Physics from Computer Science
— a position statement —

SAMSON ABRAMSKY[1⋆], BOB COECKE[1†]

Oxford University Computing Laboratory, UK

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In this statement we provide some examples of transdisciplinary journeys, from one field to another, and back. In particular, the quantum informatic endeavor is not just a matter of feeding physical theory into the general field of natural computation, but also one of using high-level methods developed in Computer Science to improve on the quantum physical formalism itself, and the understanding thereof. We highlight a seemingly contradictory phenomenon: passing to an abstract, categorical quantum informatic formalism leads directly to a simple and elegant graphical formulation of quantum theory itself, which for example makes the design of some important quantum informatic protocols completely transparent. It turns out that essentially all of the quantum informatic machinery can be recovered from this graphical calculus. But in turn, this graphical formalism provides a bridge between methods of logic and computer science, and some of the most exciting developments in the mathematics of the past two decades: namely those arising from the Jones polynomial invariant of knots and links, the Temperley-Lieb Algebra and related structures.

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⋆ email: samson.abramsky@comlab.ox.ac.uk
† email: bob.coecke@comlab.ox.ac.uk

1 WHERE SCIENCES INTERACT

We are, respectively, a computer scientist interested in the logic and semantics of computation, and a physicist interested in the foundations of quantum mechanics. Currently we are pursuing what we consider to be a very fruitful collaboration as members of the same Computer Science department. How has this come about? It flows naturally from the fact that we are working in a field of Computer Science where physical theory starts to play a key role, that is, *natural computation*, with, of course, *quantum computation* as a special case. But there is more. Our joint research is *both* research on semantics for distributed and hybrid non-von Neumann architectures, *and* on the axiomatic foundations of physical theories. This dual character of our work comes without any compromise, and proves to be very fruitful. Computer Science has *something more to offer to the other sciences than the computer*. In particular, in the mathematical and logical understanding of fundamental transdisciplinary scientific concepts such as interaction, concurrency and causality, synchrony and asynchrony, compositional modelling and reasoning, open versus closed systems, qualitative versus quantitative reasoning, operational methodologies, continuous versus discrete, hybrid systems and more, Computer Science is far ahead of many other sciences, due in part to the challenges arising from the amazing rapidity of the technology change and development it is constantly being confronted with. One could claim that computer scientists (maybe without realizing it) constitute an *avant-garde* for the sciences in terms of providing fresh paradigms and methods.

In our own recent work, we recast the standard mathematical framework of quantum mechanics (which is essentially due to John von Neumann [45]) in terms of categorical semantics [6], using formal tools which were developed in Computer Science for analyzing linearity and resource sensitivity, the geometry of interacting components, and the foundations of concurrency. At the same time, this results in a graphical calculus, which we present in Section 3. This calculus stands in a long line of work, with some pioneering ideas by Penrose [38], and extensive developments by numerous category-theorists, geometers and mathematical physicists, including Kelly, Laplaza, Joyal, Street, Freyd, Yetter, and Turaev [34, 25, 28, 44, 30]. What is particularly novel and striking about our approach is that we have shown how directly and fruitfully such a calculus can be applied to the formulation of basic Quantum Mechanics itself, and how well this fits the needs of Quantum Informatics.

But of course this is not a one-way street. Physical theories inspired by computational theories are much better tailored for Computer Science applications as compared to their low-level counterparts. For example, the current tools available for developing quantum algorithms and protocols are deficient on two main levels. Firstly, they are too low-level. Quantum algorithms are currently mainly described using the 'network model' corresponding to circuits in classical computation. One finds a plethora of ad hoc calculations with 'bras' and 'kets', normalizing constants, matrices etc. The arguments for the benefits of a high-level, conceptual approach to designing, programming and reasoning about quantum computational systems are just as compelling as for classical computation. At a more fundamental level, the von Neumann formalism is actually insufficiently comprehensive for informatic purposes. In describing a protocol such as quantum teleportation, or any quantum process in which the outcome of a measurement is used to determine subsequent actions, the von Neumann formalism does not capture the flow of information from the classical or macroscopic level, where the results of measurements of the quantum-mechanical system are recorded, back to the quantum level. This flow, and the accompanying use of 'classical information', which plays a key role in protocols such as teleportation, must therefore be handled informally. As quantum protocols and computations grow more elaborate and complex, this point is likely to prove of increasing importance. Our work yields a semantics and logic which is appropriate for developing high-level tools for quantum computation and information. It provides a candidate solution for

$$\frac{\textbf{?}}{\textbf{von Neumann quantum formalism}} \simeq \frac{\textbf{high-level language}}{\textbf{low-level language}}\,.$$

Below we proceed as follows. In the next section we discuss the field of Quantum Informatics as we see it. Then we introduce our graphical calculus, which comes hand in hand with the categorical semantics. We demonstrate the logical power of this calculus, and illustrate its potential by redesigning the quantum teleportation protocol in an almost trivial fashion. Next, we mention some emerging and very promising connections between these ideas and some of the key themes of contemporary research at the interface between mathematical physics and topology. In particular, we discuss connections with the Jones polynomial invariant of knots, and the Temperley-Lieb algebra. We conclude by suggesting some possible future developments.

