Event Structures, Stable Families and Concurrent Games

Glynn Winskel
gw104@cl.cam.ac.uk

February 2017

Preface

These notes introduce a theory of two-party games still under development. A lot can be said for a general theory to unify all manner of games found in the literature. But this has not been the main motivation. That has been the development of a generalized domain theory, to lift the methodology of domain theory and denotational semantics to address the highly interactive nature of computation we find today.

There are several arguments why the next generation of domain theory should be an intensional theory, one which pays careful attention to the ways in which output is computed from input. One is that if the theory is to be able to reason about operational concerns it had better address them, albeit abstractly. Another is that sometimes the demands of compositionality force denotations to be more intensional than one would at first expect; this occurs for example with nondeterministic dataflow—see the Introduction. These notes take seriously the idea that intensional aspects be described by strategies, and, to fit computational needs adequately, try to understand the concept of strategy very broadly.

This idea comes from game semantics where the domains and continuous functions of traditional domain theory and denotational semantics are replaced by games and strategies. Strategies supercede functions because they give a much better account of interaction extended in time. (Functions, if you like, have too clean a separation of interaction into input and output.) In traditional denotational semantics a program phrase or process term denotes a continuous function, whereas in game semantics a program phrase or process term denotes a strategy.

However, traditional game semantics is not always general enough, for instance in accounting for nondeterministic or concurrent computation. Rather than extending traditional game semantics with various bells and whistles, these notes attempt to carve out a general theory of games within a general model of nondeterministic, concurrent computation. The model chosen is the partial-order model of event structures, and for technical reasons, its enlargement to stable families. Event structures have the advantage of occupying a central position within models for concurrency, and the development here should suggest analogous developments for other 'partial-order' models such as Mazurkiewicz trace languages, Petri nets and asynchronous transition systems, and even 'interleaving' models based on transition systems or sequences.

In their present state, these notes are inadequate in several ways. First, they don't account for games with back-tracking, games where play can revisit previous positions. While a little odd from the point of view of everyday games, this feature is very important in game semantics, for instance in order to re-evaluate the argument to a function.[1] Second, the notes don't have enough examples. Third, the notes say too little on the *uses* of games and strategies in

[1]The theory has been extended to allow back-tracking and copying via event structures with symmetry, which support a rich variety of pseudo (co)monads to achieve this—see the paper on "Games with Symmetry" with Castellan and Clairambault on my homepage.

semantics, types, logic and verification. I hope to some extent to make up for these inadequacies in the lectures. What I claim the notes do do, is begin to unify a variety of approaches and provide canonical general constructions and results, which leave the student better placed to structure and analyse critically the often arcane world of games and strategies in the literature.

Such was the preface to the first version of these notes for a lecture course at Aarhus University in the late summer of 2011. The subject of concurrent games has grown since that first version of these notes. The notes ended up being my partial summary of research within the ERC-funded ECSYM project ("Events, Causality and Symmetry") concentrating on the situation as I saw it and a way to consolidate my understanding at the time. Consequently progress on the notes has often been outstripped by work done with my ECSYM colleagues. Another consequence is that the notes follow the line of discovery rather than what is possibly the most natural conceptual line. Latest developments are presented in papers on my Computer Lab home page.

Contents

Chapter 1

Introduction

Games and strategies are everywhere, in logic, philosophy, computer science, economics, in leisure and in life.

Slogan: Processes are nondeterministic concurrent strategies.

1.1 Motivation

We summarise some reasons for developing a theory of nondeterministic concurrent games and strategies.

1.1.1 What is a process?

In the earliest days of computer science it became accepted that a computation was essentially an (effective) partial function $f : \mathbb{N} \to \mathbb{N}$ between the natural numbers. This view underpins the Church-Turing thesis on the universality of computability.

As computer science matured it demanded increasingly sophisticated mathematical representations of processes. The pioneering work of Strachey and Scott in the denotational semantics of programs assumed a view of a process still as a function $f : D \to D'$, but now acting in a continuous fashion between datatypes represented as special topological spaces, 'domains' D and D'; reflecting the fact that computers can act on complicated, conceptually-infinite objects, but only by virtue of their finite approximations.

In the 1960's, around the time that Strachey started the programme of denotational semantics, Petri advocated his radical view of a process, expressed in terms of its events and their effect on local states—a model which addressed directly the potentially distributed nature of computation, but which, in common with many other current models, ignored the distinction between data and process implicit in regarding a process as a function. Here it seems that an adequate notion of process requires a marriage of Petri's view of a process and the vision of Scott and Strachey. An early hint in this direction came in answer to the following question.

What is the information order in domains? There are essentially two answers in the literature, the '*topological*,' the most well-known from Scott's work, and the '*temporal*,' arising from the work of Berry:

• *Topological*: the basic units of information are *propositions* describing finite properties; more information corresponds to more propositions being true. Functions are ordered pointwise.

• *Temporal*: the basic units of information are *events*; more information corresponds to more events having occurred over time. Functions are restricted to 'stable' functions and ordered by the intensional 'stable order,' in which common output has to be produced for the same minimal input. Berry's specialized domains 'dI-domains' are represented by event structures.

In truth, Berry developed 'stable domain theory' by a careful study of how to obtain a suitable category of domains with stable rather than all continuous functions. He arrived at the axioms for his 'dI-domains' because he wanted function spaces (so a cartesian-closed category). The realization that dI-domains were precisely those domains which could be represented by event structures, came a little later.

1.1.2 From models for concurrency

Causal models are alternatively described as: causal-dependence models; independence models; non-interleaving models; true-concurrency models; and partial-order models. They include Petri nets, event structures, Mazurkiewicz trace languages, transition systems with independence, multiset rewriting, and many more. The models share the central feature that they represent processes in terms of the events they can perform, and that they make explicit the causal dependency and conflicts between events.

Causal models have arisen, and have sometimes been rediscovered as *the* natural model, in many diverse and often unexpected areas of application:
Security protocols: for example, forms of event structure, strand spaces, support reasoning about secrecy and authentication through causal relations and the freshness of names;
Systems biology: ideas from Petri nets and event structures are used in taming the state-explosion in the stochastic simulation of biochemical processes and in the analysis of biochemical pathways;
Hardware: in the design and analysis of asynchronous circuits;
Types and proof: event structures appear as representations of propositions as types, and of proofs;
Nondeterministic dataflow: where numerous researchers have used or rediscovered causal models in providing a compositional semantics to nondeterministic dataflow;
Network diagnostics: in the patching together local of fault diagnoses of com-

munication networks;
Logic of programs: in concurrent separation logic where artificialities in Brookes' pioneering soundness proof are obviated through a Petri-net model;
Partial order model checking: following the seminal work of McMillan the unfolding of Petri nets (described below) is exploited in recent automated analysis of systems;
Distributed computation: event structures appear both classically,*e.g.* in early work of Lamport, and recently in the Bayesian analysis of trust and modelling multicore memory.

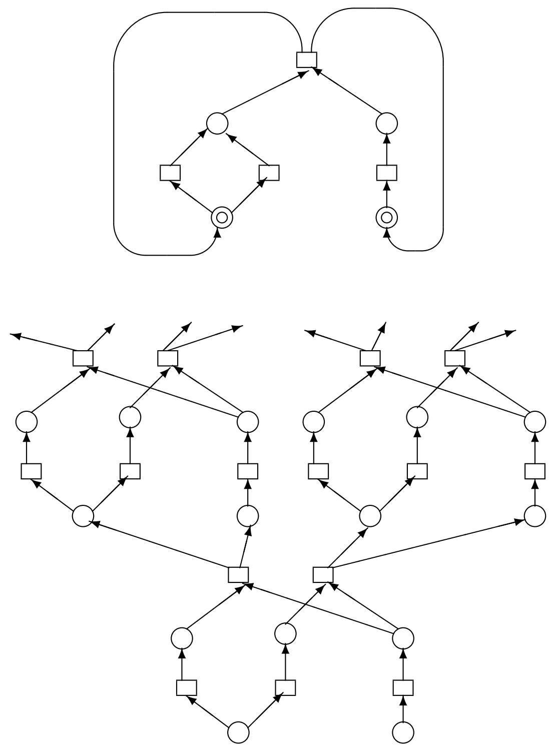
To illustrate the close relationship between Petri nets and the 'partial-order models' of occurrence nets and event structures, we sketch how a (1-safe) Petri net can be unfolded first to a net of occurrences and from there to an event structure [1]. The unfolding construction is analogous to the well-known method of unfolding a transition system to a tree, and is central to several analysis tools in the applications above. In the figure, the net on top has loops. The net below it is its *occurrence-net unfolding.* It consists of all the occurrences of conditions and events of the original net, and is infinite because of the original repetitive behaviour. The occurrences keep track of what enabled them. The simplest form of event structure, the one we shall consider here, arises by abstracting away the conditions in the occurrence net and capturing their role in relations of causal dependency and conflict on event occurrences.

The relations between the different forms of causal models are well understood [2]. Despite this and their often very successful, specialized applications, causal models lack a *comprehensive* theory which would support their systematic use in giving semantics to a broad range of programming and process languages, in particular we lack an expressive form of '*domain theory*' for causal models with rich higher-order type constructions needed by mathematical semantics.

