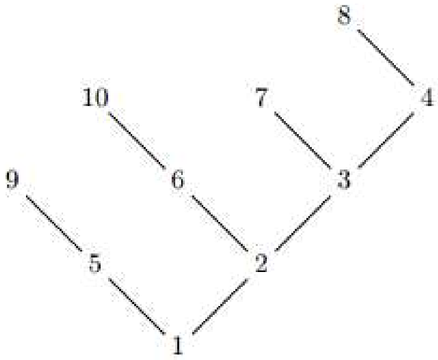
Sophia Yakoubov: Paris 2013

Sophia Yakoubov: Paris 2013   Pattern Avoidance on Combs



[1]

Motivation

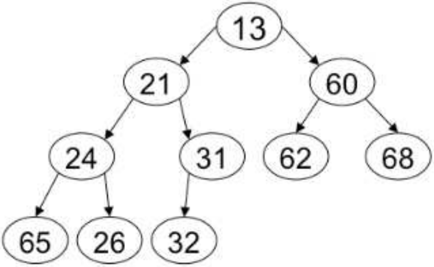


Something like combs, but not combs

Heaps!

Something like combs, but not combs

Heaps!



Heaps

Definition

A *complete binary tree* is a tree where each node has 2 or fewer children, all levels except possibly the last are completely full, and the last level has all its nodes to the left side.

Heaps

Definition

A *heap* is a complete binary tree labelled with $\{1, \ldots, n\}$ such that every child has a larger label than its parent.

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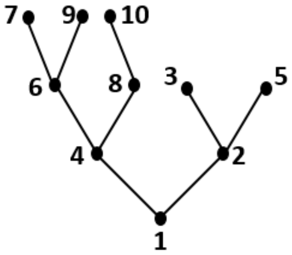
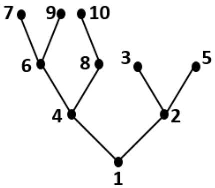


Where's the pattern?



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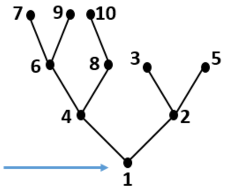

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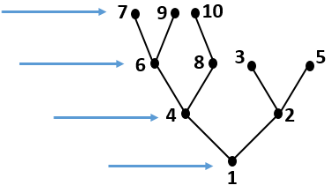
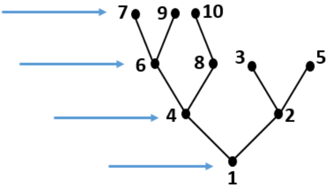


1 4 2 6 8 3 5 7 9 10

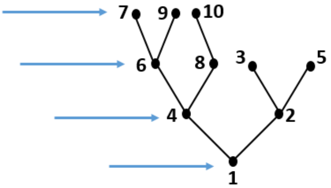
Where's the pattern?



1 4 2 6 8 3 5 7 9 10

Notice this heap contains 123, 132, 213, 231, 312.

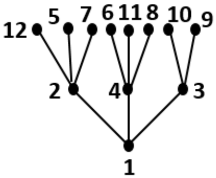
Where's the pattern?



1 4 2 6 8 3 5 7 9 10

Notice this heap contains 123, 132, 213, 231, 312.
But it avoids 321.

k ary Heaps



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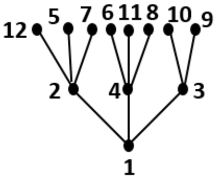
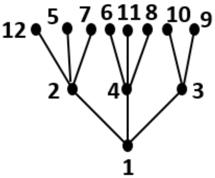


1 2 4 3 12 5 7 6 11 8 10 9

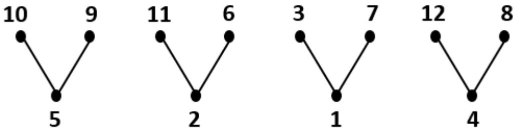
k ary Heaps



1 2 4 3 12 5 7 6 11 8 10 9

This heap avoids 231.