2 THE NEED FOR HIGH-LEVEL QUANTUM INFORMATICS

The development of Quantum Informatics is both a matter of *necessity* and one of many *opportunities*:

a. As the scale of the miniaturization of IT components reaches the quantum domain, taking quantum phenomena into account will become unavoidable.

b. On the other hand, the emerging field of Quantum Informatics has brought new computational possibilities to light, some of which endanger current cryptographic encoding schemes, but some of which at the same time provide the corresponding remedy in terms of secure quantum cryptographic and communication schemes.

Quantum Informatics emerged from the recognition that quantum phenomena and "quantum weirdness"—Einstein's "spooky action at a distance"—should be seen not as a *bug* but as a *feature*. Some first fruits of this were the BB84 and Ekert 91 public key distribution schemes, the Deutch-Jozsa, Shor and Grover algorithms, the quantum teleportation protocol and several variants [36]. But while the attitude has changed, many of the methods remained the same, and the manipulations of complex numbers, vectors and matrices in "computational bases" built from *kets* $|0\rangle$ and $|1\rangle$ bear some comparison with the acrobatics with bits and bytes in the early days of computer programming. On the other hand, many important questions on Quantum Informatics remain unanswered, and it is unlikely that the current *low-level* methods of Quantum Informatics will provide the capabilities to answer them. For example, new quantum computational models such as the measurement-based *one-way-model* of Briegel et al. [39] challenge the whole conception of what a quantum computation fundamentally *is*, and hence what its limits are. In particular, a deep and clear structural understanding of the algorithmic speed-up and the informatic quantum-classical interaction has not yet been achieved. Also, while logic has taken a prominent place in (non-quantum) Computer Science, the quest for a *quantum logic* has (until very recently) been largely a story of failure. Our (admittedly ambitious) ultimate intentions are:

1. To open up Quantum Informatics research to a wider community, as compared to its current profile of being hard and completely inaccessible to the uninitiated. This requires an *intuitive*, *simple* and *easily communicable* formalism for Quantum Informatics, and hence for quantum mechanics itself.
2. To turn Quantum Informatics research into a *systematic discipline*, which can provide a sound basis for *automated design and development tools*. This requires a quantum formalism which admits analogues to the currently available *high-level methods* from Computer Science such as

types, semantically well-founded calculi, program logics etc.

3. To blend Quantum Informatics research with the currently available and successful high-level methods for dealing with *distributed*, *hybrid* and *embedded systems*. This requires compatibility of the high-level quantum concepts with their classical counterparts.

Addressing these challenges requires new insights into the structure of quantum information, of its flow, and of its interaction with other computational resources such as classical information flow, space, time, agents, knowledge/belief etc. Our work in [6] initiated a new kind of answer to the question:

- *What is the structure of quantum mechanics?*

This forms the basis of an ongoing research programme [5].

