

Journal of Inequalities in Pure and Applied Mathematics

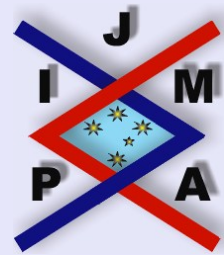
AROUND APÉRY'S CONSTANT

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©2000 Victoria University
ISSN (electronic): 1443-5756
020-06



volume 7, issue 1, article 35,
2006.

*Received 04 January, 2006;
accepted 18 January, 2006.*

Communicated by: A. Lupaş

Abstract

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Abstract

In this note we deal with some aspects of Apéry's constant $\zeta(3)$.

2000 Mathematics Subject Classification: 49J40, 90C33, 47H10.

Key words: Apéry's constant, Harmonic numbers, Infinite sums, Integrals.

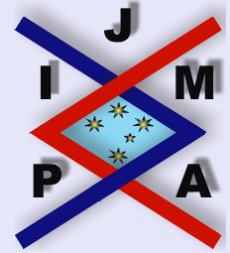
This research is partially supported by grant 04-39-3265-2/03 (11.XII.2003) of the Federal Ministry of Education and Sciences, Bosnia and Herzegovina.

Dedicated to Professor Gerd Baron on the occasion of his 65th birthday.

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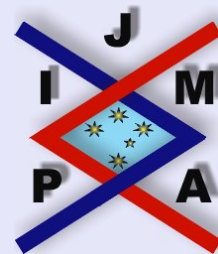
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1. Introduction

Since Apéry's miraculous proof (1979) that the value $\zeta(3)$ of Riemann's ζ -function is irrational, *Apéry's constant* $\zeta(3)$ has been the focus of attention for many mathematicians. (An extensive list of results and references are found in Section 1.6 of the highly recommended encyclopedic book [2].)

It is the purpose of this note to extend some of these results. Thereby we will also obtain a new infinite sum rapidly converging to $\zeta(3)$.

At the end of this note we raise two questions for further investigation.



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2. Two Multisums

Recently in [1] the proof of

$$(2.1) \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij(i+j)} = 2\zeta(3),$$

where $\zeta(3) = 1.202056903\dots$, was posed as a problem. Although this result was published earlier (see [2, p. 43]) it is worthwhile reconsidering in the following more general way.

Theorem 2.1. For $r \geq 1$ the multisum

$$S_r = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \frac{1}{k_1 \cdots k_r (k_1 + \cdots + k_r)}$$

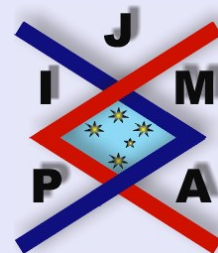
attains the value $r!\zeta(r+1)$.

Proof. Firstly we rewrite the multisum as an integral as follows

$$S_r = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \frac{1}{k_1 \cdots k_r} \int_0^1 x^{k_1 + \cdots + k_r - 1} dx$$

that is (upon interchanging of summation and integration),

$$S_r = \int_0^1 \frac{1}{x} \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \cdots \sum_{k_r=1}^{\infty} \frac{x^{k_r}}{k_r} dx.$$



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Due to

$$\sum_{j=1}^{\infty} \frac{x^j}{j} = -\ln(1-x),$$

we get

$$S_r = (-1)^r \int_0^1 \frac{\ln(1-x)^r}{x} dx.$$

Substituting $x = 1 - t$ yields

$$S_r = (-1)^r \int_0^1 \frac{\ln(t)^r}{1-t} dt.$$

This and the known result ([2, p. 47])

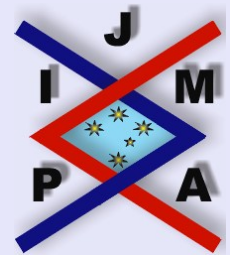
$$\int_0^1 \frac{\ln(t)^r}{1-t} dt = (-1)^r r! \zeta(r+1)$$

readily yield the claim. □

Subsequently we will deal with a ‘relative’ of S_r , namely the multisetum

$$T_r = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \frac{(-1)^{k_1+\cdots+k_r}}{k_1 \cdots k_r (k_1 + \cdots + k_r)}.$$

As it will turn out, matters are here more involved. Indeed, we will prove now



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Theorem 2.2. For $r \geq 1$,

$$(2.2) \quad T_r = (-1)^r \left(r! \zeta(r+1) - \frac{r}{r+1} (\ln 2)^{r+1} - \sum_{k=1}^{\infty} \sum_{m=1}^r \frac{r(r-1)\dots(r-m+1)}{2^k k^{m+1}} (\ln 2)^{r-m} \right)$$

holds.