Forests of Heaps



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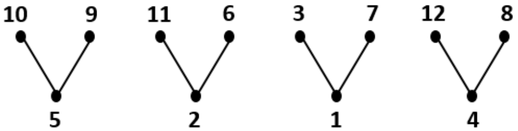

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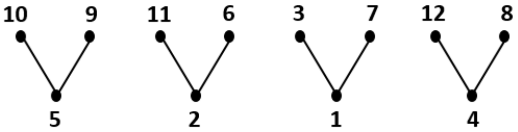

5 10 9 2 11 6 1 3 7 4 12 8

Notice each tree avoids 213, 231, 312, 321, but the forest contains 213, 231, 312, 321.

Heaps Avoiding:	Sequence	OEIS#
$\emptyset$	$1, 1, 2, 3, 8, 20, 80, 210, 896 \ldots$	
123	$1, 1, 1, 0, 0, 0, 0, 0 \ldots$	
132	$1, 1, 1, 1, 1, 1, 1, 1 \ldots$	
213	$1, 1, 2, 2, 5, 5, 14, 14, 42 \ldots$	
$231 = 312$	$1, 1, 2, 3, 7, 14, 37, 80, 222, \ldots$	
321	$1, 1, 2, 3, 7, 16, 45, 111, 318 \ldots$	
$\{213, 231\} = \{213, 312\}$	$1, 1, 2, 2, 4, 4, 8, 8, 16 \ldots$	
$\{213, 321\}$	$1, 1, 2, 2, 4, 4, 7, 7, 11 \ldots$	
$\{231, 312\} = \{231, 321\}$	$1, 1, 2, 3, 6, 11, 22, 42, 84 \ldots$	
$\{231, 312, 321\}$	$1, 1, 2, 3, 5, 8, 13 \ldots$	

Crunch the numbers, cross your fingers

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213	$1, 1, 2, 2, 5, 5, 14, 14, 42 \ldots$	$A208355$
$231 = 312$	$1, 1, 2, 3, 7, 14, 37, 80, 222, \ldots$	
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Heaps Avoiding $(231, 312)$

Narayana-Zidek-Capell Numbers:
$1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, \ldots$

Given by relation:
$a_n = 2a_{n-1}$ if n even
$a_n = 2a_{n-1} - a_{\frac{n-1}{2}}$ if n odd

Heaps Avoiding $(231, 312)$

Narayana-Zidek-Capell Numbers:
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Insertion argument:

Heaps Avoiding $(231, 312)$

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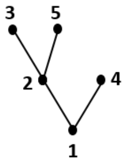

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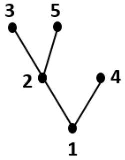

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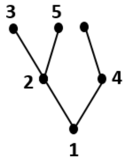

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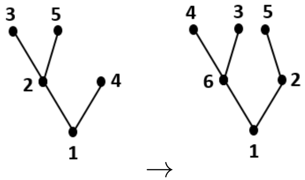

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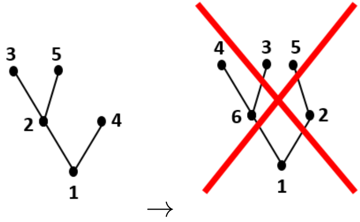

Heaps Avoiding $(231, 312)$

Lemma

The vertex labelled n is always a leaf. After insertion, the vertex labelled $n+1$ is always a leaf.

Lemma

In order to avoid 231 and 312, $n+1$ must be inserted directly before n or at the end.

Heaps Avoiding $(231, 312)$: $n + 1$ must be right before n, or at end

Proof of Lemma:

- $n + 1$ at last leaf: OK
- $n + 1$ right before n: OK

Heaps Avoiding $(231, 312)$: $n+1$ must be right before n, or at end

Proof of Lemma:

- If $n+1$ is inserted more before n, we create a 312.