3 CONCRETE PICTURES FROM ABSTRACT CATEGORIES

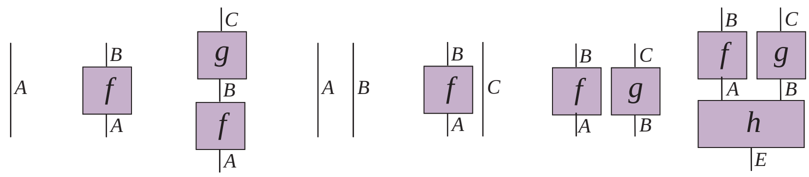
We have benefited from the currently available categorical semantics for Girard's *linear logic* [42], a resource sensitive logic developed in the late eighties. A key distinction between classical and quantum computation is indeed the inability to copy and delete unknown quantum states [37, 47], and the ability to take such an inability into account was exactly the conceptual core of linear logic. At a more refined level, we also relied on the study of a particular categorical structure called *compact closed categories* [34] which had initially been introduced for purely mathematical reasons — but which had subsequently been found useful by one of us for modelling (classical) concurrent computation [9]. Surprisingly, after *refining* compact closure to *strong compact closure* [6, 7], we were able to recover the key quantum mechanical notions of *inner-product, unitarity, full and partial trace, Hilbert-Schmidt inner-product and map-state duality, projection, positivity, measurement*, and *Born rule* (which provides the quantum *probabilities*), axiomatically at this high level of abstraction and generality. Moreover, we were able to derive the correctness of protocols such as quantum teleportation, entanglement swapping and logic-gate teleportation [12, 27, 48] in a transparent and very conceptual fashion. Also, while at this level of abstraction there is no underlying field of complex numbers, there *is* still an intrinsic notion of 'scalar', and we could still make sense of *transposition vs. adjoint* [6, 7], *global phase and elimination thereof*, *vectorial vs. projective formalism* [17]. Peter Selinger recovered *mixed state, complete positivity* and *Jamiolkowski map-state duality* [43]. Recently, in collaboration with Dusko Pavlovic and Eric Paquette, *decoherence*, *generalized measurements* and *Naimark's theorem* have been recovered [20, 19], and Ross Duncan has exposed some foundational structures for *Measurement-Based quantum computation* [23] — cf. [39, 40, 21].

This high-level of *abstraction* also comes with an intuitive and simple *graphical calculus/notation*. This "strongly compact closed graphical calculus" can be seen as a very substantial 2-dimensional extension of Dirac's *bra-ket* notation [22], and relying on category-theoretic results on free constructions of these categories [3, 28, 34, 43] one can show that an equational statement is derivable in the graphical calculus if and only if it is derivable from categorical algebra. An informal introduction to this calculus is provided in [15, 16, 18].

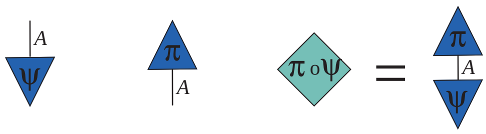
In the graphical calculus we depict physical processes by boxes, and we label the inputs and outputs of these boxes by *types* which tell on which kind of system these boxes act cf. one qubit, several qubits, classical data etc. Sequential composition (in time) is depicted by connecting matching outputs and inputs by wires, and parallel composition (tensor) by locating entities side by side e.g.

$$1_A : A \to A \quad f : A \to B \quad g\circ f \quad 1_A\otimes 1_B \quad f\otimes 1_C \quad f\otimes g \quad (f\otimes g)\circ h$$

for $g : B \to C$ and $h : E \to A \otimes B$ are respectively depicted as:



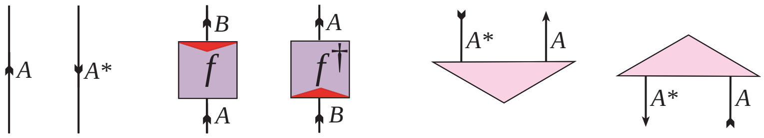
— i.e. the 'upward' vertical direction represents progress of time. A special role is played by boxes with either no input or no output, respectively called *states* and *costates* (cf. Dirac's kets and bras [22]) which we depict by triangles. Finally, we also need to consider diamonds which arise by post-composing a state with a matching costate (cf. inner-product or Dirac's bra-ket):



that is, algebraically,

$$\psi : \mathrm{I} \to A \qquad \pi : A \to \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \to \mathrm{I}$$

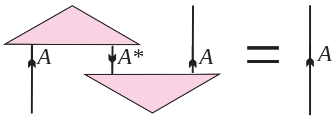
where I is the tensor unit i.e. $A\otimes \mathrm{I} \simeq A \simeq \mathrm{I}\otimes A$. Extra structure is represented by (i) assigning a direction to the wires, where reversal of this direction is denoted by $A \mapsto A^*$, (ii) allowing reversal of boxes (cf. the *adjoint* for vector spaces), and, (iii) assuming that for each type A there exists a special bipartite *Bell-state* and its adjoint *Bell-costate*:



that is, algebraically,

$$A \qquad A^* \qquad f : A \to B \qquad f^\dagger : B \to A \qquad \eta_A : \mathrm{I} \to A^*\otimes A \qquad \eta_A^\dagger : A^*\otimes A \to \mathrm{I}.$$

Hence, bras and kets are adjoint and the inner product has the form $(-)^\dagger \circ (-)$ on states. The sole *axiom* we impose is:

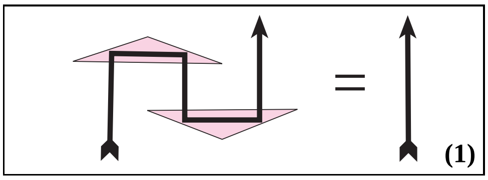


that is, algebraically,

$$(\eta_{A^*}^\dagger \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A \, .$$