Proof. Proceeding as in the previous proof we get

$$T_r = (-1)^r \int_0^1 \frac{\ln(1+x)^r}{x} dx.$$

Substitution of $x = e^{-t} - 1$ yields

$$T_r = (-1)^r \int_0^{\ln(1/2)} \frac{(-t)^r}{e^{-t} - 1} (-e^{-t}) dt,$$

that is,

$$T_r = \int_{\ln(1/2)}^0 \frac{t^r}{1 - e^t} dt.$$

Upon expanding $\frac{1}{1-e^t}$ as a geometric series we arrive at

$$T_r = \sum_{k=0}^{\infty} \int_{-\ln 2}^0 t^r e^{kt} dt.$$



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Integration by parts leads to the identity (we suppress integration constants)

$$\int t^r e^{kt} dt = e^{kt} \left(\frac{t^r}{k} + \sum_{m=1}^r (-1)^m \frac{r(r-1)\dots(r-m+1)}{k^{m+1}} t^{r-m} \right),$$

where $k > 0$.

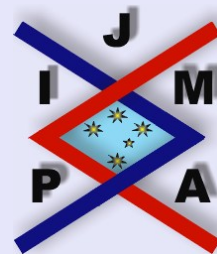
Therefore a straightforward simplification yields

$$T_r = (-1)^r \left(\frac{(\ln 2)^{r+1}}{r+1} + \sum_{k=1}^{\infty} \frac{r!}{k^{r+1}} - \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{(\ln 2)^r}{k} + \sum_{m=1}^r \frac{r(r-1)\dots(r-m+1)}{k^{m+1}} (\ln 2)^{r-m} \right) \right).$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k2^k} = \ln 2,$$

we finally get the claimed identity (2.2). □



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3. A New Formula for Apéry's Constant

Theorem 2.2 enables us to obtain a new way to express $\zeta(3)$ by a fast converging series.

Indeed, letting $r = 2$ we get

$$T_2 = 2 \left(\zeta(3) - \frac{(\ln 2)^3}{3} - \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{\ln 2}{k^2} + \frac{1}{k^3} \right) \right).$$

Furthermore [2, p. 43], reports

$$T_2 = \frac{1}{4} \zeta(3).$$

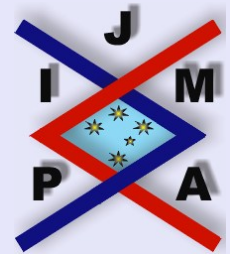
Therefore the following holds.

Theorem 3.1.

$$(3.1) \quad \zeta(3) = \frac{8}{7} \left(\frac{(\ln 2)^3}{3} + \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{\ln 2}{k^2} + \frac{1}{k^3} \right) \right).$$

This formula should be compared with the following one (see [5])

$$\zeta(3) = \frac{2}{3} (\ln 2)^3 + 4 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 2^k \binom{2k}{k}}.$$



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4. Further Observations

- From $S_2 + T_2 = \frac{9}{4}\zeta(3)$ we infer

$$2 \sum_{\substack{i,j \geq 1 \\ i+j \text{ even}}} \frac{1}{ij(i+j)} = \frac{9}{4}\zeta(3)$$

that is (we put $i + j = 2k$),

$$\sum_{k=1}^{\infty} \sum_{j=1}^{2k-1} \frac{1}{2(2k-j)jk} = \frac{9}{8}\zeta(3),$$

i.e.

$$\sum_{k=1}^{\infty} \frac{1}{2k} \sum_{j=1}^{2k-1} \frac{1}{2k} \left(\frac{1}{j} + \frac{1}{2k-j} \right) = \frac{9}{8}\zeta(3).$$

This can be summarized as

$$\sum_{k=1}^{\infty} \frac{1}{k^2} H_{2k-1} = \frac{9}{4}\zeta(3),$$

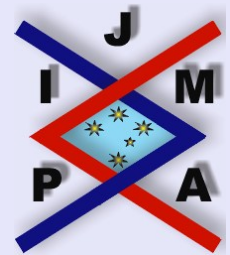
where

$$H_n = \sum_{j=1}^n \frac{1}{j}$$

denotes the n -th harmonic number.