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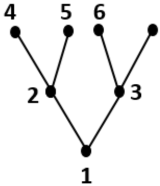

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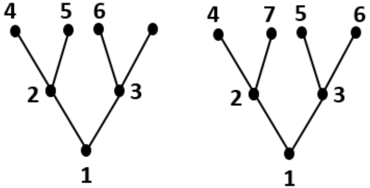

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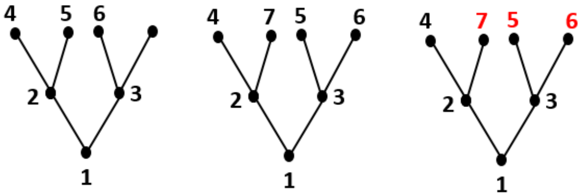

Heaps Avoiding $(231, 312)$: $n+1$ must be right before n, or at end

Proof of Lemma:

- If $n+1$ is inserted after n, but not at the end, we create a 231.

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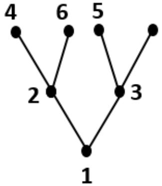

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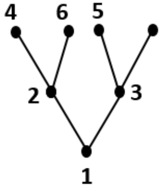

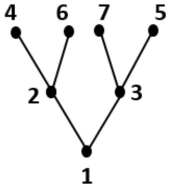

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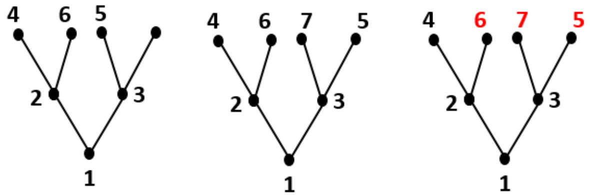

Easy Case: n is even. So the new leaf is the sibling of a current leaf. Internal nodes stay internal, leaves stay leaves.

Heaps Avoiding $(231, 312)$

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We can put $n+1$ at the (new) last leaf, or we can insert it right before n and push everything along.

Heaps Avoiding $(231, 312)$

Easy Case: n is even. So the new leaf is the sibling of a current leaf. Internal nodes stay internal, leaves stay leaves.

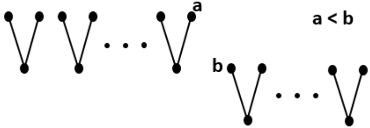
We can put $n+1$ at the (new) last leaf, or we can insert it right before n and push everything along.

$a_n = 2a_{n-1}$.

Second Case: n is odd. So the new leaf is child of a former leaf. Is it possible that we push a small label to be a child over a larger label that was a leaf?

Inserting $n+1$ at the last leaf:
Still OK!

Inserting $n+1$ anywhere except the first or last leaf:



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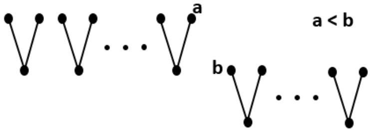

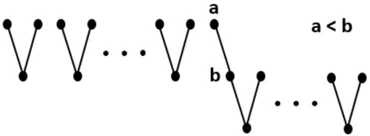

Heaps Avoiding $(231, 312)$

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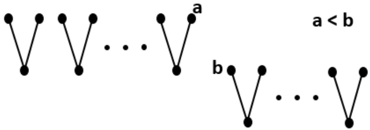
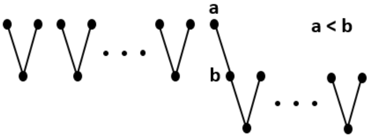
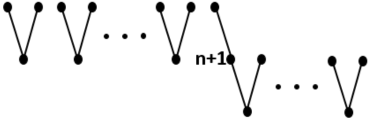




We must already have had a 231, namely $b\ n\ a$.

Inserting $n+1$ at the first leaf:



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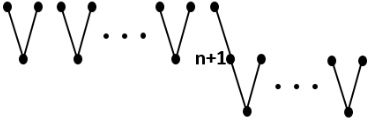

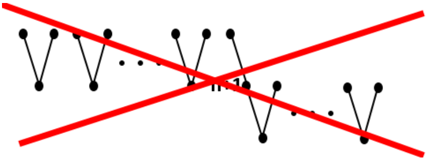

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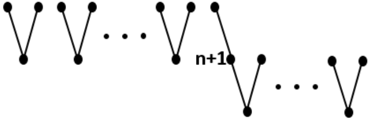

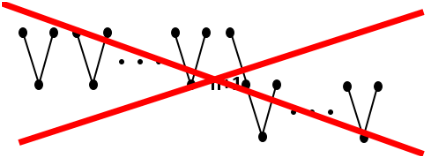

We don't want to count this, even though it may still avoid 231, 312.