1.1.3 From semantics

Denotational semantics and domain theory of Scott and Strachey set the standard for semantics of computation. The theory provided a global mathematical setting for sequential computation, and thereby placed programming languages in connection with each other; connected with the mathematical worlds of algebra, topology and logic; and inspired programming languages, type disciplines and methods of reasoning. Despite the many striking successes it has become very clear that many aspects of computation do not fit within the traditional framework of denotational semantics and domain theory. In particular, classical domain theory has not scaled up to the more intricate models used in interactive/distributed computation. Nor has it been as operationally informative as one could hope.

While, as Kahn was early to show, deterministic dataflow is a shining application of simple domain theory, nondeterministic dataflow is beyond its scope. The compositional semantics of nondeterministic dataflow needs a form of generalized relation which specifies the *ways* input-output pairs are realized.A compelling example comes from the early work of Brock and Ackerman who were

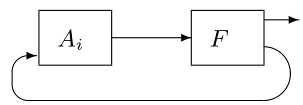


A Petri net and its occurrence-net unfolding

the first to emphasize the difficulties in giving a compositional semantics to nondeterministic dataflow, though our example is based on simplifications in the later work of Rabinovich and Trakhtenbrot, and Russell.

Nondeterministic dataflow—Brock-Ackerman anomaly

$C[A_i] =$



There are two simple nondeterministic processes A_1 and A_2, which have the same input-output relation, and yet behave differently in the common feedback context $C[-]$, illustrated above. The context consists of a fork process F (a process that copies every input to two outputs), through which the output of the automata A_i is fed back to the input channel, as shown in the figure. Process A_1 has a choice between two behaviours: either it outputs a token and stops, *or* it outputs a token, waits for a token on input and then outputs another token. Process A_2 has a similar nondeterministic behaviour: Either it outputs a token and stops, *or* it waits for an input token, then outputs two tokens. For both automata, the input-output relation relates empty input to the eventual output of one token, and non-empty input to one or two output tokens. But $C[A_1]$ can output two tokens, whereas $C[A_2]$ can only output a single token. Notice that A_1 has two ways to realize the output of a single token from empty input, while A_2 only has one. It is this extra way, not caught in a simple input-output relation, that gives A_1 the richer behaviour in the feedback context.

Over the years there have been many solutions to giving a compositional semantics to nondeterministic dataflow. But they all hinge on some form of generalized relation, to distinguish the different ways in which output is produced from input. A compositional semantics can be given using *stable spans* of event structures, an extension of Berry's stable functions to include nondeterminism [3]—see Section 6.2.1.

How are we to extend the methodology of denotational semantics to the much broader forms of computational processes we need to design, understand and analyze today? How are we to maintain clean algebraic structure and abstraction alongside the operational nature of computation?

Game semantics advanced the idea of replacing the traditional continuous functions of domain theory and denotational semantics by strategies. The reason for doing this was to obtain a representation of interaction in computation that was more faithful to operational reality. It is not always convenient or mathematically tractable to assume that the environment interacts with a computation in the form of an input argument. It is built into the view of a process as a strategy that the environment can direct the course of evolution of a process throughout its duration. Game semantics has had many dramatic successes. But it has developed from simple well-understood games, based on alternating sequences of player and opponent moves, to sometimes arcane extensions and generalizations designed to fit the demands of a succession of additional programming or process features. It is perhaps time to stand back and see how games fit within a very general model of computation, to understand better what current features of games in computer science are simply artefacts of the particular history of their development.

1.1.4 From logic

An informal understanding of games and strategies goes back at least as far as the ancient Greeks where truth was sought through debate using the dialectic method; a contention being true if there was an argument for it that could survive all counter-arguments. Formalizing this idea, logicians such as Lorenzen and Blass investigated the meaning of a logical assertion through strategies in a game built up from the assertion. These ideas were reinforced in game semantics which can provide semantics to proofs as well as programs. The study of the mathematics and computational nature of proof continues. There are several strands of motivation for games in logic. Along with automata games constitute one of the tools of logic and algorithmics; often a logical or algorithmic question can be reduced to the question of whether a particular game has a winning/optimal strategy or counterstrategy. Games are used in verification and, for example, the central equivalence of bisimulation on processes has a reading in terms of strategies.

Chapter 2

Event structures

Event structures are a fundamental model of concurrent computation and, along with their extension to stable families, provide a mathematical foundation for the course.

2.1 Event structures

Event structures are a model of computational processes. They represent a process, or system, as a set of event occurrences with relations to express how events causally depend on others, or exclude other events from occurring. In one of their simpler forms they consist of a set of events on which there is a consistency relation expressing when events can occur together in a history and a partial order of causal dependency—writing $e' \leq e$ if the occurrence of e depends on the previous occurrence of e'.

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$
\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con} \;\&\; e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.
\end{aligned}
$$

The events are to be thought of as event occurrences without significant duration; in any history an event is to appear at most once. We say that events e, e' are *concurrent*, and write $e \; co \; e'$ if $\{e, e'\} \in \mathrm{Con} \;\&\; e \not\leq e' \;\&\; e' \not\leq e$. Concurrent events can occur together, independently of each other. The relation of *immediate* dependency $e \rightarrowtriangle e'$ means e and e' are distinct with $e \leq e'$ and no event in between. Clearly $\leq$ is the reflexive transitive closure of $\rightarrowtriangle$.

An event structure represents a process. A configuration is the set of all events which may have occurred by some stage, or history, in the evolution of the process. According to our understanding of the consistency relation and causal dependency relations a configuration should be consistent and such that if an event appears in a configuration then so do all the events on which it causally depends.

The *configurations* of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and

Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

We shall largely work with *finite* configurations, written $\mathcal{C}(E)$. Write $\mathcal{C}^\infty(E)$ for the set of *finite and infinite* configurations of the event structure E.

The configurations of an event structure are ordered by inclusion, where $x \subseteq x'$, *i.e.* x is a sub-configuration of x', means that x is a sub-history of x'. Note that an individual configuration inherits an order of causal dependency on its events from the event structure so that the history of a process is captured through a partial order of events. The finite configurations correspond to those events which have occurred by some finite stage in the evolution of the process, and so describe the possible (finite) states of the process.

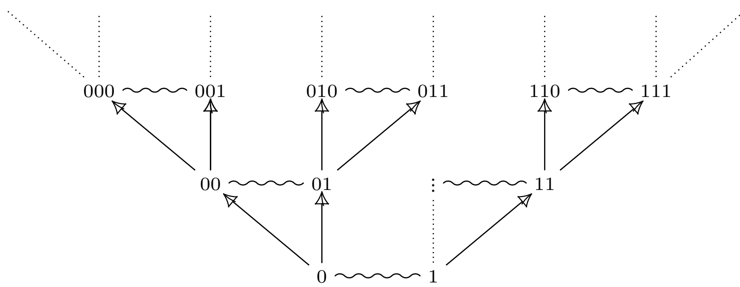
For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X. The axioms on the consistency relation ensure that the down-closure of any finite set in the consistency relation s a finite configuration, and that any event appears in a configuration: given $X \in \mathrm{Con}$ its down-closure $\{e' \in E \mid \exists e \in X.\ e' \leq e\}$ is a finite configuration; in particular, for an event e, the set $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e. We shall sometimes write $[e) =_{\mathrm{def}} \{e' \in E \mid e' < e\}$.

When the consistency relation is determined by the pairwise consistency of events we can replace it by a binary relation or, as is more usual, by a complementary binary conflict relation on events (written as $\#$ or $\smile$).

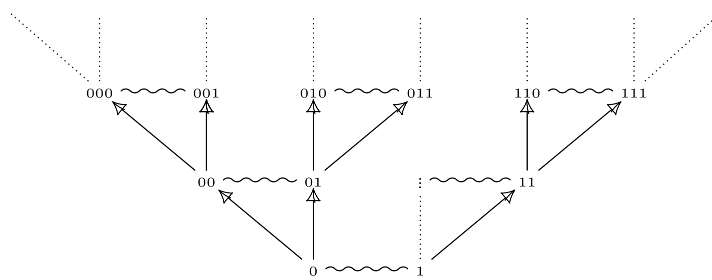
Remark on the use of "cause." In an event structure $(E, \leq, \mathrm{Con})$ the relation $e' \leq e$ means that the occurrence of e depends on the previous occurrence of the event e'; if the event e has occurred then the event e' must have occurred previously. In informal speech cause is also used in the forward-looking sense of one thing arising because of another. Often when used in this way the history of events is understood or presupposed. According to the history around my life, the meeting of my parents caused my birth. But the history might have been very different: in an alternative world the meeting of my parents might not have led to my birth. More formally, w.r.t. a configuration x in which an event e occurs while it seems sensible to talk about the events $[e)$ causing e, it is so only by virtue of the understood configuration x.

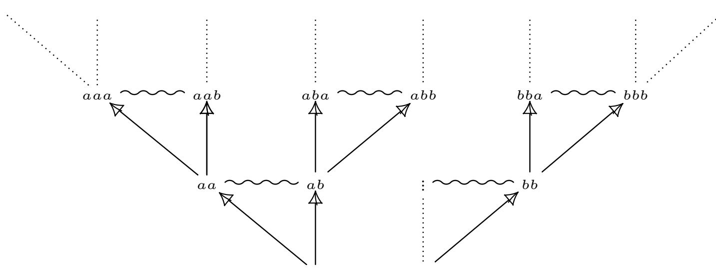
We also encounter events which in a history may have been caused in more than one way. There are generalisations of the current event structures which do this—see Chapter 16, on "disjunctive causes." But for now we will work with the simple definition above in which an event, or really an event occurrence, e is causally dependent on a unique set of events $[e)$. Much of the mathematics we develop around these simpler forms of event structures (sometimes called prime event structures in the literature) will be reusable when we come to consider events with several causes. Roughly the simpler event structures will suffice in considering nondeterministic strategies. Where their limitations will first show up is in our treatment of probabilistic strategies.