In a similar way $S_2 - T_2 = \frac{7}{4}\zeta(3)$ implies the formula

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} H_{2k} = \frac{7}{16}\zeta(3).$$



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From these two formulae we get easily

Theorem 4.1.

$$(4.1) \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} H_{2k-1} = \frac{21}{16} \zeta(3)$$

and

$$(4.2) \quad \sum_{k=1}^{\infty} \frac{1}{(2k)^2} H_{2k} = \frac{11}{16} \zeta(3).$$

Adding these two identities yields

$$(4.3) \quad \sum_{k=1}^{\infty} \frac{1}{k^2} H_k = 2\zeta(3),$$

a result already known to L. Euler.

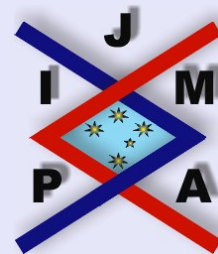
- [4, p. 499, item 2.6.9.14] reads

$$T_2 = \int_0^1 \frac{\ln(1+x)^2}{x} dx = 2 \sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k^2} - \frac{\pi^2 \gamma}{6},$$

where $\psi(z) = (\ln \Gamma(z))'$ and γ denote the digamma function and the Euler-Mascheroni constant, resp.

Therefore there holds the curious identity

$$\sum_{k=1}^{\infty} \frac{(-1)^k \psi(k)}{k^2} = \frac{1}{8} \zeta(3) + \frac{\pi^2 \gamma}{12}.$$



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Because of $\psi(k) = -\gamma + H_{k-1}$ it reads in equivalent form

$$(4.4) \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} H_k = \frac{5}{8} \zeta(3)$$

- Recently [3] posed the problem of proving the identity

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = 7 \int_0^{\pi/4} \frac{\ln(\cos x) \ln(\sin x)}{\cos x \sin x} dx.$$

We show that it implies a remarkable result for two doublesums.

Indeed, we firstly note

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = \sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{1}{(2k)^3},$$

that is

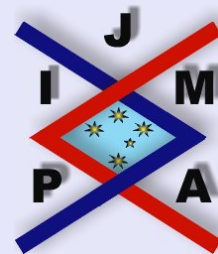
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = \frac{7}{8} \zeta(3).$$

Therefore, the identity under consideration in fact means

$$\zeta(3) = 8 \int_0^{\pi/4} \frac{\ln(\cos x)}{\cos x} \cdot \frac{\ln(\sin x)}{\sin x} dx.$$

Letting $f(x) = \frac{\ln(\cos x)}{\cos x}$ and setting $z = \pi/2 - x$ we obtain

$$\int_0^{\pi/4} f(x) f\left(\frac{\pi}{2} - x\right) dx = \int_{\pi/4}^{\pi/2} f\left(\frac{\pi}{2} - z\right) f(z) dz$$



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whence

$$\zeta(3) = 4 \int_0^{\pi/2} \frac{\ln(\cos x)}{\cos x} \cdot \frac{\ln(\sin x)}{\sin x} dx.$$

Next, we substitute $\sin x = \sqrt{w}$.

From $\cos x dx = \frac{1}{2\sqrt{w}} dw$ and $\cos x = \sqrt{1-w}$ we get $dx = \frac{1}{2\sqrt{w}\sqrt{1-w}} dw$.

This in turn yields

$$\zeta(3) = 4 \int_0^1 \frac{\ln(\sqrt{1-w})}{\sqrt{1-w}} \cdot \frac{\ln(\sqrt{w})}{\sqrt{w}} \cdot \frac{1}{2\sqrt{w}\sqrt{1-w}} dw,$$

that is

$$\zeta(3) = \frac{1}{2} \int_0^1 \frac{\ln(1-w)}{1-w} \cdot \frac{\ln w}{w} dw.$$

Upon rewriting this as

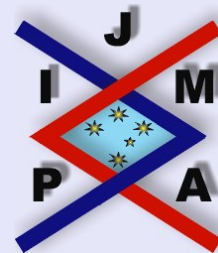
$$\zeta(3) = \frac{1}{2} \int_0^1 \frac{\ln(1-w)}{w} \cdot \frac{\ln w}{1-w} dw$$

and developing the two factors of the integrand we get

$$\zeta(3) = \frac{1}{2} \int_0^1 \left(-\sum_{i=1}^{\infty} \frac{w^{i-1}}{i} \right) \left(-\sum_{j=1}^{\infty} \frac{(1-w)^{j-1}}{j} \right) dw,$$

that is

$$\zeta(3) = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij} \int_0^1 w^{i-1} (1-w)^{j-1} dw.$$



