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AROUND APÉRY'S CONSTANT

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Abstract

In this note we deal with some aspects of Apéry's constant $\zeta(3)$.

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Dedicated to Professor Gerd Baron on the occasion of his 65th birthday.

Contents

1. Introduction

Since Apéry's miraculous proof (1979) that the value $\zeta(3)$ of Rieman's ζ-function is irrational, *Apéry's constant* $\zeta(3)$ has been the focus of attention for many mathematicians. (An extensive list of results and references are found in Section 1.6 of the highly recommended encyclopedic book [2].)

It is the purpose of this note to extend some of these results. Thereby we will also obtain a new infinite sum rapidly converging to $\zeta(3)$.

At the end of this note we raise two questions for further investigation.

2. Two Multisums

Recently in [1] the proof of

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)}=2\zeta(3), \tag{2.1}$$

where $\zeta(3)=1.202056903\ldots$, was posed as a problem. Although this result was published earlier (see [2, p. 43]) it is worthwhile reconsidering in the following more general way.

Theorem 2.1. *For $r\geq 1$ the multisum*

$$S_r=\sum_{k_1=1}^{\infty}\cdots\sum_{k_r=1}^{\infty}\frac{1}{k_1\cdots k_r(k_1+\cdots+k_r)}$$

attains the value $r!\zeta(r+1)$.

Proof. Firstly we rewrite the multisum as an integral as follows

$$S_r=\sum_{k_1=1}^{\infty}\cdots\sum_{k_r=1}^{\infty}\frac{1}{k_1\cdots k_r}\int_0^1 x^{k_1+\cdots+k_r-1}dx$$

that is (upon interchanging of summation and integration),

$$S_r=\int_0^1\frac{1}{x}\sum_{k_1=1}^{\infty}\frac{x^{k_1}}{k_1}\cdots\sum_{k_r=1}^{\infty}\frac{x^{k_r}}{k_r}dx.$$

Due to

$$\sum_{j=1}^{\infty} \frac{x^j}{j} = -\ln(1-x),$$

we get

$$S_r = (-1)^r \int_0^1 \frac{\ln(1-x)^r}{x} dx.$$

Substituting $x = 1 - t$ yields

$$S_r = (-1)^r \int_0^1 \frac{\ln(t)^r}{1-t} dt.$$

This and the known result ([2, p. 47])

$$\int_0^1 \frac{\ln(t)^r}{1-t} dt = (-1)^r r! \zeta(r+1)$$

readily yield the claim. □

Subsequently we will deal with a 'relative' of S_r, namely the multisum

$$T_r = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \frac{(-1)^{k_1+\cdots+k_r}}{k_1 \cdots k_r (k_1 + \cdots + k_r)}.$$

As it will turn out, matters are here more involved. Indeed, we will prove now

Theorem 2.2. *For $r \geq 1$,*

$$T_r = (-1)^r \left(r!\zeta(r+1) - \frac{r}{r+1}(\ln 2)^{r+1} - \sum_{k=1}^{\infty} \sum_{m=1}^{r} \frac{r(r-1)\dots(r-m+1)}{2^k k^{m+1}} (\ln 2)^{r-m} \right) \tag{2.2}$$

holds.

Proof. Proceeding as in the previous proof we get

$$T_r = (-1)^r \int_0^1 \frac{\ln(1+x)^r}{x} dx.$$

Substitution of $x = e^{-t} - 1$ yields

$$T_r = (-1)^r \int_0^{\ln(1/2)} \frac{(-t)^r}{e^{-t} - 1} (-e^{-t}) dt,$$

that is,

$$T_r = \int_{\ln(1/2)}^0 \frac{t^r}{1 - e^t} dt.$$

Upon expanding $\frac{1}{1-e^t}$ as a geometric series we arrive at

$$T_r = \sum_{k=0}^{\infty} \int_{-\ln 2}^0 t^r e^{kt} dt.$$

Integration by parts leads to the identity (we suppress integration constants)

$$\int t^r e^{kt} dt = e^{kt} \left(\frac{t^r}{k} + \sum_{m=1}^{r} (-1)^m \frac{r(r-1)\dots(r-m+1)}{k^{m+1}} t^{r-m} \right),$$

where $k > 0$.
Therefore a straightforward simplification yields

$$T_r = (-1)^r \left(\frac{(\ln 2)^{r+1}}{r+1} + \sum_{k=1}^{\infty} \frac{r!}{k^{r+1}} - \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{(\ln 2)^r}{k} + \sum_{m=1}^{r} \frac{r(r-1)\dots(r-m+1)}{k^{m+1}} (\ln 2)^{r-m} \right) \right).$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k 2^k} = \ln 2,$$

we finally get the claimed identity (2.2). □

3. A New Formula for Apéry's Constant

Theorem 2.2 enables us to obtain a new way to express $\zeta(3)$ by a fast converging series.
Indeed, letting $r = 2$ we get

$$T_2 = 2\left(\zeta(3) - \frac{(\ln 2)^3}{3} - \sum_{k=1}^{\infty} \frac{1}{2^k}\left(\frac{\ln 2}{k^2} + \frac{1}{k^3}\right)\right).$$