Example 2.1. The diagram below illustrates an event structure representing streams of 0s and 1s:



Above we have indicated conflict (or inconsistency) between events by $\sim\!\sim\!\sim$. The event structure representing pairs of 0/1-streams and a/b-streams is represented by the juxtaposition of two event structures:





Exercise 2.2. *Draw the event structure of the occurrence net unfolding in the introduction.* □

2.2 Maps of event structures

Let E and E' be event structures. A *(partial) map* of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}^\infty(E)$ its direct image $fx \in \mathcal{C}^\infty(E')$ and

$$\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2) \text{ (with both defined), then } e_1 = e_2.$$

(Those maps defined is unaffected if we replace possibly infinite configurations $\mathcal{C}^\infty(E)$ by finite configurations $\mathcal{C}(E)$ above; this is because any configuration is the union of finite configurations and direct image preserves such unions.) The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. The map f respects the instantaneous nature of events: two distinct event occurrences which are consistent with each other cannot both coincide with the occurrence of a common event in the image. Partial maps of event structures compose as partial functions, with identity maps given by identity functions.

Proposition 2.3. *Let $f : E \to E'$ be a map of event structures. Then,*
(i) f locally reflects causal dependency: whenever $e, e' \in x$, a configuration of E, and $f(e)$ and $f(e')$ are both defined with $f(e') \le f(e)$, then $e' \le e$;
(ii) f preserves the concurrency relation, when defined: if e co e' in E and $f(e)$ and $f(e')$ are both defined then $f(e)$ co $f(e')$.

Proof. (i) Let $x \in \mathcal{C}^\infty(E), e, e' \in E$ with $f(e') \le f(e)$ (both being defined). The map $f : E \to E'$ must send the configuration $[e]$ to the configuration $f[e]$. As $f[e]$ is down-closed there must be $e'' \in [e]$ such that $f(e'') = f(e')$. But because f is locally injective on x and both $e', e'' \in x$ we see that $e' = e''$ so $e' \in [e]$, *i.e.* $e' \le e$. Consequently the map f locally reflects causal dependency: whenever $e, e' \in x$, a configuration of E, and $f(e)$ and $f(e')$ are both defined with $f(e') \le f(e)$, then $e' \le e$.

(ii) Suppose e *co* e' in E and $f(e)$ and $f(e')$ are both defined. Then $\{e, e'\} \in \mathrm{Con}_E$. Hence their down-closure $[e, e'] \in \mathcal{C}(E)$. It follows that $f[e, e'] \in \mathcal{C}(E')$ making $\{f(e), f(e')\} \in \mathrm{Con}_{E'}$ with $f(e)$ and $f(e')$ incomparable w.r.t. $\le_{E'}$ by (i); this ensures $f(e)$ *co* $f(e')$. □

We will say the map is *total* if the function f is total. Notice that for a total map f the condition on maps now says it is *locally injective*, in the sense that w.r.t. any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective. Say a total map of event structures is *rigid* when it preserves causal dependency.

Proposition 2.4. *Let $f : E \to E'$ be a total map of event structures. Then, for $e_1, e_2 \in E$,*

$$e_1 \rightarrowtriangle e_2 \implies f(e_1) \text{ co } f(e_2) \text{ or } f(e_1) \rightarrowtriangle f(e_2)\,.$$

Proof. Assume $e_1 \rightarrowtriangle e_2$ and not $f(e_1)$ *co* $f(e_2)$. Then as $\{f(e_1), f(e_2)\} \in \mathrm{Con}$, we have $f(e_1) \le f(e_2)$ or $f(e_2) \le f(e_1)$. As f reflects causal dependency locally w.r.t. the configuration $[e_2]$, the dependency $f(e_2) \le f(e_1)$ would entail the $e_2 \le e_1$, contradicting $e_1 \rightarrowtriangle e_2$. Hence $f(e_1) \le f(e_2)$. As a consequence,

$$f(e_1) \rightarrowtriangle \cdots \rightarrowtriangle f(e_2)$$

for some chain of immediate causal dependencies in E'. As f is total and reflects causal dependency locally w.r.t. the configuration $[e_2]$, we obtain a chain

$$e_1 \rightarrowtriangle \cdots \rightarrowtriangle e_2$$

in E of equal length. However, $e_1 \rightarrowtriangle e_2$ so the chain must be of length one, ensuring $f(e_1) \rightarrowtriangle f(e_2)$. □

Definition 2.5. Write $\mathcal{E}$ for the category of event structures with (partial) maps. Write $\mathcal{E}_t$ and $\mathcal{E}_r$ for the categories of event structures with total, respectively rigid, maps.

Exercise 2.6. *Show a map $f : A \rightharpoonup B$ of $\mathcal{E}$ is mono if the function $\mathcal{C}(A) \to \mathcal{C}(B)$ taking configuration x to its direct image fx is injective. [Recall a map $f : A \to B$ is mono iff for all maps $g, h : C \to A$ if $fg = fh$ then $g = h$.] Show the converse does not hold, that it is possible for a map to be mono but not injective on configurations.* □

Proposition 2.7. *Let E and E' be event structures. Suppose*

$$\theta_x : x \cong \theta_x x, \text{ indexed by } x \in \mathcal{C}(E),$$

is a family of bijections such that whenever $\theta_y : y \cong \theta_y y$ is in the family then its restriction $\theta_z : z \cong \theta_z z$ is also in the family, whenever $z \in \mathcal{C}(E)$ and $z \subseteq y$. Then, $\theta =_{\mathrm{def}} \bigcup_{x \in \mathcal{C}(E)} \theta_x$ is the unique total map of event structures from E to E' such that $\theta\, x = \theta_x x$ for all $x \in \mathcal{C}(E)$.

Proof. The conditions ensure that $\theta =_{\mathrm{def}} \bigcup_{x \in \mathcal{C}(A)} \theta_x$ is a function $\theta : A \to B$ such that the image of any finite configuration x of A under θ is a configuration of B and local injectivity holds. □

2.2.1 Partial-total factorisation

Let $(E, \le, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \le_V, \mathrm{Con}_V)$, where $v \le_V v'$ iff $v \le v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$.

Consider a partial map of event structures $f : E \to E'$. Let

$$V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\}.$$

Then f clearly factors into the composition

$$E \xrightarrow{f_0} E{\downarrow}V \xrightarrow{f_1} E'$$

of f_0, a partial map of event structures taking $e \in E$ to itself if $e \in V$ and undefined otherwise, and f_1, a total map of event structures acting like f on V. We call f_1 the *defined part* of the partial map f. We say a map $f : E \to E'$ is a *projection* if its defined part is an isomorphism.

The factorisation is characterised to within isomorphism by the following universal characterisation: for any factorisation

$$E \xrightarrow{g_0} E_1 \xrightarrow{g_1} E'$$

where g_0 is partial and g_1 is total there is a (necessarily total) unique map $h : E{\downarrow}V \to E_1$ such that

$$\begin{tikzcd} E \arrow[r, "f_0"] \arrow[dr, "g_0"'] & E{\downarrow}V \arrow[r, "f_1"] \arrow[d, dashed, "h"] & E' \\ & E_1 \arrow[ur, "g_1"'] & \end{tikzcd}$$

commutes.

2.3 Rigid maps

Recall a map f is *rigid* iff it is total and f preserves causal dependency, *i.e.*, if $e' \leq e$ in E then $f(e') \leq f(e)$ in E'.

Proposition 2.8. *A total map $f : E \to E'$ of event structures is rigid iff for all $x \in \mathcal{C}(E)$ and $y \in \mathcal{C}(E')$*

$$y \subseteq f(x) \implies \exists z \in \mathcal{C}(E).\ z \subseteq x \text{ and } fz = y\ .$$

The configuration z is necessarily unique by the local injectivity of f. (The class of maps would be unaffected if we allow all configurations in the definition above.)

Proof. "*Only if*": Total maps reflect causal dependency. So, if f preserves causal dependency, then for any configuration x of E, the bijection $f : x \to fx$ preserves and reflects causal dependency. Hence for any subconfiguration y of fx, the bijection restricts to a bijection $f : z \to y$ with z a down-closed subset of x. But then z must be a configuration of E. "*If*": Let $e \in E$. Then $[f(e)] \subseteq f[e]$. Hence there is a subconfiguration z of $[e]$ such that $fz = [f(e)]$. By local injectivity, $e \in z$, so $z = [e]$. Hence $f[e] = [f(e)]$. It follows that if $e' \leq e$ then $f(e') \leq f(e)$. □

A rigid map of event structures preserves the causal dependency relation "rigidly," so that the causal dependency relation on the image fx is a copy of that on a configuration x of E—in this sense f is a local isomorphism. This is not so for general maps where x may be augmented with extra causal dependency over that on fx.