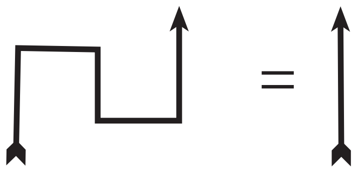
If we extend the graphical notation of Bell-(co)states to:

we obtain a clear graphical interpretation for the axiom:*



* Underlying this graphical presentation is the formal definition of a strongly compact closed category: it is a symmetric monoidal category in which there is (i) an involution $A \mapsto A^*$ on objects, (ii) a strict identity-on-objects contravariant monoidal involution $f \mapsto f^\dagger$, (iii) a given morphism $\eta_A : \mathrm{I} \to A^* \otimes A$ for each object A, such that the equivalent diagram to picture **(1)** commutes (which can be found in [7]). We assume moreover that all the natural isomorphisms of the structure are *unitary*, i.e. $U \circ U^\dagger = U^\dagger \circ U = 1$. Examples of these categories can be found in [3, 6, 7].

which now tells us that we are allowed to *yank* the black line:



— we called this line the *quantum information flow* [16]. The intuitive graphical calculus is an important benefit of the categorical axiomatics. Other advantages can be found in [6, 2, 5].

4 QUANTUM NON-LOGIC VS. QUANTUM HYPER-LOGIC

The term *quantum logic* is usually understood in connection with the 1936 Birkhoff-von Neumann proposal [13, 41] to consider the (closed) linear subspaces of a Hilbert space ordered by inclusion as the formal expression of the logical distinction between quantum and classical physics. While in classical logic we have deduction, the linear subspaces of a Hilbert space form a non-distributive lattice and hence there is no obvious notion of implication or deduction. Quantum logic was therefore always seen as logically very weak, or even a non-logic. In addition, it has never given a satisfactory account of compound systems and entanglement.

On the other hand, *compact closed logic* in a sense goes beyond ordinary logic in the principles it admits. Indeed, while in ordinary categorical logic "logical deduction" implies that *morphisms internalize as elements* (which above we referred to above as *states*) i.e.

$$B \xrightarrow{f} C \quad \overset{\simeq}{\longleftrightarrow} \quad \mathrm{I} \xrightarrow{\ulcorner f \urcorner} B \Rightarrow C$$

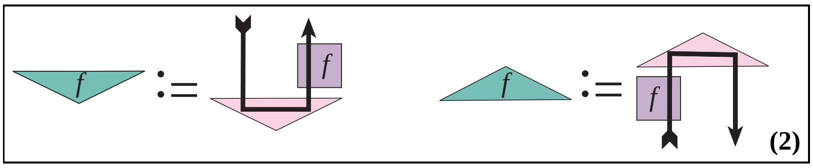
(where I is the $\otimes$-unit), in *compact closed logic* they internalize *both* as states *and* as costates i.e.

$$B \otimes C^* \xrightarrow{\llcorner f \lrcorner} \mathrm{I} \quad \overset{\simeq}{\longleftrightarrow} \quad B \xrightarrow{f} C \quad \overset{\simeq}{\longleftrightarrow} \quad \mathrm{I} \xrightarrow{\ulcorner f \urcorner} B^* \otimes C$$

where we introduce the following notation:

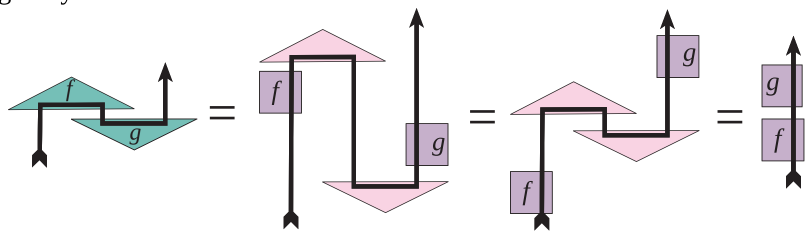
$$\ulcorner f \urcorner = (1_{A^*} \otimes f) \circ \eta_A : \mathrm{I} \to A^* \otimes B \quad \llcorner f \lrcorner = \eta_{B^*}^{\dagger} \circ (f \otimes 1_{B^*}) : A \otimes B^* \to \mathrm{I}\,.$$

It is exactly this dual internalization which allows the *yanking axiom* in picture **(1)** to be expressed. In the graphical calculus this is witnessed by the fact that we can define both a state and a costate

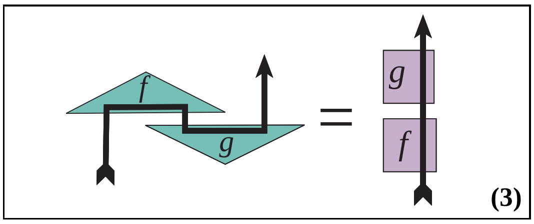


for each operation f. Physically, costates form the (destructive parts of) *projectors*, i.e. branches of projective measurements.

