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Fifteen problems in number theory

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Abstract. In this paper we collected problems, which was either proposed or follow directly from results in our papers.

1 Introduction

In this paper, which is based on a talk delivered at the Winter School on Explicit Methods in Number Theory, Debrecen, January 29, 2009 we collected problems, which we proposed and/or tried to solve. The problems are dealing with perfect powers in linear recursive sequences, solutions of parametrized families of Thue equations, patterns in the set of solutions of norm form equations and generalized radix representations.

In each case we give a short description of the background information, cite some relevant paper, especially papers, where the problem appeared at the first time. Sometime we present our feeling about the hardness of the problem and how one could solve it. The collection is subjective.

2 Powers in linear recursive sequences

To find perfect powers and polynomial values in linear recursive sequences is one of my favorite topics. A long standing problem was to prove that $0, 1, 8$

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and 144 are the only powers in the Fibonacci sequence. This was proved finally by Bugeaud, Mignotte and Siksek in 2006 [9].

In 1996 at The Seventh International Research Conference on Fibonacci Numbers and Their Applications I proposed the following [17]

Problem 1 *The sequence of tribonacci numbers is defined by* $T_0 = T_1 = 0, T_2 = 1$ *and* $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ *for* $n \geq 0$. *Are the only squares* $T_0 = T_1 = 0, T_2 = T_3 = 1, T_5 = 4, T_{10} = 81, T_{16} = 3136 = 56^2$ *and* $T_{18} = 10609 = 103^2$ *among the numbers* T_n*?*

By using the sieve method from [16] with the moduli $3, 7, 11, 13, 29, 41, 43, 53, 79, 101, 103, 131, 239, 97, 421, 911, 1021$ and 1123 one can show that this is true for $n \leq 2 \cdot 10^6$, but known methods do not seem to be applicable for its solution.

The problem is still unsolved, although in the edited version of the second part of that talk [18] combining results of Shorey and Stewart [23] with that of Corvaja and Zannier [10] I proved

Theorem 1 *Let* G_n *be a third order LRS. For the roots* $\alpha_i, i = 1, 2, 3$ *of the characteristic polynomial of* G_n *assume that* $|\alpha_1| > |\alpha_2| \geq |\alpha_3|$ *and non of them is a root of unity. Then there are only finitely many perfect powers in* G_n.

As the characteristic polynomial of the tribonacci sequence $x^3 - x^2 - x - 1$ is irreducible with one dominating real root ≈ 1.839286755 it follows that there exist finitely many perfect powers in it. Unfortunately the proof of Theorem 1 is only partially effective. We have an effective bound for the exponent of the possible perfect powers, but no effective bound for the size of a fixed power, e.g., for squares.

I think that Theorem 1 can be generalized at least in the following form:

Problem 2 *Let* G_n *be an LRS such that its characteristic polynomial is irreducible and has a dominating root, then there is only finitely many perfect powers in it.*

By a result of Shorey and Stewart [23] the exponent of perfect powers can be bounded effectively. The problem is to handle the powers with bounded exponent. Combining this with the result of Corvaja and Zannier [10] and with the combinatorics of the roots, like in Pethő [18], one can probably settle this conjecture.

Like the Fibonacci sequence, we can continue the tribonacci sequence in "negative direction", and get $T_{-n} = -T_{-n+1} - T_{-n+2} + T_{-n+3}$ with initial terms $T_0 = 0, T_{-1} = 1, T_{-2} = -1$. We call this sequence n-tribonacci. One can ask again, which are the perfect powers in this sequence. After a simple search we find: $T_0 = T_{-3} = T_{-16} = 0, T_{-1} = T_{-6} = -T_{-2} = 1, T_{-7} = 2^2, T_{-8} = (-2)^3, T_{-13} = 3^2, T_{-29} = 3^4, T_{-32} = 56^2, T_{-33} = 103^2$ and $T_{-62} = 6815^2$. It is interesting to observe that $T_{10} = T_{-29}, T_{16} = T_{-32}$ and $T_{18} = T_{-33}$.