Proposition 2.9. *The inclusion functor $\mathcal{E}_r \hookrightarrow \mathcal{E}_t$ has a right adjoint. The category $\mathcal{E}_t$ is isomorphic to the Kleisli category of the monad for the adjunction.*

Proof. The right adjoint's action on objects is given as follows. Let B be an event structure. For $x \in \mathcal{C}(B)$, an *augmentation* of x is a partial order (x, α) where $\forall b, b' \in x.\ b \leq_B b' \implies b\,\alpha\,b'$. We can regard such augmentations as elementary event structures in which all subsets of events are consistent. Order all augmentations by taking $(x, \alpha) \sqsubseteq (x', \alpha')$ iff $x \subseteq x'$ and the inclusion $i : x \hookrightarrow x'$ is a rigid map $i : (x, \alpha) \to (x', \alpha')$. Augmentations under $\sqsubseteq$ form a prime algebraic domain; the complete primes are precisely the augmentations with a top element. Define $aug(B)$ to be its associated event structure.

There is an obvious total map of event structures $\epsilon_B : aug(B) \to B$ taking a complete prime to the event which is its top element. It can be checked that post-composition by ϵ_B yields a bijection

$$\epsilon_B \circ \_ : \mathcal{E}_r(A, aug(B)) \cong \mathcal{E}(A, B)\ .$$

Hence aug extends to a right adjoint to the inclusion $\mathcal{E}_r \hookrightarrow \mathcal{E}_t$.

Write aug also for the monad induced by the adjunction and $Kl(aug)$ for its Kleisli category. Under the bijection of the adjunction

$$Kl(aug)(A, B) =_{\mathrm{def}} \mathcal{E}_r(A, aug(B)) \cong \mathcal{E}(A, B)\ .$$

The categories $Kl(aug)$ and $\mathcal{E}$ share the same objects, and so are isomorphic. □

2.3.1 Rigid image

Rigid maps $f : A \to B$ have a useful image given by restricting the causal dependency of B to the set of events in the image of A under f and taking a finite set of events to be consistent if they are the image of a consistent set in A. More generally, a total map $f : A \to B$ has a *rigid image* given by the image of its corresponding Kleisli map, the rigid map $\overline{f} : A \to aug(B)$. A total map $f : A \to B$ has a *rigid image* comprising

$$\begin{tikzcd} A \arrow[r, two heads, "f_0"] \arrow[dr, "f"'] & B_0 \arrow[d, "f_1"] \\ & B\,, \end{tikzcd}$$

where f_0 is rigid epi and f_1 is a total map, with the universal property summarised in the diagram below:

$$\begin{tikzcd} A \arrow[r, two heads, "f_0'"] \arrow[rr, bend left, two heads, "f_0"] \arrow[dr, "f"'] & B' \arrow[r, dashed, "h"] \arrow[d, "f_1'"] & B_0 \arrow[dl, "f_1"] \\ & B & \end{tikzcd}$$

for a unique rigid h; the map h is necessarily also epi. If we don't specify further we shall take the rigid image of a total map $f : A \to B$ to be a substructure of $aug(B)$. By a substructure of B we mean an event structure B_0 with events included in those of B so that the inclusion is a map.

2.3.2 Rigid embeddings and inclusions

Special forms of rigid maps appeared as *rigid embeddings* in Kahn and Plotkin's work on concrete domains [**?**]. Their extension to event structures can be used in defining event structures recursively.

A total map $f : E \to E'$ is a *rigid embedding* iff it is rigid and an injective function on events for which the inverse relation f^{op} is a (partial) map of event structures $f^{\mathrm{op}} : E' \to E$. (There are several alternative equivalent definitions.)

Rigid embeddings include as a special case those in which the function f is an inclusion. These give the well-known approximation order $\trianglelefteq$ on event structures:

$$
\begin{aligned}
(E', \leq', \mathrm{Con}') \trianglelefteq (E, \leq, \mathrm{Con}) \iff\ & E' \subseteq E\ \& \\
& \forall e' \in E'.\ [e']' = [e']\ \& \\
& \forall X' \subseteq E'.\ X' \in \mathrm{Con}' \iff X \in \mathrm{Con}\,.
\end{aligned}
$$

The order $\trianglelefteq$ forms a 'large cpo,' with bottom the empty event structure, and is useful when defining event structures recursively [4, **?**, 2]. With some care in defining the precise constructions on event structures they can be ensured to be continuous w.r.t. $\trianglelefteq$; for this it suffices to check that they are $\trianglelefteq$-monotonic and continuous on event sets. Further details can be found in [4, **?**].

2.3.3 Rigid families

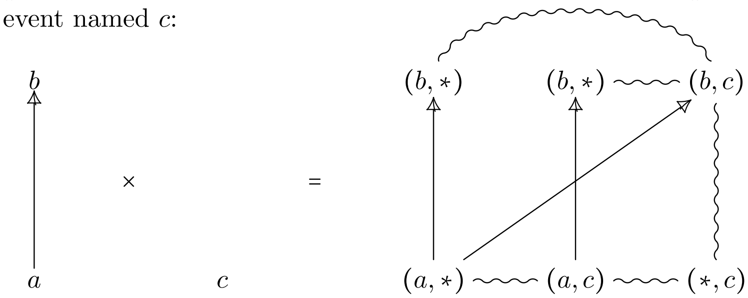
It is occasionally useful to build an event structure out of a non-empty family $\mathcal{Q}$ of finite partial orders. We can do so provided the family is *rigid*.

For $\mathcal{Q}$ to be a rigid family we require that its is closed under rigid inclusions, or equivalently, that any down-closed subset of any element q, with order the restriction of that of q, is itself an element of $\mathcal{Q}$. (In this case rigid inclusions coincide with rigid embeddings.)

From a rigid family $\mathcal{Q}$ we construct an event structure as follows. Its events are those partial orders in $\mathcal{Q}$ with a top element. Its causal dependency is given by rigid inclusion. We say a finite subset of partial orders with top is consistent iff all its members are rigidly included in a common member of $\mathcal{Q}$.

2.4 Products of event structures

The category of event structures has products, which essentially allow arbitrary synchronizations between their components. For example, here is an illustration of the product of two event structures $a \rightarrowtriangle b$ and c, the later comprising just a single event named c:



The original event b has split into three events, one a synchronization with c, another b occurring unsynchronized after an unsynchronized a, and the third b occurring unsynchronized after a synchronizes with c. The splittings correspond to the different histories of the event.

It can be awkward to describe operations such as products, pullbacks and synchronized parallel compositions directly on the simple event structures here, essentially because an event determines its whole causal history. One closely related and more versatile, though perhaps less intuitive and familiar, model is that of stable families. Stable families will play an important technical role in establishing and reasoning about constructions on event structures.

Chapter 3

Stable families

Stable families support a form of disjunctive causes in which an event may be enabled in several different but incompatible ways. Stable families, their basic properties and relations to event structures are developed.[1]

3.1 Stable families

The notion of stable family extends that of finite configurations of an event structure to allow an event can occur in several incompatible ways.

Notation 3.1. *Let $\mathcal{F}$ be a family of subsets. Let $X \subseteq \mathcal{F}$. We write $X\uparrow$ for $\exists y \in \mathcal{F}.\ \forall x \in X. x \subseteq y$ and say X is* compatible*. When $x, y \in \mathcal{F}$ we write $x \uparrow y$ for $\{x, y\}\uparrow$.*

A *stable family* comprises $\mathcal{F}$, a nonempty family of finite subsets, satisfying:
Completeness: $\forall Z \subseteq \mathcal{F}.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}$;
Stability: $\forall Z \subseteq \mathcal{F}.\ Z \neq \varnothing\ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}$;
Coincidence-freeness: For all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,

$$\exists y \in \mathcal{F}.\ y \subseteq x\ \&\ (e \in y \iff e' \notin y)\,.$$

We call the elements of $\bigcup \mathcal{F}$ of a stable family $\mathcal{F}$ its *events*.

An alternative characterisation of stable families:

Proposition 3.2. *A* stable family *comprises $\mathcal{F}$, a family of finite subsets, satisfying:*
Completeness: *$\varnothing \in \mathcal{F}\ \&\ \forall x, y \in \mathcal{F}.\ x \uparrow y \implies x \cup y \in \mathcal{F}$;*
Stability: *$\forall x, y \in \mathcal{F}.\ x \uparrow y \implies x \cap y \in \mathcal{F}$;*
Coincidence-freeness: *For all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,*

$$\exists y \in \mathcal{F}.\ y \subseteq x\ \&\ (e \in y \iff e' \notin y)\,.$$

[1] A useful reference for stable families is the report "Event structure semantics for CCS and related languages," a full version of the ICALP'82 article, available from `www.cl.cam.ac.uk/~gw104`, though its terminology can differ from that here.

Proof. Simple inductions show that the reformulations of "Completeness" and "Stability" are equivalent to their original formulations. ☐

Proposition 3.3. *The family of finite configurations of an event structure forms a stable family.*

On the other hand stable families are more general than finite configurations of an event structure, as the following example shows.