Compositionality. The semantics is obviously compositional, both with respect to sequential composition of operations and parallel composition of types and operations, allowing the description of systems to be built up from smaller components. But we also have something more specific: a form of compositionality with direct applications to the analysis of compound entangled systems. Since we have:

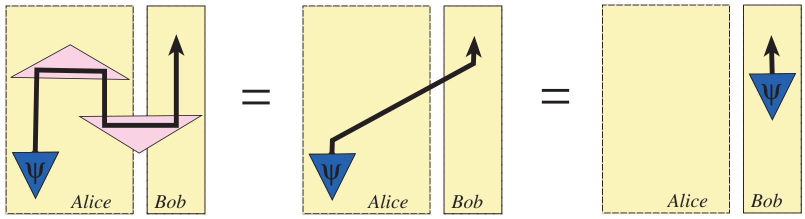


we obtain:

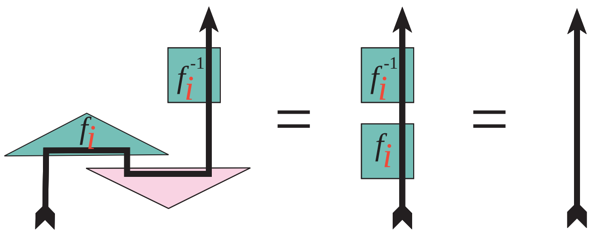


i.e. composition of operations *internalizes* in the behavior of entangled states and costates, and note in particular the interesting phenomenon of 'apparant reversal of the causal order' which is the source of many quite mystical interpretations of quantum teleportation in terms of 'traveling backward in time' — cf. [35]. Indeed, while on the left, physically, we first prepare the state labeled g and then apply the costate labeled f, the global effect is *as if* we first applied f itself first, and only then g.

Derivation of quantum teleportation. This is the most basic application of compositionality in action. Immediately from picture **(1)** we can read the quantum mechanical potential for teleportation:



This is not quite the whole story, because of the non-deterministic nature of measurements. But it suffices to introduce a unitary correction. Using picture **(3)** the full description of teleportation becomes:

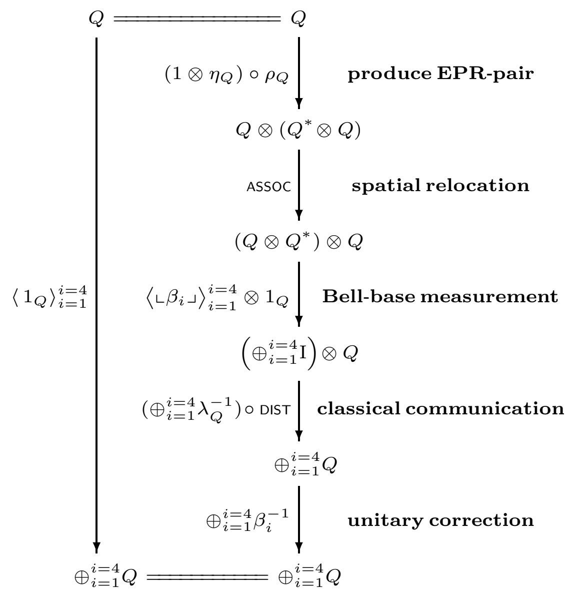


where the *classical communication* is now implicit in the fact that the index i is both present in the costate (= measurement-branch) and the correction, and hence needs to be send from Alice to Bob.

Related work. In [14] Braunstein, D'Adriano, Milburn and Sacchi extend Dirac notation to obtain results similar to the compositionality result expressed in picture **(3)**, in the concrete setting of Hilbert spaces. In [32] Louis Kauffman relies on very similar topological ideas to derive the teleportation protocol. In [11] John Baez discusses structures close to strong compact closure and compares these to models of topological quantum field theories.