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Keeping in mind that

$$\int_0^1 w^{i-1}(1-w)^{j-1}dw = \frac{(i-1)!(j-1)!}{(i+j-1)!},$$

we arrive at the formula

$$(4.5) \quad \zeta(3) = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij^2 \binom{i+j-1}{j}}.$$

Equation (4.5) and $\zeta(3) = \frac{1}{2}S_2$ give the two noteworthy identities

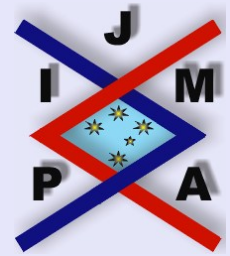
$$(4.6) \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij^2 \binom{i+j-1}{j}} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij(i+j)}$$

and

$$(4.7) \quad \int_0^1 \frac{\ln(1-z)}{z} \cdot \frac{\ln z}{1-z} dz = \int_0^1 \frac{\ln(1-z)}{z} \ln(1-z) dz.$$

• Finally, Theorem 4.1 enables us to prove the following finite analog of the initial formula (2.1) of the present note, namely

$$(4.8) \quad \sum_{i=1}^{\infty} \sum_{j=1}^i \frac{1}{ij(i+j)} = \frac{5}{4}\zeta(3).$$



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Indeed, (4.2) and (4.3) imply

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \left(\frac{1}{k+1} + \cdots + \frac{1}{2k} \right) = \frac{3}{4} \zeta(3).$$

However,

$$\frac{1}{k^2} \left(\frac{1}{k+1} + \cdots + \frac{1}{2k} \right) = \frac{1}{k} \sum_{j=1}^k \frac{1}{k(k+j)}$$

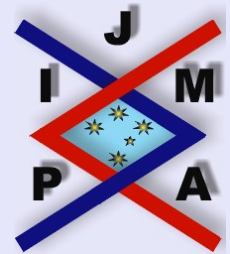
and

$$\frac{1}{k} \sum_{j=1}^k \frac{1}{k(k+j)} = \frac{1}{k} \sum_{j=1}^k \frac{1}{j} \left(\frac{1}{k} - \frac{1}{k+j} \right) = \frac{1}{k^2} H_k - \sum_{j=1}^k \frac{1}{kj(k+j)}$$

readily lead to

$$2\zeta(3) - \sum_{k=1}^{\infty} \sum_{j=1}^k \frac{1}{kj(k+j)} = \frac{3}{4} \zeta(3)$$

as claimed.



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5. Two Questions for Further Research

- The results of Theorem 4.1 may be regarded as special cases of the more general sums

$$S_{a,b} = \sum_{k=1}^{\infty} \frac{1}{(ak - b)^2} H_{ak-b},$$

where $0 \leq b < a$ are entire numbers.

Problem 1. Determine $S_{a,b}$ for $a \geq 3$ in terms of 'familiar' expressions.

- Let, in analogy to S_r and T_r , $U_{r,s}$ denote the multisum

$$U_{r,s} = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \frac{(-1)^{k_1+\cdots+k_s}}{k_1 \cdots k_r (k_1 + \cdots + k_r)},$$

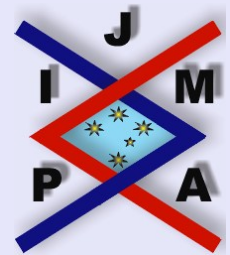
where $r \geq 1$ and $0 \leq s \leq r$.

Problem 2. Determine $U_{r,s}$ in the spirit of Theorem 2.2.

In other words evaluate the integrals

$$I_{r,s} = \int_0^1 \frac{\ln(1-x)^s \ln(1+x)^{r-s}}{x} dx$$

for $r \geq 1$ and $0 \leq s \leq r$.



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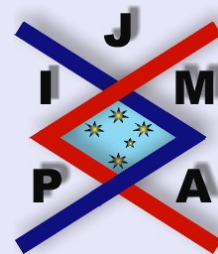
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