Example 3.4. Let $\mathcal{F}$ be the stable family, with events $E = \{0, 1, 2\}$,

$$\begin{tikzcd}
\{0,2\} & & \{0,1\} & & \{1,2\} \\
\{0\} \arrow[u, phantom, sloped, "\subset"] \arrow[urr, phantom, sloped, "\subset"] & & & & \{1\} \arrow[u, phantom, sloped, "\subset"] \arrow[ull, phantom, sloped, "\subset"] \\
& & \varnothing \arrow[ull, phantom, sloped, "\subset"] \arrow[urr, phantom, sloped, "\subset"] & &
\end{tikzcd}$$

or equivalently

$$\begin{tikzcd}
\{0,2\} & & \{0,1\} & & \{1,2\} \\
\{0\} \arrow[u, hook] \arrow[urr, hook] & & & & \{1\} \arrow[u, hook] \arrow[ull, hook] \\
& & \varnothing \arrow[ull, hook] \arrow[urr, hook] & &
\end{tikzcd}$$

where $\longrightarrow\!\!\!\subset$ is the covering relation representing an occurrence of one event. The events 0 and 1 are concurrent, neither depends on the occurrence or non-occurrence of the other to occur. The event 2 can occur in two incompatible ways, either through event 0 having occurred or event 1 having occurred. This possibility can make stable families more flexible to work with than event structures.

A (partial) map of stable families $f : \mathcal{F} \to \mathcal{G}$ is a partial function f from the events of $\mathcal{F}$ to the events of $\mathcal{G}$ such that for all $x \in \mathcal{F}$,

$$fx \in \mathcal{G} \ \& \ (\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2)\,.$$

Maps of stable families compose as partial functions, with identity maps given by identity functions. We call a map $f : \mathcal{F} \to \mathcal{G}$ of stable families *total* when it is total as a function; the f restricts to a bijection $x \cong fx$ for all $x \in \mathcal{F}$.

Definition 3.5. Let $\mathcal{F}$ be a stable family. We use $x \longrightarrow\!\!\!\subset y$ to mean y covers x in $\mathcal{F}$, *i.e.* $x \subset y$ in $\mathcal{F}$ with nothing in between, and $x \xrightarrow{e}\!\!\!\subset y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{F}$ and event $e \notin x$. We sometimes use $x \xrightarrow{e}\!\!\!\subset$, expressing that event e is enabled at configuration x, when $x \xrightarrow{e}\!\!\!\subset y$ for some y.

Exercise 3.6. Let $\mathcal{F}$ be a nonempty family of finite sets satisfying the Completeness axiom in the definition of stable families. Show $\mathcal{F}$ is coincidence-free iff

$$\forall x, y \in \mathcal{F}.\ x \subsetneq y \implies \exists x_1, e_1.\ x \overset{e_1}{\longrightarrow\!\!\subset} x_1 \subseteq y\,.$$

[Hint: For 'only if' use induction on the size of $y \setminus x$.] □

3.1.1 Stable families and event structures

Finite configurations of an event structure form a stable family. Conversely, a stable family determines an event structure:

Proposition 3.7. *Let x be a configuration of a stable family $\mathcal{F}$. For $e, e' \in x$ define*

$$e' \le_x e \text{ iff } \forall y \in \mathcal{F}.\ y \subseteq x \ \&\ e \in y \implies e' \in y.$$

When $e \in x$ define the prime *configuration*

$$[e]_x = \bigcap \{y \in \mathcal{F} \mid y \subseteq x \ \&\ e \in y\}\ .$$

Then $\le_x$ is a partial order and $[e]_x$ is a configuration such that

$$[e]_x = \{e' \in x \mid e' \le_x e\}.$$

Moreover the configurations $y \subseteq x$ are exactly the down-closed subsets of $\le_x$.

Exercise 3.8. *Prove Proposition 3.7.* □

Lemma 3.9. *Let $\mathcal{F}$ be a stable family. Then,*

$$[e]_x \subseteq z \iff [e]_x = [e]_z$$

whenever $e \in x$ and z in $\mathcal{F}$.

Proof. "⇒" From $e \in [e]_x \subseteq z$ we get $[e]_z \subseteq [e]_x$. Hence $e \in [e]_z \subseteq x$ ensuring the converse inclusion $[e]_x \subseteq [e]_z$, so $[e]_x = [e]_z$. "⇐" Trivial. □

Proposition 3.10. *Let $\mathcal{F}$ be a stable family. Then,* $\mathrm{Pr}(\mathcal{F}) =_{\mathrm{def}} (P, \mathrm{Con}, \le)$ *is an event structure where:*

$$\begin{aligned} &P = \{[e]_x \mid e \in x \ \&\ x \in \mathcal{F}\}\ ,\\ &Z \in \mathrm{Con} \textit{ iff } Z \subseteq P \ \&\ \bigcup Z \in \mathcal{F} \quad \textit{and,}\\ &p \le p' \textit{ iff } p, p' \in P \ \&\ p \subseteq p'\,. \end{aligned}$$

There is an order isomorphism

$$\theta : (\mathcal{C}(\mathrm{Pr}(\mathcal{F})), \subseteq) \cong (\mathcal{F}, \subseteq)$$

where $\theta(y) = \bigcup y$ for $y \in \mathcal{C}(\mathrm{Pr}(\mathcal{F}))$; its mutual inverse is ϕ where $\phi(x) = \{[e]_x \mid e \in x\}$ for $x \in \mathcal{F}$.

Proof. It is easy to check that $\mathrm{Pr}(\mathcal{F})$ is an event structure. Clearly, both θ and ϕ preserve $\subseteq$.

Firstly, $\theta\phi(x) = \bigcup\{[e]_x \mid e \in x\} = x$, for all $x \in \mathcal{F}$, by an obvious argument.

Secondly, $\phi\theta(y) = \{[e]_{\bigcup y} \mid e \in \bigcup y\}$, for $y \in \mathcal{C}(\mathrm{Pr}(\mathcal{F}))$. To show $rhs = y$ we use Lemma 3.9 repeatedly:

$$[e]_x \subseteq z \iff [e]_x = [e]_z\,,$$

whenever $e \in x$ and z in $\mathcal{F}$.
From $e \in [e]_x \subseteq z$ we get $[e]_z \subseteq [e]_x$. Hence $e \in [e]_z \subseteq x$ ensuring the converse inclusion $[e]_x \subseteq [e]_z$, so $[e]_x = [e]_z$.
"$y \subseteq rhs$": $[e]_x \in y \Rightarrow [e]_x \subseteq \bigcup y \Rightarrow [e]_x = [e]_{\bigcup y} \in rhs$.
"$rhs \subseteq y$: Assume $p \in rhs$. Then $p = [e]_{\bigcup y}$ with $e \in \bigcup y$. We have $e \in [e']_x \in y$ for some e', x with $e' \in x$. So $[e]_x \subseteq [e']_x \in y$ ensuring $[e]_x \in y$. As $[e]_x \subseteq \bigcup y$ we obtain $p = [e]_{\bigcup y} = [e]_x$, so $p \in y$. □

Remark The above proposition ensures that the partial orders comprising stable families ordered by inclusion and the orders of configurations of event structures are the same to within isomorphism; both coincide with the orders of finite elements of "prime algebraic domains" in which every finite, or isolated, element dominates only finitely many elements.

The operation Pr is right adjoint to the "inclusion" functor, taking an event structure E to the stable family $\mathcal{C}(E)$. The unit of the adjunction at an event structure E is a map $E \to \mathrm{Pr}(\mathcal{C}(E))$ which takes an event e to the prime configuration $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$. The counit at a stable family $\mathcal{F}$ is a map $top_{\mathcal{F}} : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ which takes a prime configuration $[e]_x$ to e; this is well-defined as a function by coincidence-freeness (see the proof of Theorem 3.11.

Theorem 3.11. *There is a map* $top_{\mathcal{F}} : \mathrm{Pr}(\mathcal{F}) \to \mathcal{F}$ *given by* $top_{\mathcal{F}}([e]_x) = e$ *for* $e \in x \in \mathcal{F}$. *In fact,* $\mathrm{Pr}(\mathcal{F})$*,* $top_{\mathcal{F}}$ *is cofree over* $\mathcal{F}$ *i.e. for any map* $g : \mathcal{C}(E') \to \mathcal{F}$ *of stable families with* E' *a prime event structure, there is a unique map* $f : E' \to \mathrm{Pr}(\mathcal{F})$ *such that* $g = top_{\mathcal{F}} f$.

Proof. By Proposition 3.10, $\mathrm{Pr}(\mathcal{F})$ is a prime event structure. We require that $top_{\mathcal{F}} : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ above is a map. Firstly we need top is well-defined as a function $top : P \to E$ where $P = \{[e]_x \mid e \in x \in \mathcal{F}\}$. Suppose $[e]_x = [e']_y$ for $e \in x$ and $x \in \mathcal{F}$ and $e' \in y$ and $y \in \mathcal{F}$. Then by the coincidence-freeness of $\mathcal{F}$ we have $e = e'$, giving top well-defined as a (total) function. From the definition, if z is a configuration of $\mathrm{Pr}(\mathcal{F})$ then $z = \{[e]_x \mid e \in x\}$ for some $x \in \mathcal{F}$; thus $top(z) = \bigcup z = x \in \mathcal{F}$. Let z be a configuration of $\mathrm{Pr}(\mathcal{F})$ so $p, p' \in z$ and $top(p) = top(p') = e$ say. Then $p = p' = [e]_{\bigcup z}$. Thus top is a map of stable families.