Heaps Avoiding $(231, 312)$

How many shouldn't we count?
Since n was on the first leaf, all other leaf labels are in decreasing order.

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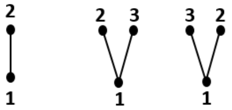
Thus $a_n = 2a_{n-1} - a_{\frac{n-1}{2}}$ when n is odd.
So we have the same recurrence relation as the Narayana-Zidek-Capell numbers.

Test yourself!

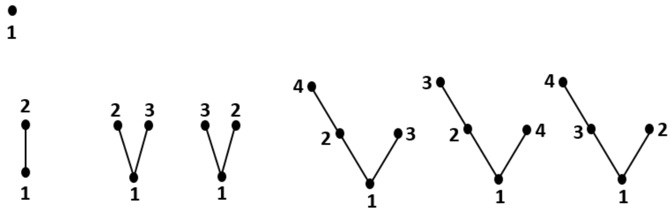
Heaps Avoiding 231

Heaps Avoiding 231

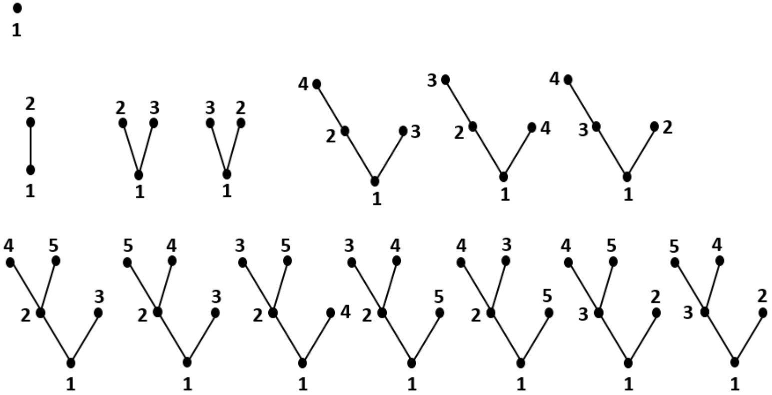
Heaps Avoiding 231



Heaps Avoiding 231



Heaps Avoiding 231



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the on-line encyclopedia of integer sequences®

founded in 1964 by N. J. A. Sloane

Many excellent designs for a new banner were submitted. We will use the best of them in rotation.

1,1,2,3,7,14, 37, 80, 222 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,1,2,3,7,14,37,80,222**

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

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Let's look at where the n appears. (Definitely on a leaf)

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All the labels before n are less than all labels after n.

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The labels on the leaves after n can be arranged in Catalan many ways.

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The subheap before n is a smaller case of a heap avoiding 231.

Let b_n be the number of heaps with n nodes avoiding 231.

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Let i be the number of leaves after n. $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

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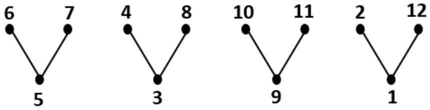
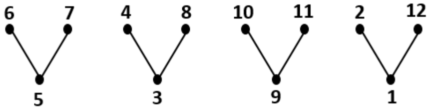
Let i be the number of leaves after n. $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

$b_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} C_i b_{n-i-1}$

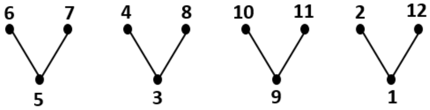
k ary forests avoiding 132





Obvious statements: Each tree avoids 132.

k ary forests avoiding 132



Obvious statements: Each tree avoids 132.
The roots of each tree avoid 132.

k ary forests avoiding 132

Lemma

Knowing the roots is enough!

k ary forests avoiding 132

Theorem

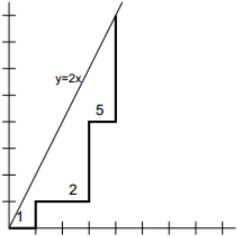
The number of forests of t heaps each with v vertices that avoid 132 is given by $\frac{1}{vt+1}\binom{(v+1)t}{t}$.