Problem 3 *Are all perfect powers of the n-tribonacci sequence listed above? Are there only finitely many perfect powers in the n-tribonacci sequence?*

The answer seems to be very difficult, because the characteristic polynomial of the n-tribonacci sequence has two conjugate complex roots of the same absolute value and its real root is less than one. Thus the result of Shorey and Stewart is not applicable.

Let $a, b \in \mathbb{Z}$ and $\delta \in \{1, -1\}$ such that $a^2 - 4(b - 2\delta) \neq 0$, $b\delta \neq 2$ and if $\delta = 1$ then $b \neq 2a - 2$. Let further the sequence $G_n = G_n(a, b, \delta)$, $n \geq 0$ defined by the initial terms $G_0 = 0$, $G_1 = 1$, $G_2 = a$, $G_3 = a^2 - b - \delta$ and by the recursion

$$G_{n+4} = aG_{n+3} - bG_{n+2} + \delta a G_{n+1} - G_n, \quad n \geq 0. \tag{1}$$

I proved in [19] that these are divisibility sequences, i.e., $G_n | G_m$, whenever $n | m$. More precisely, the roots of the characteristic polynomial of G_n can be numbered so that they are $\eta, \frac{\delta}{\eta}, \vartheta, \frac{\delta}{\vartheta}$ and

$$G_n = \frac{\eta^n - \vartheta^n}{\eta - \vartheta} \frac{1 - \left(\frac{\delta}{\eta\vartheta}\right)^n}{1 - \frac{\delta}{\eta\vartheta}}$$

Here we ask again to prove

Problem 4 *For fixed a, b there are only finitely many perfect powers in G_n.*

We can again bound the exponent by the result of Shorey and Stewart [23], but can not treat the equation $G_n = x^q$ for fixed $q > 1$. Especially complicated seems the case $q = 2$, because the greatest common divisor of the algebraic numbers $\frac{\eta^n - \vartheta^n}{\eta - \vartheta}$ and $\frac{1 - \left(\frac{\delta}{\eta\vartheta}\right)^n}{1 - \frac{\delta}{\eta\vartheta}}$ can be arbitrary large.

3 Thue equations

After the work of E. Thomas [24] several paper appeared about the solutions of parametrized families of Thue equations. With Halter-Koch, Lettl and Tichy we proved [13] the following:

Theorem 2 *Let* $n \geq 3$, $a_1 = 0, a_2, \dots, a_{n-1}$ *be distinct integers and* $a_n = a$ *an integral parameter. Let* $\alpha = \alpha(a)$ *be a zero of* $P(x) = \prod_{i=1}^{n}(x - a_i) - d$ *with* $d = \pm 1$ *and suppose that the index* I *of* $\langle \alpha - a_1, \dots, \alpha - a_{n-1} \rangle$ *in* $U_{\mathcal{O}}$*, the group of units of* $\mathcal{O}$*, is bounded by a constant* $J = J(a_1, \dots, a_{n-1}, n)$ *for every* a *from some subset* $\Omega \subset \mathbb{Z}$*. Assume further that the Lang-Waldschmidt conjecture is true. Then for all but finitely many values* $a \in \Omega$ *the diophantine equation*

$$\prod_{i=1}^{n}(x - a_i y) - dy^n = \pm 1 \tag{2}$$

only has trivial solutions, except when $n = 3$ *and* $|a_2| = 1$*, or when* $n = 4$ *and* $(a_2, a_3) \in \{(1, -1), (\pm 1, \pm 2)\}$*, in which cases (2) has exactly one more general solution.*

The assumption on the index I is technical, the essential assumption is the Lang-Waldschmidt conjecture. In the cited paper we formulated:

Problem 5 *The last theorem is true for all large enough parameter value without further assumptions.*

A weaker version of this conjecture was formulated by E. Thomas [25]. He assumed that $a_i = p_i(a), i = 2, \dots, n-1$ and $0 < \deg p_2 < \dots < \deg p_{n-1}$, where p_i denotes monic polynomial with integer coefficients. This weaker conjecture was proved by C. Heuberger [14] under some technical conditions on the degree of the polynomials.