We show $\mathrm{Pr}(\mathcal{F})$, $top_{\mathcal{F}}$ is cofree over $\mathcal{F}$. Let $g : \mathcal{C}(E') \to \mathcal{F}$ be a map of stable families where E' is a prime event structure $E' = (E', \mathrm{Con}', \leq')$. We require a unique map $f : E' \to \mathrm{Pr}(\mathcal{F})$ s.t. the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} & \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \arrow[l, "top"'] \\ & \mathcal{C}(E') \arrow[ul, "g"] \arrow[u, dashed, "f"'] \end{tikzcd}$$

Define $f : E' \to P$ by

$$f(e') = \begin{cases} [g(e')]_{g[e']} & \text{if } g(e') \text{ is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Above $[e']$ is the downwards closure of e' in E'. Let $x \in \mathcal{C}(E')$. Then

$$\begin{aligned} fx &= \{[g(e')]_{g[e']} \mid e' \in x \ \&\ g(e') \text{ is defined}\} \\ &= \{[e]_{gx} \mid e \in gx\} \end{aligned}$$

where we have observed that $[g(e')]_{g[e']} \subseteq gx$ when $e' \in x$, so $[g(e')]_{g[e']} = [g(e')]_{gx}$. Hence fx is a configuration of $\mathrm{Pr}(F)$. If $e, e' \in x$ and $f(e) = f(e')$ (both defined) then $g(e) = g(e')$ (both defined) so $e = e'$, as g is a map. Thus f is a map. Clearly $topf = g$ so f makes the diagram commute.

Let $f' : E' \to \mathrm{Pr}(\mathcal{F})$ be a map such that the diagram commutes *i.e.* $topf = g$. We require $f' = f$. Let $e' \in E'$. Firstly note if $g(e')$ is defined then because *top* is a total function we must have $f'(e)$ defined which agrees with f. So suppose that $g(e)$ defined. Then $f'(e)$ is a prime configuration of F s.t. $top(f'(e)) = g(e)$. Now *top* is just union so using the assumed commutation we get

$$f'(e) \subseteq \bigcup f'[e] = topf'[e] = g[e]$$

As $f'(e)$ is a prime configuration in $g[e]$ and $top(f'(e)) = g(e)$ we have $f'(e) = [g(e)]_{g[e]}$, *i.e.* $f'(e) = f(e)$.

Consequently f is the unique map making the diagram commute. □

Theorem 3.11 gives a bijection between maps maps $g : \mathcal{C}(E) \to \mathcal{F}$ of stable families and maps $f : E \to \mathrm{Pr}(\mathcal{F})$ of event structures where E is an event structure and $\mathcal{F}$ is a stable family. The bijection is natural in E. As is well-known there is a unique extension of Pr to a functor so that the bijection is also natural in $\mathcal{F}$. Once extended in this way we obtain the natural bijection of an adjunction.

Corollary 3.12. *The functor $\mathcal{C}(\_)$ from the category of event structures to the category of stable families has a right adjoint the functor which acts as* Pr *on stable families and as follows on a map $f : \mathcal{A} \to \mathcal{B}$ of stable families: the map $\mathrm{Pr}(f) : \mathrm{Pr}(\mathcal{A}) \to \mathrm{Pr}(\mathcal{B})$ takes $[a]_x$, an event of $\mathrm{Pr}(\mathcal{A})$, where $a \in x \in \mathcal{A}$, to the event $[f(a)]_{fx}$ of $\mathrm{Pr}(\mathcal{B})$ if $f(a)$ is defined, and to undefined otherwise.*

The unit of the adjunction at an event structure E is the isomorphism $E \cong \mathrm{Pr}(\mathcal{C}(E))$ taking e to $[e]$. The counit at a stable family $\mathcal{F}$ is given by $top_{\mathcal{F}} : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$.

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be a map of stable families. We must first be sure that $\mathrm{Pr}(f)$ is well-defined as a partial function. Suppose $[a]_x = [a']_y$ for $a \in x \in \mathcal{A}$ and $b \in y \in \mathcal{B}$. We require $\mathrm{Pr}(f)([a]_x) = \mathrm{Pr}(f)([a']_y)$ when either is defined. Firstly, $a = a'$ by the coincidence-freeness of $\mathcal{A}$. Suppose $f(a)$ is defined. Then,

$$[f(a)]_{fx} \subseteq f[a]_x = f[a]_y \subseteq fy\,.$$

Hence by Lemma 3.9, $[f(a)]_{fx} = [f(a)]_{fy}$, *i.e.* $\mathrm{Pr}(f)([a]_x) = \mathrm{Pr}(f)([a']_y)$.

We should check that $\mathrm{Pr}(f)$ is a map of event structures. By Proposition 3.10, a configuration y of $\mathrm{Pr}(\mathcal{A})$ has the form $\{[a]_x \mid a \in x\}$ for some $x \in \mathcal{A}$. Under $\mathrm{Pr}(f)$ it is sent to

$$\{[f(a)]_{fx} \mid a \in x \ \&\ f(a) \text{ is defined}\} = \{[b]_{fx} \mid b \in fx\}\,,$$

a configuration of $\mathrm{Pr}(\mathcal{B})$. Moreover, if $[a]_x, [a']_{x'} \in y$ and $\mathrm{Pr}(f)([a]_x) = \mathrm{Pr}(f)([a']_{x'})$, then $[f(a)]_{fx} = [f(a')]_{fx'}$. But now $f(a) = f(a')$ as $\mathcal{B}$ is coincidence-free and $a, a' \in \bigcup y \in \mathcal{A}$ which implies $a = a'$. As $[a]_x, [a]_{x'} \subseteq \bigcup y$ from Lemma 3.9 we deduce $[a]_x = [a]_{\bigcup y} = [a]_{x'}$, as required.

The map $\mathrm{Pr}(f)$ clearly makes the diagram

$$\begin{tikzcd}
\mathcal{B} & \mathcal{C}(\mathrm{Pr}(\mathcal{B})) \arrow[l, "top_{\mathcal{B}}"'] \\
\mathcal{A} \arrow[u, "f"] & \mathcal{C}(\mathrm{Pr}(\mathcal{A})) \arrow[l, "top_{\mathcal{A}}"'] \arrow[u, "\mathrm{Pr}(f)"']
\end{tikzcd}$$

commute Hence, $\mathrm{Pr}(f)$ gives the unique extension of Pr to a functor which makes the bijection (between maps $g : \mathcal{C}(E) \to \mathcal{F}$ of stable families and maps $f : E \to \mathrm{Pr}(\mathcal{F})$ of event structures) given by the cofreeness property of Theorem 3.11 natural, so forming an adjunction.

It is easily checked that the putative unit and counit maps do indeed correspond to the identities on $\mathcal{C}(E)$ and $\mathrm{Pr}(\mathcal{F})$, respectively, as required for their to be unit and counit. □

Remark. The fact that the unit is an isomorphism and the fact that the left adjoint is full and faithful are in fact equivalent and say that the adjunction is in a *coreflection.* Later it will play a role in obtaining products of event structures from those of stable families.

Definition 3.13. Let $\mathcal{F}$ be a stable family. W.r.t. $x \in \mathcal{F}$, write $[e)_x =_{\mathrm{def}} \{e' \in E \mid e' \leq_x e \ \&\ e' \neq e\}$. The relation of *immediate* dependence of event structures generalizes: with respect to $x \in \mathcal{F}$, the relation $e \rightarrowtriangle_x e'$ means $e \leq_x e'$ with $e \neq e'$ and no event in between. For $e, e' \in x \in \mathcal{F}$ we write $e\ co_x\ e'$ when neither $e \leq_x e'$ nor $e' \leq_x e$. Note the relations $\leq_x$, $\rightarrowtriangle_x$ and co_x, 'local' to a configuration x, coincide with the 'global' versions $\leq$, $\rightarrowtriangle$ and co when the stable family comprises the finite configurations of an event structure.

We shall use the following property of maps repeatedly, both for stable families and the special case of event structures. It says that their maps locally reflect causal dependency.

Proposition 3.14. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of stable families. Let $e, e' \in x$, a configuration of $\mathcal{F}$. If $f(e)$ and $f(e')$ are defined and $f(e) \leq_{fx} f(e')$ then $e \leq_x e'$.*

Proof. Let $e, e' \in x \in \mathcal{F}$. Suppose $f(e)$ and $f(e')$ are defined and $f(e) \leq_{fx} f(e')$. Suppose y is a subconfiguration of x, *i.e.* $y \in \mathcal{F}$ and $y \subseteq x$, which contains e'. Then clearly fy is a subconfiguration of fx which contains $f(e')$. We have $f(e) \in fy$ as $f(e) \leq_{fx} f(e')$. Hence there is e” $\in y$ such that $f(e$”$) = f(e)$. But now e, e” $\in x$ with $f(e) = f(e$”$)$, so $e = e$”. We deduce $e \in y$. The argument was for an arbitrary y, so $e \leq_x e'$ as required. □

The next two propositions relate immediate causal dependency between events to the covering relation between configurations.