Theorem

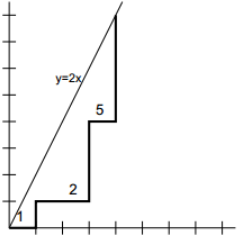
The number of forests of t heaps each with v vertices that avoid 132 is given by $\frac{1}{vt+1}\binom{(v+1)t}{t}$.

Proof method: Bijection to the number of paths under the line $y = vx$ from $(0, 0)$ to (t, vt) using steps $(0, 1)$ and $(1, 0)$.

Examples of bijection

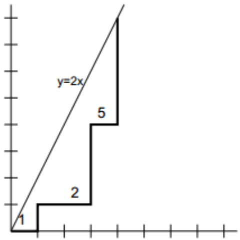


Examples of bijection



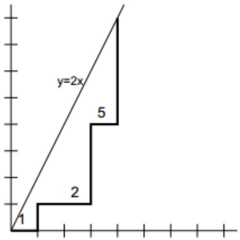
Make a string of the level step heights in reverse order: 4 1 1 0

Examples of bijection



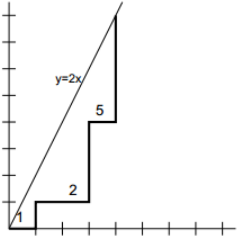
Make a string of the level step heights in reverse order: 4 1 1 0
Add one to each element of the string. 5 2 2 1

Examples of bijection

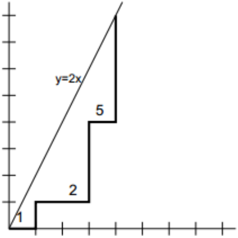


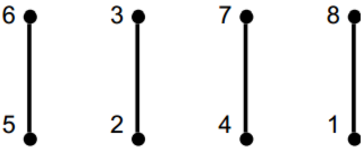
Make a string of the level step heights in reverse order: 4 1 1 0
Add one to each element of the string. 5 2 2 1
The smallest number currently unused in the forest that is greater than or equal to the next element of the string gives the next root.

Examples of bijection

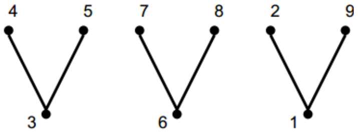


Examples of bijection

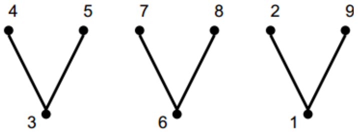




Examples of bijection



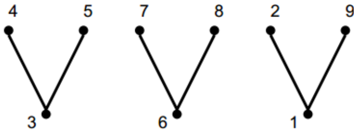
Examples of bijection



Make a string, starting with the root of the first heap.

3

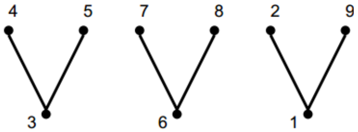
Examples of bijection



Make a string, starting with the root of the first heap.
Repeat the same number in the string for each root as long as the permutation is increasing.

$3 \rightarrow 3\ 3$

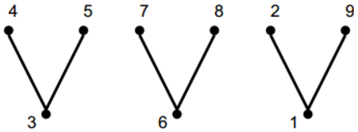
Examples of bijection



Make a string, starting with the root of the first heap.
Repeat the same number in the string for each root as long as the permutation is increasing.
If the permutation has a descent, the next root is the next entry in the string.

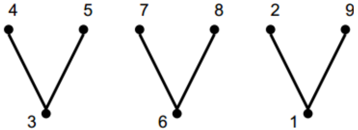
$$3 \longrightarrow 3\ 3 \longrightarrow 3\ 3\ 1$$