4 Progressions in the set of solutions of norm form equations

Let $\mathbb{K}$ be an algebraic number field of degree k, and let $\alpha_1, \dots, \alpha_n$ be linearly independent elements of $\mathbb{Z}_{\mathbb{K}}$ over $\mathbb{Q}$. Let m be a non-zero integer and consider the norm form equation

$$N_{\mathbb{K}/\mathbb{Q}}(x_1\alpha_1 + \ldots + x_n\alpha_n) = m \tag{3}$$

in integer vectors $(x_1, \dots, x_n)$. Let H denote the solution set of (3) and $|H|$ the size of H. Note that if the $\mathbb{Z}$-module generated by $\alpha_1, \dots, \alpha_n$ contains a submodule, which is a full module in a subfield of $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ different from the imaginary quadratic fields and $\mathbb{Q}$, then equation (3) can have infinitely many solutions (see e.g. Schmidt [22]).

Arranging the elements of H in an $|H| \times n$ array $\mathcal{H}$, one may ask at least two natural questions about arithmetical progressions appearing in H. The "horizontal" one: do there exist infinitely many rows of $\mathcal{H}$, which form arithmetic progressions; and the "vertical" one: do there exist arbitrary long arithmetic progressions in some column of $\mathcal{H}$? Note that the first question is meaningful only if $n > 2$.

We are now presenting an example. Let $\mathbb{K} := \mathbb{Q}(\alpha)$ with $\alpha^5 = 3$. Then

$$\begin{aligned} N_{\mathbb{K}/\mathbb{Q}}(x_1 + x_2\alpha + \cdots + x_5\alpha^4) = {} & 9x_3^5 + 81x_5^5 + x_1^5 + 27x_4^5 + 3x_2^5 - 135x_5^3x_4x_1 + \\ & +45x_5x_4^2x_1^2 + 135x_2x_4^2x_5^2 - 45x_2x_4^3x_1 + 45x_5^2x_3x_1^2 - 45x_2x_3^3x_4 + \\ & +135x_3^2x_5^2x_4 + 45x_1x_5^2x_2^2 - 45x_4x_2^3x_5 + 45x_4^2x_2^2x_3 + 45x_4^2x_1x_3^2 - \\ & -15x_4x_1^3x_3 + 15x_4x_1^2x_2^2 + 15x_2x_3^2x_1^2 + 45x_5x_2^2x_3^2 - 15x_5x_1^3x_2 - \\ & -135x_5x_3x_4^3 - 135x_2x_5^3x_3 - 45x_5x_3^3x_1 - 15x_2^3x_3x_1 - 45x_2x_5x_3x_4x_1. \end{aligned}$$

The next table contains a finite portion of the set of solutions of the equation

$$N_{\mathbb{K}/\mathbb{Q}}(x_1 + x_2\alpha + \cdots + x_5\alpha^4) = 1.$$

x_1	x_2	x_3	x_4	x_5
4	-5	4	**-2**	0
1	2	-1	**-1**	0
4	2	0	**0**	1
1	1	0	**1**	0
1	5	1	**2**	2
-17	1	-6	**3**	8
7	**6**	**5**	**4**	**3**
-2	-1	1	1	0
-11	-5	5	6	0
-2	0	1	-1	1
-8	-8	1	6	2
28	16	4	3	8
10	12	12	4	9
...	...	...	...	...