5 CATEGORICAL ALGEBRA

A purely algebraic category-theoretic version of our picture story is in [6], where the 'branching due to measurements' is captured by *biproducts*. In this approach, the right-hand side of the diagram



gives a *complete description of the teleportation protocol*, as the *sequence of operations*:

$$(1 \otimes \eta_Q) \circ \rho_Q \,;\, \mathsf{ASSOC} \,;\, \left\langle \llcorner \beta_i \lrcorner \right\rangle_{i=1}^{i=4} \otimes 1_Q \,;\, (\oplus_{i=1}^{i=4} \lambda_Q^{\dagger}) \circ \mathsf{DIST} \,;\, \oplus_{i=1}^{i=4} \beta_i^{\dagger}$$

where $\oplus$ is the biproduct connective and $\langle - \rangle$ the corresponding *pairing* operation. In particular, the propagation of classical information from Alice to Bob on the outcome of the measurement is expressed by *distributivity* of the tensor product over the biproduct:

$$\mathsf{DIST} : (A_1 \otimes A_2) \oplus B \xrightarrow{\cong} (A_1 \otimes B) \oplus (A_2 \otimes B)\,.$$

This then allows the dependence of the subsequent unitary correction on the outcome of the measurement to be expressed directly in the formalism, e.g. by the further (quantum) action

$$(1 \otimes U_1) \oplus (1 \otimes U_2).$$

The left-hand side of the teleportation diagram expresses the *intended behavior*, which is the identity in each of the four pictures, so that the qubit is successfully transmitted in all cases, whatever the result of the measurement. In [6] we proved correctness, i.e. *the diagram commutes*, using the usual diagram-chasing methods of category theory. A similar derivation of Gottesman-Chuang logic-gate teleportation, both graphically and category-theoretically, uses the full power of **(3)** since in this case g will be the teleported gate. A logical approach to proving such facts, using *normalization of proof-nets*, is developed in [8]. An alternative axiomatization, which uses coalgebraic structure instead of biproducts, was recently developed in [20].

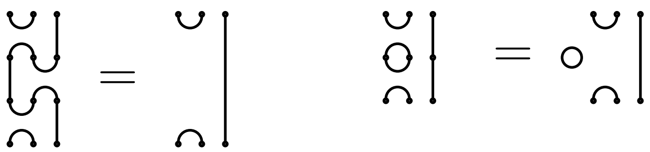
6 FROM CUT-ELIMINATION TO KNOT THEORY VIA QUANTUM MECHANICS

We shall also mention briefly some beautiful connections which are beginning to emerge between our categorical approach to quantum mechanics and quantum information, and work of the past two decades relating knot theory, topological quantum field theory, and statistical physics [31]. A huge swathe of mathematical developments linking all these fields and more were initiated by Vaughan Jones' discovery of his new polynomial invariant of knots and links. Jones' approach was algebraic; a central role was played by what has come to be known as the *Temperley-Lieb algebra*. (The original work of Temperley and Lieb was in discrete lattice models of statistical physics. In finding exact solutions for a certain class of systems, they had identified the same class of relations which Jones, quite independently, found later in his work). This was originally presented, rather forbiddingly, in terms of abstract generators and relations. It was recast in beautifully elementary and conceptual terms by Louis Kauffman as a *planar diagram algebra*.

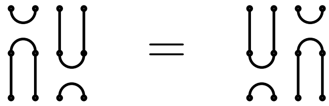
Generators:

U_1 $\qquad\qquad$ U_{n-1}

Relations:



$U_1 U_2 U_1 = U_1$ $\qquad\qquad$ $U_1^2 = \delta U_1$

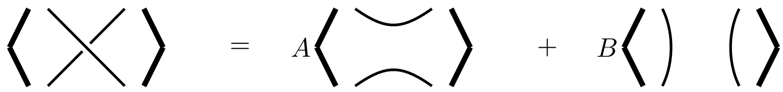


$$U_1 U_3 = U_3 U_1$$

We start with two parallel rows of n dots. The general form of an element of the algebra (actually of the basic multiplicative monoid: the algebra is then constructed freely over this as the "monoid algebra") is obtained by "joining up the dots" in a planar fashion. Multiplication xy is defined by identifying the bottom row of x with the top row of y. In general loops may be formed — these are "scalars", which can float freely across these figures, represented symbolically by δ above.

It should be clear that this diagram algebra is closely related to the graphical calculus described above. In fact, it arises by taking a non-symmetric version of the calculus (no crossings), with only one basic "generating" type A, which is taken to be self-dual: $A = A^*$. The "cups" and "caps" of the Temperley-Lieb algebra correspond to the basic triangles of the graphical calculus.