Examples of bijection

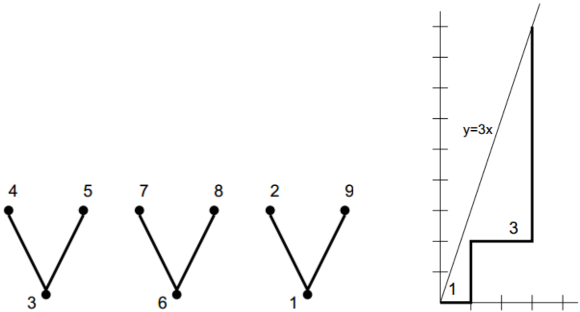


Make a string, starting with the root of the first heap.
Repeat the same number in the string for each root as long as the permutation is increasing.
If the permutation has a descent, the next root is the next entry in the string.
Subtract 1 from each element of the string. These are your sequence of level steps. 3$\rightarrow$ 3 3$\rightarrow$ 3 3 1$\rightarrow$ 2 2 0

Examples of bijection



Examples of bijection



Corollary

Let σ be a permutation of length nm composed of a concatenation of m increasing sequences of length n. The number of such σ that avoid 132 is $\frac{1}{nm+1}\binom{(n+1)m}{nm+1}$.

Summary

Heaps Avoiding:	Sequence	OEIS#
$\emptyset$	$1, 1, 2, 3, 8, 20, 80, 210, 896 \ldots$	$A056971$
123	$1, 1, 1, 0, 0, 0, 0, 0 \ldots$	$A000004$
132	$1, 1, 1, 1, 1, 1, 1, 1 \ldots$	$A000012$
213	$1, 1, 2, 2, 5, 5, 14, 14, 42 \ldots$	$A208355$
$231 = 312$	$1, 1, 2, 3, 7, 14, 37, 80, 222, \ldots$	Soon in OEIS!
321	$1, 1, 2, 3, 7, 16, 45, 111, 318 \ldots$	OPEN
$\{213, 231\} = \{213, 312\}$	$1, 1, 2, 2, 4, 4, 8, 8, 16 \ldots$	$A016116$
$\{213, 321\}$	$1, 1, 2, 2, 4, 4, 7, 7, 11 \ldots$	$A000124$
$\{231, 312\} = \{231, 321\}$	$1, 1, 2, 3, 6, 11, 22, 42, 84 \ldots$	$A002083$
$\{231, 312, 321\}$	$1, 1, 2, 3, 5, 8, 13 \ldots$	$A000045$

What about k ary?

Heaps Avoiding:	Sequence	OEIS#
$\emptyset$	$1, 1, 2, 3, 8, 20, 80, 210, 896\ldots$	$A056971$
123	$1, 1, 1, 0, 0, 0, 0, 0\ldots$	$A000004$
132	$1, 1, 1, 1, 1, 1, 1, 1\ldots$	$A000012$
213	$1, 1, 2, 2, 5, 5, 14, 14, 42\ldots$	$A208355$
$231 = 312$	$1, 1, 2, 3, 7, 14, 37, 80, 222, \ldots$	Soon to be in OEIS!
321	$1, 1, 2, 3, 7, 16, 45, 111, 318\ldots$	OPEN
$\{213, 231\} = \{213, 312\}$	$1, 1, 2, 2, 4, 4, 8, 8, 16\ldots$	$A016116$
$\{213, 321\}$	$1, 1, 2, 2, 4, 4, 7, 7, 11\ldots$	$A000124$
$\{231, 312\} = \{231, 321\}$	$1, 1, 2, 3, 6, 11, 22, 42, 84\ldots$	$A002083$
$\{231, 312, 321\}$	$1, 1, 2, 3, 5, 8, 13\ldots$	$A000045$

What next?

- Trees that aren't heaps

What next?

- Trees that aren't heaps
- Unary-binary, binary, k-ary

What next?

- Trees that aren't heaps
- Unary-binary, binary, k-ary
- Slightly different question: How many permutations avoid σ can be realized as trees?

Thank You!

- Anant Godbole
- Permutation Patterns 2014 Organizers
- UWEC Department of Mathematics
- UWEC Office of Research and Sponsored Programs

References

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[2] Lewis, Joel. Pattern Avoidance for Alternating Permutations and Reading Words of Tableaux. Department of Mathematics MIT (2012): 1-69. June 2012. Web. 22 Aug. 2013

[3] R. Simion and F.W. Schmidt, Restricted Permutations, Europ. J. Comb., 6, 1985, 383-406.