The bold face numbers form a five term horizontal AP and a seven terms vertical AP. The "horizontal" problem was treated by Bérczes and Pethő [7] by proving that if $\alpha_i = \alpha^{i-1}$ $(i = 1, \ldots, n)$ then in general $\mathcal{H}$ contains only finitely many effectively computable "horizontal" AP's and they were able to localize the possible exceptional cases. The following question remains unanswered:

Problem 6 *Does there exist infinitely many quartic algebraic integers* α *such that* $\frac{4\alpha^4}{\alpha^4-1} - \frac{\alpha}{\alpha-1}$ *is a quadratic algebraic number.*

We were able to found only one example with defining polynomial $x^4+2x^3+5x^2+4x+2$ such that the corresponding element is a real quadratic number. It is a root of x^2-4x+2. Allowing however α not to be integral we can obtain a lot of examples.

The investigation of the "vertical" AP's is much more difficult. In this direction Bérczes, Hajdu and Pethő [6] proved

Theorem 3 *Let* $(x_1^{(j)}, \ldots, x_n^{(j)})$ $(j = 1, \ldots, t)$ *be a sequence of distinct elements in* H *such that* $x_i^{(j)}$ *is a non-zero arithmetic progression for some* $i \in \{1, \ldots, n\}$. *Then we have* $t \le c_1$, *where* $c_1 = c_1(k, m)$ *is an explicitly computable constant.*

It is interesting to note that c_1 depends only on the degree of the norm form and not on its coefficients. One can probably strengthen this result such that the upper bound for the length of the AP's depend not on m, but only on the number of its prime divisors. It is even possible that the bound depends only on k.

Earlier Pethő and Ziegler [21] as well as Dujella, Pethő and Tadić [11] investigated the AP's on Pell equations, which are quadratic norm form equations. We proved that for all but one non-constant AP of integers of length four y_1, y_2, y_3, y_4 there exist infinitely many integers d, m for which $x_i^2 - dy_i^2 = m, i = 1, 2, 3, 4$ with some integers $x_i = x_i(d, m, y_1, \ldots, y_4), i = 1, 2, 3, 4$. In contrast, five term AP's are lying on only finitely many Pell equations.

Problem 7 *Prove analogous result for norm form equations over cubic number fields. More specifically: let* $y^{(i)}, i = 1, \ldots, 5$ *an AP of integers. Then there exist infinitely many* $m \in \mathbb{Z}$ *and* $\mathbb{Q}$*-independent algebraic integers* $\alpha_1, \alpha_2, \alpha_3$ *such that* $\mathbb{K} = \mathbb{Q}(\alpha_2, \alpha_3)$ *has degree three and (3) holds for* $(x_1^{(i)}, x_2^{(i)}, y^{(i)}), i = 1, \ldots, 5$ *with some* $x_1^{(i)}, x_2^{(i)} \in \mathbb{Z}$. *Can 5 be replaced with a larger number?*

In the above mentioned papers we worked out a systematic method to find Pell equations having long AP's. For example the AP $-7, -5, -3, -1, 1, 3, 5, 7$ is lying on the equation $x^2 - 570570y^2 = 4406791$ and $-461, -295, -129, 37, 203, 369, 535$ on $x^2 + 1245y^2 = 375701326$.

Problem 8 *Find a systematic method to construct cubic norm form equations with long AP. Do the same for higher degree norm form equations.*

Problem 9 *Prove analogous results for geometric progressions.*

5 Polynomials

Problem 10 *Let* K *be a algebraically closed field of characteristic zero. Characterize all* $P(X) \in K[X], Q(Y) \in K[Y], R(X, Y) \in K[X, Y]$ *such that the set of zeroes of* $P(X)$ *and* $Q(Y)$ *coincide, provided* $R(X, Y) = 0$.