How does this connect to knots? Again, a key conceptual insight is due to Kauffman, who saw how to recast the Jones polynomial in elementary combinatorial form in terms of his *bracket polynomial*. The basic idea of the bracket polynomial is expressed by the following equation:



Each over-crossing in a knot or link is evaluated to a weighted sum of the two possible planar smoothings. With suitable choices for the coefficients A and B (as Laurent polynomials), this is invariant under the second and third Reidemeister moves. With an ingenious choice of normalizing factor, it becomes invariant under the first Reidemeister move — and yields the Jones polynomial! What this means algebraically is that the braid group has a representation in the Temperley-Lieb algebra — the above bracket evaluation shows how the basic generators of the braid group are mapped into the Temperley-Lieb algebra. Every knot arises as the closure of a braid; the invariant arises by mapping the open braid into the Temperley-Lieb algebra, and taking the trace there.

Moreover, it turns out that this connection can itself carry interesting information between the Computer Science ideas and the geometry and algebra. Indeed, using Computer Science methods it is possible to give the first *direct presentation* (no quotients) of the Temperley-Lieb algebra, using logical methods. In fact, the elements of the Temperley-Lieb algebra are completely determined by the relations they induce on the "dots"; and *planarity* can be characterized using only the ordering relations on the two rows of dots. Moreover, the multiplication of the algebra can be described as a form of Cut-Elimination, using the methods developed in the "Geometry of Interaction" [26, 1, 3].

We give a brief indication of the ideas. A diagram joining up a row of n dots with a row of m dots is formalized as a fixed-point free involution on $[n]^{\mathsf{op}} \lhd [m]$, where $[n]$ is the linear order

$$1 < 2 < \cdots < n$$

and $P \lhd Q$ is the concatenation of linear orders. Planarity is captured by the following axiom:

$$i < j < f(i) \quad \Rightarrow \quad i < f(j) < f(i).$$

Composition. Consider a map $f : [n] + [m] \longrightarrow [n] + [m]$. Each input lies in *either* $[n]$ *or* $[m]$ (exclusive or), and similarly for the corresponding output. This leads to a decomposition of f into four *disjoint partial maps*:

$$\begin{array}{ll} f_{n,n} : [n] \longrightarrow [n] & \qquad f_{n,m} : [n] \longrightarrow [m] \\ f_{m,n} : [m] \longrightarrow [n] & \qquad f_{m,m} : [m] \longrightarrow [m] \end{array}$$

so that f can be recovered as the disjoint union of these four maps. If f is an involution, then these maps will be partial involutions.

Now suppose we have maps $f : [n]+[m] \to [n]+[m]$ and $g : [m]+[p] \to [m] + [p]$. We write the decompositions of f and g as above in matrix form:

$$f = \begin{pmatrix} f_{n,n} & f_{n,m} \\ f_{m,n} & f_{m,m} \end{pmatrix} \qquad\qquad g = \begin{pmatrix} g_{m,m} & g_{m,p} \\ g_{p,m} & g_{p,p} \end{pmatrix}$$

The "Execution Formula". We can view these maps as *binary relations* on $[n] + [m]$ and $[m] + [p]$ respectively, and use relational algebra (union $R \cup S$, relational composition $R; S$ and reflexive transitive closure R^*) to define a **new relation** θ on $[n] + [p]$. If we write

$$\theta = \begin{pmatrix} \theta_{n,n} & \theta_{n,p} \\ \theta_{p,n} & \theta_{p,p} \end{pmatrix}$$

so that θ is the disjoint union of these four components, then we can define it component-wise as follows:

$$\begin{array}{lcl}
\theta_{n,n} & = & f_{n,n} \ \cup \ \ f_{n,m}; g_{m,m}; (f_{m,m}; g_{m,m})^*; f_{m,n} \\
\theta_{n,p} & = & f_{n,m}; (g_{m,m}; f_{m,m})^*; g_{m,p} \\
\theta_{p,n} & = & g_{p,m}; (f_{m,m}; g_{m,m})^*; f_{m,n} \\
\theta_{p,p} & = & g_{p,p} \ \cup \ \ g_{p,m}; f_{m,m}; (g_{m,m}; f_{m,m})^*; g_{m,p}.
\end{array}$$

Thus for example $\theta_{n,n}$ specifies which dots in the top row will be joined up after we multiply the two diagrams. This happens *either* if they were joined up by f (the first term of the union), *or* if f joins the ith dot in the top row to some dot j_1 in the middle row, g joins j_1 to j_2 in the middle row, ..., and so on until f joins j_k (k even) to a dot in the top row. The other components of θ can be read similarly.

This form of composition is standard in the Geometry of Interaction literature, and arises in a canonical way in constructing the free compact closed category from a traced monoidal category [29, 1].