Furthermore [2, p. 43], reports

$$T_2 = \frac{1}{4}\zeta(3).$$

Therefore the following holds.

Theorem 3.1.

$$\zeta(3) = \frac{8}{7}\left(\frac{(\ln 2)^3}{3} + \sum_{k=1}^{\infty} \frac{1}{2^k}\left(\frac{\ln 2}{k^2} + \frac{1}{k^3}\right)\right). \tag{3.1}$$

This formula should be compared with the following one (see [5])

$$\zeta(3) = \frac{2}{3}(\ln 2)^3 + 4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 2^k \binom{2k}{k}}.$$

4. Further Observations

- From $S_2 + T_2 = \frac{9}{4}\zeta(3)$ we infer

$$2 \sum_{\substack{i,j\geq 1\\ i+j \text{ even}}} \frac{1}{ij(i+j)} = \frac{9}{4}\zeta(3)$$

that is (we put $i + j = 2k$),

$$\sum_{k=1}^{\infty} \sum_{j=1}^{2k-1} \frac{1}{2(2k-j)jk} = \frac{9}{8}\zeta(3),$$

i.e.

$$\sum_{k=1}^{\infty} \frac{1}{2k} \sum_{j=1}^{2k-1} \frac{1}{2k} \left(\frac{1}{j} + \frac{1}{2k-j}\right) = \frac{9}{8}\zeta(3).$$

This can be summarized as

$$\sum_{k=1}^{\infty} \frac{1}{k^2} H_{2k-1} = \frac{9}{4}\zeta(3),$$

where

$$H_n = \sum_{j=1}^{n} \frac{1}{j}$$

denotes the n-th harmonic number.
In a similar way $S_2 - T_2 = \frac{7}{4}\zeta(3)$ implies the formula

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} H_{2k} = \frac{7}{16}\zeta(3).$$

From these two formulae we get easily

Theorem 4.1.

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} H_{2k-1} = \frac{21}{16}\zeta(3) \tag{4.1}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} H_{2k} = \frac{11}{16}\zeta(3). \tag{4.2}$$

Adding these two identities yields

$$\sum_{k=1}^{\infty} \frac{1}{k^2} H_k = 2\zeta(3), \tag{4.3}$$

a result already known to L. Euler.

- [4, p. 499, item 2.6.9.14] reads

$$T_2 = \int_0^1 \frac{\ln(1+x)^2}{x} dx = 2\sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k^2} - \frac{\pi^2\gamma}{6},$$

where $\psi(z) = (\ln \Gamma(z))'$ and γ denote the digamma function and the Euler-Mascheroni constant, resp.
Therefore there holds the curious identity

$$\sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k^2} = \frac{1}{8}\zeta(3) + \frac{\pi^2\gamma}{12}.$$

Because of $\psi(k) = -\gamma + H_{k-1}$ it reads in equivalent form

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} H_k = \frac{5}{8}\zeta(3) \tag{4.4}$$

- Recently [3] posed the problem of proving the identity

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = 7\int_0^{\pi/4} \frac{\ln(\cos x)\ln(\sin x)}{\cos x \sin x} dx.$$

We show that it implies a remarkable result for two doublesums. Indeed, we firstly note

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = \sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{1}{(2k)^3},$$

that is

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = \frac{7}{8}\zeta(3).$$

Therefore, the identity under consideration in fact means

$$\zeta(3) = 8\int_0^{\pi/4} \frac{\ln(\cos x)}{\cos x} \cdot \frac{\ln(\sin x)}{\sin x} dx.$$

Letting $f(x) = \frac{\ln(\cos x)}{\cos x}$ and setting $z = \pi/2 - x$ we obtain

$$\int_0^{\pi/4} f(x) f\left(\frac{\pi}{2} - x\right) dx = \int_{\pi/4}^{\pi/2} f\left(\frac{\pi}{2} - z\right) f(z) dz$$

whence

$$\zeta(3) = 4\int_0^{\pi/2} \frac{\ln(\cos x)}{\cos x} \cdot \frac{\ln(\sin x)}{\sin x} dx.$$