The case $R(X, Y) = Y - A(X)$ was solved completely by Fuchs, Pethő and Tichy [12]. They proved

Theorem 4 *Assume that* $P(X)$ *has* k *different zeroes. Then there exist* $a, b, c \in K, a, c \neq 0$ *such that:*
if $k = 1$ *then*

$$P(X) = a(X - b)^{\deg P} \text{ and } A(X) = c(X - b)^{\deg A} + b;$$

if $k \geq 2$ *then either* $A(X) = X$ *or* $A(X) = aX + b, a \neq 1$ *and in this case*

$$P(X) = c\left(X + \frac{b}{a-1}\right)^s \prod_{i=1}^{r} \prod_{j=0}^{\ell-1} \left(X - a^j x_i - b\frac{a^j - 1}{a-1}\right),$$

where $x_1, \ldots, x_r$ *are all different and* ℓ *is the multiplicative order of* a.

6 Shift radix systems

For $(r_1, \ldots, r_d) = \mathbf{r} \in \mathbb{R}^d$ and $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ let $\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -\lfloor \mathbf{ra} \rfloor)^T$, where $\mathbf{ra}$ denotes the scalar product. This nearly linear mapping was introduced by Akiyama, Borbély, Brunotte, Thuswaldner and myself [1]. We proved that it can be considered as a common generalization of canonical number systems (CNS) and β-expansions.

We also defined the sets

$$\begin{aligned} \mathcal{D}_d &= \{\mathbf{r} : \{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k=0}^{\infty} \quad \text{is bounded for all} \quad \mathbf{a} \in \mathbb{Z}^d\}, \\ \mathcal{D}_d^0 &= \{\mathbf{r} : \{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k=0}^{\infty} \quad \text{is ultimately zero for all} \quad \mathbf{a} \in \mathbb{Z}^d\} \end{aligned}$$

and $\mathcal{E}_d$, which is the set of real monic polynomials, whose roots lie in the closed unit disc. We proved in the same paper that if $\mathbf{r} \in \mathcal{D}_d$ then $R(X) = X^d + r_d X^{d-1} + \cdots + r_2 X + r_1 \in \mathcal{E}_d$ and if $R(X)$ is lying in the interior of $\mathcal{E}_d$ then $\mathbf{r} \in \mathcal{D}_d$.

We called $\tau_{\mathbf{r}}$ a shift radix system (SRS), if $\mathbf{r} \in \mathcal{D}_d^0$ and gave an algorithm, which decides whether $\mathbf{r} \in \mathbb{Q}^d$ is a SRS. However this algorithm is exponential, moreover we are not able to give a polynomial time verification for $\mathbf{r} \notin \mathcal{D}_d^0 \cap \mathbb{Q}^d$. We found points $\mathbf{r} \in \mathbb{Q}^2$ such that $\mathbf{r} \notin \mathcal{D}_2^0$, but the cycles proving this can be arbitrary long. Computational experiments, see e.g. [1, 15] support the following :

Problem 11 *Prove that the SRS problem can not be solved by a polynomial time algorithm. Stronger statement is that it does not belong to the NP complexity class.*

The structure of $\mathcal{D}_d^0$, especially near to its boundary, is very complicated, see [2] for $d = 2$. On the other hand we know [1], that the closure of $\mathcal{D}_d$ is $\mathcal{E}_d$. However the investigation of the boundary points of $\mathcal{E}_d$ leads to interesting and hard problems. The case $d = 2$ was studied by Akiyama et al. in [2]. They proved that $\mathcal{D}_2$ is equal to the closed triangle with vertices $(-1, 0), (1, -2), (1, 2)$, but without the points $(1, -2), (1, 2)$, the line segment $\{(x, -x - 1) : 0 < x < 1\}$ and, possibly, some points of the line segment $\{(1, \lambda) : -2 < \lambda < 2\}$. Write in the last case $\lambda = 2\cos\alpha$ and $\omega = \cos\alpha + i\sin\alpha$. It is easy to see, that if $\lambda = 0, \pm 1$ (i.e., $\alpha = 0, \pm\pi/2$) then $(1, \lambda)$ belongs to $\mathcal{D}_2$ and we conjectured in [2] that this is true for all points of this line segment. In [4] the conjecture was proved for the golden mean, i.e., for $\lambda = \frac{1+\sqrt{5}}{2}$ and in [5] for those ω, which are quadratic algebraic numbers. The conjecture has the following nice arithmetical form:

Problem 12 *Let $|\lambda| < 2$ be a real number. If the sequence of integers $\{a_n\}$ satisfies the relation*

$$0 \le a_{n-1} + \lambda a_n + a_{n+1} < 1$$

then it is periodic.

