Theorem: A subset of a metric space is compact if and only if it is sequentially compact.

Proof:

$\boxed{\Rightarrow}$ Suppose that $(\mathbb{X}, d)$ is a compact metric space. Further, suppose that it is not sequentially compact.

- If $\mathbb{X}$ is not sequentially compact, there exists a sequence (x_n) in $\mathbb{X}$ that has no convergent subsequence. Since there is no convergent subsequence, (x_n) must contain an infinite number of distinct points. (If there were only a finite number of distinct points, the sequence would eventually become constant and would therefore be convergent and thus all subsequences would be convergent!)
- Let $x \in \mathbb{X}$. If, for every $\varepsilon > 0$, the ball $B_\varepsilon(x)$ contains a point in the sequence (x_n) that is distinct from x, the x will be the limit of a subsequence since we would be able to choose points from (x_n) from shrinking balls around x. So, there is a $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x)$ contains no points from (x_n), except possibly x itself.
- The collection of open balls $\{B_{\varepsilon_x}(x) : x \in \mathbb{X}\}$ is an open cover of $\mathbb{X}$.
- The union of every finite number of these balls contains at most n terms in the sequence. Because there are an infinite number of distinct terms in the sequence, no finite subcollection of these balls will cover $\mathbb{X}$ since no finite subcollection will even cover the terms of the sequence (x_n) in $\mathbb{X}$.
- So, we have found an open cover of $\mathbb{X}$ that has no finite subcover. This contradicts that $\mathbb{X}$ is compact. Therefore, $\mathbb{X}$ must be sequentially compact.

$\boxed{\Leftarrow}$ Now suppose that $(\mathbb{X}, d)$ is sequentially compact. Let $\{G_\alpha\}$ be an arbitrary open cover of $\mathbb{X}$.

- From the Lemma at the beginning of this solutions, $\mathbb{X}$ is separable which means that $\mathbb{X}$ contains a countable dense subset A.
- Let $\mathcal{B}$ be the collection of open balls with rational radius and center in A. Since A is countable and the rationals are countable, $\mathcal{B}$ is countable.
- Let $\mathcal{C}$ be the subcollection of balls in $\mathcal{B}$ that are contained in at least one of the open sets in the cover $\{G_\alpha\}$. Since $\mathcal{C}$ is a subset of $\mathcal{B}$ and $\mathcal{B}$ is countable, $\mathcal{C}$ is countable.
- For every $x \in \mathbb{X}$ there is a G_α such that $x \in G_\alpha$. Since G_α is open, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq G_\alpha$.
- Since A is dense in $\mathbb{X}$, there exists a point $y \in A$ that is within $\varepsilon/3$ of x. Note then that $x \in B_{\varepsilon/3}(y)$ and that $B_{2\varepsilon/3}(y) \subseteq G_\alpha$.

- Take $q \in \mathbb{Q}$ such that $\varepsilon/3 < q < 2\varepsilon/3$. Then $x \in B_q(y) \subseteq B_{2\varepsilon/3}(y) \subseteq G_\alpha$. Since $B_q(y)$ has rational radius and center in A it is a ball in $\mathcal{B}$. Furthermore, since it is a ball in $\mathcal{B}$ that is contained in a G_α, it is in the collection $\mathcal{C}$.

- Thus, every $x \in \mathbb{X}$ belongs to a ball in $\mathcal{C}$. So, $\mathcal{C}$ is a countable open cover of $\mathbb{X}$!

- Every ball $B \in \mathcal{C}$ is in at least one set G_α in $\{G_\alpha\}$. Pick an index α_B such that $B \subseteq G_{\alpha_B}$. Since $\mathcal{C}$ is countable and covers $\mathbb{X}$ and since $\{G_{\alpha_B} | B \in \mathcal{C}\}$ covers $\mathcal{C}$, $\{G_{\alpha_B} | B \in \mathcal{C}\}$ countable subcover (of the open cover $\{G_\alpha\}$) of $\mathbb{X}$.

- We wanted to show that an open cover of a sequentially compact space has a finite subcover. So far, we have shown that it has a countable subcover. We will now show that a countable open cover of a sequentially compact space has a finite subcover.

  Ignore all of the previous notation and assume that $\{G_n\}$ is a countable open cover of $\mathbb{X}$. Assume that there is no finite subcover. We are going to construct a sequence in $\mathbb{X}$ that has no convergent subsequence, thereby contradicting that $\mathbb{X}$ is sequentially compact.

- Since $\{G_n\}$ has no finite subcover, $\cup_{k=1}^{n} G_k$ does not contain $\mathbb{X}$ for any n.

  Construction of the sequence:

  - Choose $x_1 \in \mathbb{X}$. Since $\{G_n\}$ covers $\mathbb{X}$, there exists an n_1 such that $x_1 \in G_{n_1}$.
  - Choose $x_2 \in \mathbb{X}$ such that $x_2 \notin \cup_{n=1}^{n_1} G_n$. We can do this because we have assumed that $\mathbb{X}$ can not be covered by a finite subset of $\{G_n\}$. Since $\{G_n\}$ covers $\mathbb{X}$, there exists an n_2 such that $x_2 \in G_{n_2}$.
  - Choose $x_3 \in \mathbb{X}$ such that $x_3 \notin \cup_{n=1}^{n_3} G_n$. Choose n_3 so that $x_3 \in G_{n_3}$.
  - Et cetera! Note that

    $$x_k \in G_{n_k} \qquad \text{and} \qquad x_k \notin \cup_{n=1}^{n_k - 1} G_n.$$

    So, G_{n_k} is not equal to G_n for any $n = 1, 2, \ldots, n_{k-1}$, and the sequence (n_k) is strictly increasing.

- Since $\mathbb{X}$ is sequentially compact, (x_n) must have a subsequence that converges to a point $x \in \mathbb{X}$. Since $\{G_n\}$ covers $\mathbb{X}$, $x \in G_n$ for some n.

- However, by construction of our sequence, there exists an integer K_n such that $x_k \notin G_n$ for all $k \geq K_n$.

- $x \in G_n$ yet the sequence (x_n), and hence any subsequence of (x_n) can not be in G_n after some point. This contradicts the statement that (x_n) must have a subsequence converging to x and the sequential compactness of $\mathbb{X}$.

- Therefore, the open cover $\{G_n\}$ must have a finite subcover and $\mathbb{X}$ is compact.