Proposition 3.15. *Let $\mathcal{F}$ be a stable family. Let $e, e' \in x \in \mathcal{F}$.*

$$\exists y, y_1 \in \mathcal{F}.\ y, y_1 \subseteq x \ \&\ y \overset{e}{\longrightarrow\!\!\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\!\!\subset} \iff e \rightarrowtriangle_x e' \ \text{ or } \ e\ co_x\ e'\,, \qquad (i)$$

$$\text{and} \quad e \rightarrowtriangle_x e' \iff \exists y, y_1 \in \mathcal{F}.\ y, y_1 \subseteq x \ \&\ y \overset{e}{\longrightarrow\!\!\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\!\!\subset} \ \&\ \neg\, e\ co_x\ e' \qquad (ii)$$

$$\iff \exists y, y_1 \in \mathcal{F}.\ y, y_1 \subseteq x \ \&\ y \overset{e}{\longrightarrow\!\!\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\!\!\subset} \ \&\ \neg\, y \overset{e'}{\longrightarrow\!\!\!\!\subset}\,. \qquad (iii)$$

The proposition simplifies in the special case of event structures:

Proposition 3.16. *Let E be an event structure. Let $e, e' \in E$.*

$$\exists y, y_1 \in \mathcal{C}^\infty(E).\ y \overset{e}{\longrightarrow\!\!\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\!\!\subset} \iff e \rightarrowtriangle e' \ \text{ or } \ e\ co\ e'\,,$$

$$\text{and} \quad e \rightarrowtriangle e' \iff \exists y, y_1 \in \mathcal{C}^\infty(E).\ y \overset{e}{\longrightarrow\!\!\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\!\!\subset} \ \&\ \neg\, e\ co\ e'\,,$$

$$\iff \exists y, y_1 \in \mathcal{C}^\infty(E).\ y \overset{e}{\longrightarrow\!\!\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\!\!\subset} \ \&\ \neg\, y \overset{e'}{\longrightarrow\!\!\!\!\subset}\,.$$

3.2 Completed stable families

We can extend a stable family to include infinite configurations, by constructing its "ideal completion."

Definition 3.17. Let $\mathcal{F}$ be a stable family. Define $\mathcal{F}^\infty$, a completed stable family, to comprise all $\bigcup I$ where $I \subseteq \mathcal{F}$ is an ideal (*i.e.*, I is a nonempty subset of $\mathcal{F}$ closed downwards w.r.t. $\subseteq$ in $\mathcal{F}$ and such that if $x, y \in I$ then $x \cup y \in I$).

Exercise 3.18. *For an event structure E, show $\mathcal{C}^\infty(E) = \mathcal{C}(E)^\infty$.* □

Exercise 3.19. *Let $\mathcal{F}$ be a stable family. Show $\mathcal{F}^\infty$ satisfies:*

Completeness: $\forall Z \subseteq \mathcal{F}^\infty.(\forall X \subseteq_{\text{fin}} Z.\ X\uparrow) \implies \bigcup Z \in \mathcal{F}^\infty$ *;*
Stability: $\forall Z \subseteq \mathcal{F}^\infty.\ Z \neq \emptyset \ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}^\infty$*;*
Coincidence-freeness: *For all* $x \in \mathcal{F}^\infty$, $e, e' \in x$ *with* $e \neq e'$,

$$\exists y \in \mathcal{F}^\infty.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y)\,;$$

Finiteness: *For all* $x \in \mathcal{F}^\infty$,

$$\forall e \in x \exists y \in \mathcal{F}.\ e \in y \ \&\ y \subseteq x \ \&\ y \text{ is finite}\ .$$

Show that $\mathcal{F}$ *consists of precisely the finite sets in* $\mathcal{F}^\infty$*.* □

Remark Above the conditions of Finiteness and Coincidence-freeness together can be replaced by the equivalent condition
Secured: if $e \in x \in \mathcal{F}$ then there exists a securing chain $e_1, \cdots, e_n = e$ in x s.t. $\{e_1, \cdots, e_i\} \in \mathcal{F}$ for all $i \le n$.

3.3 Process constructions

3.3.1 Products

Let $\mathcal{A}$ and $\mathcal{B}$ be stable families with events A and B, respectively. Their product, the stable family $\mathcal{A} \times \mathcal{B}$, has events comprising pairs in $A \times_* B =_{\text{def}} \{(a, *) \mid a \in A\} \cup \{(a, b) \mid a \in A \ \&\ b \in B\} \cup \{(*, b) \mid b \in B\}$, the product of sets with partial functions, with (partial) projections π_1 and π_2—treating $*$ as 'undefined'—with configurations

$x \in \mathcal{A} \times \mathcal{B}$ iff
x is a finite subset of $A \times_* B$ such that
(a) $\pi_1 x \in \mathcal{A} \ \&\ \pi_2 x \in \mathcal{B}$,
(b) $\forall e, e' \in x.\ \pi_1(e) = \pi_1(e')$ or $\pi_2(e) = \pi_2(e') \Rightarrow e = e'$,&
(c) $\forall e, e' \in x.\ e \neq e' \Rightarrow \exists y \subseteq x.\ \pi_1 y \in \mathcal{A} \ \&\ \pi_2 y \in \mathcal{B} \ \&\ (e \in y \iff e' \notin y)$.

Note how (a) and (b) express that the projections are maps while (c) says the structure $\mathcal{A} \times \mathcal{B}$ is coincidence-free.

In checking that $\mathcal{A} \times \mathcal{B}$, π_1, π_2 is a product in the category of stable families we shall use the following lemma showing that the direct image under a partial function preserves intersections when the function is locally injective.

Lemma 3.20. *Let* $\theta : E_0 \rightharpoonup E_1$ *be a partial function between sets* E_0 *and* E_1*. Let* $X \subseteq \mathcal{P}(E_0)$*. Then if*

$$\forall e, e' \in \bigcup X\ .\ \theta(e) = \theta(e') \implies e = e'$$

then $\theta \bigcap X = \bigcap \theta X$*.*

Proof. Suppose $\theta(e) = \theta(e')$ (both defined) implies $e = e'$ for every $e, e' \in \bigcup x$. Clearly θ is monotonic w.r.t. $\subseteq$ so $\theta \bigcap X \subseteq \bigcap \theta X$. Take $e \in \bigcap \theta X$ and $x \in X$. For some $e' \in x$ we have $\theta(e') = e$. Take $y \in X$. Then for some $e_y \in y$ we have $\theta(e_y) = e$. However $e_y, e \in \bigcup X$ and $\theta(e_y) = \theta(e')$. Thus by hypothesis $e_y = e'$. Therefore $e' \in \bigcap X$ so $e \in \theta \bigcap X$. This establishes the converse inclusion; so $\theta \bigcap X = \bigcap \theta X$, as required. □

Theorem 3.21. *For stable families $\mathcal{A}$ and $\mathcal{B}$ the construction $\mathcal{A} \times \mathcal{B}$ with projections π_1 and π_2 described above is the product in the category of stable families.*

Proof. Suppose $x \subseteq \mathcal{A} \times \mathcal{B}$ and $e, e' \in x$. We shall say "y is a separating set for e, e' in x" when $y \subseteq x$ and $\pi_1(y) \in \mathcal{A}$ and $\pi_2(y) \in \mathcal{B}$ and $e \in y \iff e' \notin y$.

We first check $\mathcal{F} =_{\text{def}} \mathcal{A} \times \mathcal{B}$ is a stable family.

Complete. Suppose $X \subseteq \mathcal{F}$ and $X \uparrow$. We require $\bigcup X$ satisfies (a)-(c) in the definition of product.

(a) Clearly $\pi_i \bigcup X = \bigcup \pi_i X$. As X is compatible in F so are $\pi_1 X$ in $\mathcal{A}$ and $\pi_2 X \in \mathcal{B}$. Thus $\pi_1(\bigcup X) \in \mathcal{A}$ and $\pi_2(\bigcup X) \in \mathcal{B}$.

(b) By the compatibility of X, if $e, e' \in \bigcup X$ and $\pi_i(e) = \pi_i(e')$, both being defined, for $i = 1$ or 2, then $e = e'$.

(c) Suppose $e, e' \in \bigcup X$ and $e \neq e'$. Then $\exists x, y \in X \,.\, e \in x \;\&\; e' \in y$. If either $e \notin y$ or $e' \notin x$ we have respectively either y or x is a separating set for e, e' in $\bigcup X$. Otherwise $e, e' \in x$ or $e, e' \in y$. Then as both x and y satisfy (c) we obtain the required separating set.

Stable. Suppose $\varnothing \neq X \subseteq \mathcal{F}$ and $X \uparrow$. We require X satisfies (a)-(c).

(a) By lemma 3.20, $\pi_i \bigcap X = \bigcap \pi_i X$. But $\bigcap \pi_1 X \in \mathcal{A}$, as $\pi_1 X$ is a compatible set in $\mathcal{A}$, and similarly $\bigcap \pi_2 X \in \mathcal{B}$, so we have $\pi_1(\bigcap X) \in \mathcal{A}$ and $\pi_2(\bigcap X) \in \mathcal{B}$.

(b) As any $x \in X$ satisfies (b) and $\bigcap X \subseteq x$ certainly $\bigcap X$ satisfies (b).

(c) Suppose $e, e' \in \bigcap X$ and $e \neq e'$. Choose $x \in X$. Because $x \in \mathcal{F}$ there is a separating set y for e, e' in x. Take $v = y \cap \bigcap X$. Clearly $y, \bigcap X \subseteq x$ so because $\mathcal{A}$ and $\mathcal{B}$ are stable, by lemma 3.20*** $\pi_1 v = \pi_1 y \cap \pi_1 \bigcap X) \in \mathcal{A}$ and $\pi_2 v = \pi_2 y \cap \pi_2 \bigcap X \in \mathcal{B}$. This makes v a separating set for e, e' in $\bigcap X$.