Next, we substitute $\sin x = \sqrt{w}$.
From $\cos x dx = \frac{1}{2\sqrt{w}} dw$ and $\cos x = \sqrt{1-w}$ we get $dx = \frac{1}{2\sqrt{w}\sqrt{1-w}} dw$.
This in turn yields

$$\zeta(3) = 4\int_0^1 \frac{\ln(\sqrt{1-w})}{\sqrt{1-w}} \cdot \frac{\ln(\sqrt{w})}{\sqrt{w}} \cdot \frac{1}{2\sqrt{w}\sqrt{1-w}} dw,$$

that is

$$\zeta(3) = \frac{1}{2}\int_0^1 \frac{\ln(1-w)}{1-w} \cdot \frac{\ln w}{w} dw.$$

Upon rewriting this as

$$\zeta(3) = \frac{1}{2}\int_0^1 \frac{\ln(1-w)}{w} \cdot \frac{\ln w}{1-w} dw$$

and developing the two factors of the integrand we get

$$\zeta(3) = \frac{1}{2}\int_0^1 \left(-\sum_{i=1}^{\infty} \frac{w^{i-1}}{i}\right)\left(-\sum_{j=1}^{\infty} \frac{(1-w)^{j-1}}{j}\right) dw,$$

that is

$$\zeta(3) = \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij}\int_0^1 w^{i-1}(1-w)^{j-1} dw.$$

Keeping in mind that

$$\int_0^1 w^{i-1}(1-w)^{j-1}dw = \frac{(i-1)!(j-1)!}{(i+j-1)!},$$

we arrive at the formula

$$\zeta(3) = \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij^2\binom{i+j-1}{j}}. \tag{4.5}$$

Equation (4.5) and $\zeta(3) = \frac{1}{2}S_2$ give the two noteworthy identities

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij^2\binom{i+j-1}{j}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)} \tag{4.6}$$

and

$$\int_0^1 \frac{\ln(1-z)}{z}\cdot\frac{\ln z}{1-z}dz = \int_0^1 \frac{\ln(1-z)}{z}\ln(1-z)dz. \tag{4.7}$$

- Finally, Theorem 4.1 enables us to prove the following finite analogon of the initial formula (2.1) of the present note, namely

$$\sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{1}{ij(i+j)} = \frac{5}{4}\zeta(3). \tag{4.8}$$

Indeed, (4.2) and (4.3) imply

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{1}{k+1}+\cdots+\frac{1}{2k}\right)=\frac{3}{4}\zeta(3).$$

However,

$$\frac{1}{k^2}\left(\frac{1}{k+1}+\cdots+\frac{1}{2k}\right)=\frac{1}{k}\sum_{j=1}^{k}\frac{1}{k(k+j)}$$

and

$$\frac{1}{k}\sum_{j=1}^{k}\frac{1}{k(k+j)}=\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}\left(\frac{1}{k}-\frac{1}{k+j}\right)=\frac{1}{k^2}H_k-\sum_{j=1}^{k}\frac{1}{kj(k+j)}$$

readily lead to

$$2\zeta(3)-\sum_{k=1}^{\infty}\sum_{j=1}^{k}\frac{1}{kj(k+j)}=\frac{3}{4}\zeta(3)$$

as claimed.

5. Two Questions for Further Research

- The results of Theorem 4.1 may be regarded as special cases of the more general sums

$$S_{a,b} = \sum_{k=1}^{\infty} \frac{1}{(ak-b)^2} H_{ak-b},$$

where $0 \leq b < a$ are entire numbers.

Problem 1. *Determine $S_{a,b}$ for $a \geq 3$ in terms of 'familiar' expressions.*

- *Let, in analogy to S_r and T_r, $U_{r,s}$ denote the multisum*

$$U_{r,s} = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \frac{(-1)^{k_1+\cdots+k_s}}{k_1 \cdots k_r (k_1 + \cdots + k_r)},$$

where $r \geq 1$ and $0 \leq s \leq r$.

Problem 2. *Determine $U_{r,s}$ in the spirit of Theorem 2.2.*
In other words evaluate the integrals

$$I_{r,s} = \int_0^1 \frac{\ln(1-x)^s \ln(1+x)^{r-s}}{x} dx$$

for $r \geq 1$ and $0 \leq s \leq r$.

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