If ω, defined above, is a root of unity then the problem may be easier as in the general case. On the other hand from the point of view of arithmetic the cases, when λ is a rational number, e.g., $\lambda = \frac{1}{2}$ seems simpler.

If the point $\mathbf{r}$ belongs to the boundary of $\mathcal{E}_d$ then either $\mathbf{r} \in \mathcal{D}_d$ or $\mathbf{r} \notin \mathcal{D}_d$. With other words this means that the sequence $\{\tau_{\mathbf{r}}(\mathbf{a})\}$ is ultimately periodic for all $\mathbf{a} \in \mathbb{Z}^d$ as well as there exists $\mathbf{a} \in \mathbb{Z}^d$ for which $\{\tau_{\mathbf{r}}(\mathbf{a})\}$ is divergent. However we do not know any general method to distinguish between these cases. Recently I gave an algorithm [20] in the special case, when $\pm 1, \pm i$ is a simple root of $X^d + r_d X^{d-1} + \cdots + r_2 X + r_1$.

Problem 13 *Is it algorithmically decidable for* $\mathbf{r} \in \mathcal{E}_d \cap \mathbb{Q}^d$ *whether* $\mathbf{r} \in \mathcal{D}_d$*?*

I am not sure that the answer is affirmative. The problem is open even for $d = 2$. In this case, by the results of [2], the status only points of the line segment $\{(1, y) \,:\, -2 < y < 2\}$ is questionable. If the answer to Problem 9 is affirmative, which I strongly believe, then $d = 2$ would be completely solved. A related, probably easier problem is:

Problem 14 *Prove that there are no elements of* $\mathcal{D}_d^0$ *on the boundary of* $\mathcal{E}_d$*.*

This is true for $d = 2$ [2], but open for $d \geq 3$.

For each $d \in \mathbb{N}$, $d \geq 1$ define the set

$$\mathcal{B}_d = \{(b_1, \dots, b_d) \in \mathbb{Z}^d : X^d - b_1 X^{d-1} - \cdots - b_d \text{ is a Pisot or Salem polynomial}\}.$$

Further for $M \in \mathbb{N}_{>0}$ set

$$\mathcal{B}_d(M) = \left\{(b_2, \dots, b_d) \in \mathbb{Z}^{d-1} \,:\, (M, b_2, \dots, b_d) \in \mathcal{B}_d\right\}. \tag{4}$$

It is clear that $\mathcal{B}_d(M)$ is a finite set. In [3] we proved

Theorem 5 *Let* $d \geq 2$*. We have*

$$\left| \frac{|\mathcal{B}_d(M)|}{M^{d-1}} - \lambda_{d-1}(\mathcal{D}_{d-1}) \right| = O(M^{-1/(d-1)}), \tag{5}$$

where λ_{d-1} *denotes the* $(d-1)$*-dimensional Lebesgue measure.*

To fix the coefficient of the term X^{d-1} of a d-th degree monic polynomial is unusual. Generally the height, i.e., the maximum of the absolute values of its coefficients is used to measure polynomials. Having this in mind we define

$$\hat{\mathcal{B}}_d(M) = \left\{(b_1, b_2, \dots, b_d) \in \mathbb{Z}^d \cap \mathcal{B}_d \,:\, \max\{|b_1|, |b_2|, \dots, |b_d|\} \leq M\right\}.$$

and propose our last problem.

Problem 15 *Does there exist a constant* c*, such that*

$$\lim_{M\to\infty} \frac{|\hat{\mathcal{B}}_d(M)|}{M^d} = c?$$

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