Proposition 1 *If f and g are planar, so is θ.*

Cycles. Given $f \in \mathrm{P}(n,m)$, $g \in \mathrm{P}(m,p)$, we define $\chi(f,g) := f_{m,m}; g_{m,m}$. Note that $\chi(f,g)^c = (g_{m,m}; f_{m,m})$, and

$$\chi(f,g); \chi(f,g)^c \subseteq 1_{[m]}, \qquad\qquad \chi(f,g)^c; \chi(f,g) \subseteq 1_{[m]}.$$

Thus $\chi(f,g)$ is a *partial bijection*. However, in general it is neither an involution, nor fixpoint-free. The *cyclic elements* of $\chi(f,g)$ are those elements of $[m]$ which lie in the intersection

$$\chi(f,g)^+ \ \cap \ 1_{[m]}.$$

Thus if i is a cyclic element, there is a least $k > 0$ such that $\chi(f,g)^k(i) = i$. The corresponding *cycle* is

$$\{i, \ \chi(f,g)(i), \ \ldots, \ \chi(f,g)^{k-1}(i)\}.$$

Distinct cycles are disjoint. We write $Z(f,g)$ for the number of distinct cycles of $\chi(f,g)$.

The Temperley-Lieb category TL. Given $f \in \mathrm{P}(n,m)$, $g \in \mathrm{P}(m,p)$, we write $g \odot f \in \mathrm{P}(n,p)$ for the planar map constructed as above.

The objects of the Temperley-Lieb category are the natural numbers. A morphism $n \to m$ is a pair (s, f) where s is a natural number (representing the number of loops), and $f \in \mathrm{P}(n, m)$ is a planar map. Finally, we define the composition of morphisms in $\mathbf{TL}$. Given $(s, f) : n \to m$ and $(t, g) : m \to p$:

$$(t, g) \circ (s, f) = (s + t + Z(f, g), g \odot f).$$

There seems some potential here for a non-trivial interaction between geometrical and computational/logical ideas, at a foundational level. Further details will appear in a forthcoming paper [4].

7 CONCLUDING REMARKS

We see an exciting agenda for future research at the interface between Computer Science and Physics. This seems to be the right context for addressing many issues which are fundamental to future developments in Quantum Information and Computation, such as:

Q. What are the precise structural relationships between parallelism, entanglement and mixedness as quantum informatic resources?

Q. Which features of quantum mechanics account for differences in computational and informatic power as compared to classical computation?

Q. How do quantum and classical information interact with each other, and with a spatio-temporal causal structure?

Q. Which quantum control features (e.g. iteration) are possible and what additional computational power can they provide?

Q. What is the precise logical status and axiomatics of (No-)Cloning and (No-)Deleting, and more generally, of the quantum mechanical formalism as a whole?

The connections to geometry briefly sketched in the previous section also merit further investigation, and raise many interesting issues. Note firstly that the Temperley-Lieb category can be characterized as the free non-symmetric strongly compact closed category over a single, self-dual generator ($A = A^*$). (More precisely, the free *pivotal category* [25] over one self-dual generator.) This gives an immediate connection to our categorical and diagrammatic approach to Quantum Mechanics. It also leads to a number of intriguing questions:

- If we take planarity as a constraint on Geometry of Interaction, and the corresponding logics we may interpret, what impact does this have on expressiveness? For example, can we still represent all poly-time functions subject to this constraint?

- We can ask the same kind of question with respect to Quantum Informatics. it seems in practice that few naturally occurring quantum protocols require the use of the symmetry maps. How much of Quantum Informatics can be done in the plane? What is the significance of this constraint?

- Beyond the planar world we have *braiding*, which carries 3-dimensional geometric information. Does this information have some computational significance? Some ideas in this direction have been explored by Kauffman and Lomonaco [33], but no clear understanding has yet been achieved.

- Beyond this, we have the general setting of TQFT (Topological Quantum Field Theories) [46, 10] and related notions. This may be relevant to Quantum Informatic concerns in (at least) two ways:

    1. A novel and promising paradigm of *Topological Quantum Computing* has recently been proposed [24].
    2. As the issues arising from *distributed quantum computing*, *quantum security protocols* etc. are investigated, the interactions between quantum informatics and spatio-temporal structure will inevitably need to be considered.

There are a rich set of questions here, which will surely provide fertile ground for research involving both the Computer Science and Physics communities.[†]

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[†] We note that the issues raised here are also well within the scope of one of the current Grand Challenges in Computing Research, namely GC7: Journeys in Unconventional Computing, for which see the web-site

`http://www.cs.york.ac.uk/nature/gc7/`.

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