Coincidence-free. Suppose $e, e' \in x \in F$ and $e \neq e'$. As x satisfies (c) there is a separating set y for e, e' in x. We further require $y \in F$. Clearly y satisfies (a), (b). To Show y satisfies (c), assume $\epsilon, \epsilon' \in y$ and $\epsilon \neq \epsilon'$. Take a separating set v for ϵ, ϵ' in x. Take $u = v \cap y$. Then, just as in the proof of stability, part (c), we get u is a separating set for ϵ, ϵ' in x.

Thus we have shown $\mathcal{A} \times \mathcal{B}$ is a stable family. It remains to show that with projections π_1, π_2 it forms the product in the category of stable families. First note π_1 and π_2 are maps by (a), (b) in the construction of the product .

Suppose there are maps $f_1 : \mathcal{F} \to \mathcal{A}$ and $f_2 : \mathcal{F} \to \mathcal{B}$ are maps of stable families. We require a unique map h such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{A} \times \mathcal{B} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{A} & & \mathcal{B} \\
 & \mathcal{F} \arrow[ul, "f_1"] \arrow[ur, "f_2"'] \arrow[uu, dashed, "h"] &
\end{tikzcd}$$

Take h so that

$$h(e) = \begin{cases} (f_1(e), f_2(e)) & \text{if } f_1(e) \text{ is defined or } f_2(e) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

In a pair $(f_1(e), f_2(e))$ we shall identify undefined with $*$.
Obviously $\pi_i \circ h = f_i$ in the category of sets with partial functiosn, for $i = 1, 2$ so provided h is a map of stable families it is unique so the diagram commutes. To show h is a map we need:

$$\forall x \in \mathcal{F} \,.\, hx \in \mathcal{F} \tag{I}$$
$$\forall x \in \mathcal{F} \forall e, e' \in x \,.\, h(e) = h(e') \implies e = e' \tag{II}$$

We prove (II) first:

Suppose $e, e' \in x \in \mathcal{F}$. Then if $h(e) = h(e')$ then $f_i(e) = f_i(e')$, both being defined, for either $i = 1$ or $i = 2$. As each f_i is a map $e = e'$, as required to prove (II).

Now we prove (I). Let $x \in \mathcal{F}$. We need hx satisfies (a)-(c) in the construction of the product. Both (a) and (b) follow from the commutations $\pi_i \circ h = f_i$ using the map properties of f_1 and fa_2. To prove (c), suppose $e, e' \in hx$ and $e \neq e'$. Then $e = h(\epsilon)$ and $e' = h(\epsilon')$ for some $\epsilon, \epsilon' \in x$. We must have $\epsilon \neq \epsilon'$. Thus as $\mathcal{F}$ is coincidence-free we have some $y \in \mathcal{F}$ such that $y \subseteq x$ and $\epsilon \in y \iff \epsilon' \notin y$. As we know h satisfies (II) above it follows that one and only one of e, e' is in hy. The commutations $\pi_i \circ h = f_i$ give $\pi_1 hy \in \mathcal{A}$ and $\pi_2 hy \in \mathcal{B}$. Thus hy separates e, e' in x.

Thus finally we have shown $\mathcal{A} \times \mathcal{B}$ with projections π_1, π_2 is a product in the category of stable families. □

Proposition 3.22. *Let $x \in \mathcal{A} \times \mathcal{B}$, a product of stable families with projections π_1 and π_2. Then, for all $y \subseteq x$,*

$$y \in \mathcal{A} \times \mathcal{B} \iff \pi_1 y \in \mathcal{A} \;\&\; \pi_2 y \in \mathcal{B}\,.$$

Proof. Straightforwardly from the definition of $\mathcal{A} \times \mathcal{B}$. □

Right adjoints preserve products. Hence if $\mathcal{A} \times \mathcal{B}$, π_1, π_2 is a product of stable families then $\Pr(\mathcal{A}) \times \Pr(\mathcal{B})$, $\Pr(\pi_1)$, $\Pr(\pi_2)$ is a product of event structures.

Consequently we obtain a product of event structures A and B by first regarding them as stable families $\mathcal{C}(A)$ and $\mathcal{C}(B)$, forming their product

$$\mathcal{C}(A) \times \mathcal{C}(B), \pi_1, \pi_2$$

and then constructing the event structure

$$A \times B =_{\text{def}} \Pr(\mathcal{C}(A) \times \mathcal{C}(B))$$

with projections the composite maps

$$\Pi_1 : A \times B \xrightarrow{\Pr(\pi_1)} \Pr(\mathcal{C}(A)) \cong A \quad \text{and} \quad \Pi_2 : A \times B \xrightarrow{\Pr(\pi_2)} \Pr(\mathcal{C}(B)) \cong B$$

—the isomorphisms are inverses to those of the unit of the adjunction. The projections can be simplified:

Proposition 3.23. *Let A and B be event structures.*

$$A \times B =_{\text{def}} \Pr(\mathcal{C}(A) \times \mathcal{C}(B))$$

and its projections as $\Pi_1 =_{\text{def}} \pi_1 top : A \times B \to A$ and $\Pi_2 =_{\text{def}} \pi_2 top : A \times B \to B$.

Proof. For example,

$$\Pi_1 : A \times B \xrightarrow{\Pr(\pi_1)} \Pr(\mathcal{C}(A)) \cong A$$

takes an event $[e]_x \in A \times B$ via $\Pr(\pi_1)$ to $[\pi_1(e)]_{\pi_1 x}$ if $\pi_1(e)$ is defined, by Corollary 3.12, whence to $\pi_1(e)$ under the isomorphism, *i.e.* to $\pi_1 \circ top([e]_x)$. □

Exercise 3.24. *Let A be the event structure consisting of two distinct events $a_1 \leq a_2$ and B the event structure with a single event b. Following the method above describe the product of event structures $A \times B$.* □

Later we shall use the following properties of $\rightarrowtriangle$ in a product of stable families or event structures.

Lemma 3.25. *Let $x \in \mathcal{A} \times \mathcal{B}$, a product of stable families with projections π_1, π_2. Let $e, e' \in x$. If $e \rightarrowtriangle_x e'$, then*
either
(i) $\pi_1(e)$ and $\pi_1(e')$ are both defined with $\pi_1(e) \rightarrowtriangle_{\pi_1 x} \pi_1(e')$ in $\mathcal{A}$ and if $\pi_2(e)$, $\pi_2(e')$ are defined then $\pi_2(e) \rightarrowtriangle_{\pi_2 x} \pi_2(e')$ or $\pi_2(e)\ co_{\pi_2 x}\ \pi_2(e')$ in $\mathcal{B}$,
or
(ii) $\pi_2(e)$ and $\pi_2(e')$ are both defined with $\pi_2(e) \rightarrowtriangle_{\pi_2 x} \pi_2(e')$ in $\mathcal{B}$ and if $\pi_1(e)$, $\pi_1(e')$ are defined then $\pi_1(e) \rightarrowtriangle_{\pi_1 x} \pi_1(e')$ or $\pi_1(e)\ co_{\pi_1 x}\ \pi_1(e')$ in $\mathcal{A}$.

Proof. By Proposition 3.15(iii), $e \rightarrowtriangle_x e'$ iff (I) $y \overset{e}{\longrightarrow\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\subset}$ and (II) $\neg y \overset{e'}{\longrightarrow\!\!\subset}$, for subconfigurations y, y_1 of x. From (I),

(a) if $\pi_1(e)$, $\pi_1(e')$ are defined then $\pi_1 y \overset{\pi_1(e)}{\longrightarrow\!\!\subset} \pi_1 y_1 \overset{\pi_1(e')}{\longrightarrow\!\!\subset}$

and

(b) if $\pi_2(e)$, $\pi_2(e')$ are defined then $\pi_2 y \xrightarrow{\pi_2(e)}\!\!\subset \pi_2 y_2 \xrightarrow{\pi_2(e')}\!\!\subset$.

Suppose both ($\pi_1(e')$ defined $\Rightarrow \pi_1 y \xrightarrow{\pi_1 e'}\!\!\subset$) and ($\pi_2(e')$ defined $\Rightarrow \pi_2 y \xrightarrow{\pi_2 e'}\!\!\subset$). Then $y \cup \{e'\} \subseteq x$ with $\pi_1(y \cup \{e'\}) \in \mathcal{A}$ and $\pi_2(y \cup \{e'\}) \in \mathcal{B}$. So, by Proposition 3.22, $y \cup \{e'\} \in \mathcal{A} \times \mathcal{B}$—contradicting (II). Hence, either $\neg\, \pi_1 y \xrightarrow{\pi_1 e'}\!\!\subset$, with $\pi_1 e'$ defined, or $\neg\, \pi_2 y \xrightarrow{\pi_2 e'}\!\!\subset$, with $\pi_2 e'$ defined.

Assume the case $\neg\, \pi_1 y \xrightarrow{\pi_1 e'}\!\!\subset$, with $\pi_1 e'$ defined. Supposing $\pi_1(e)$ is undefined, from (I) we obtain the contradictory $\pi_1 y = \pi_1 y_1 \xrightarrow{\pi_1 e'}\!\!\subset$. Hence, in this case, both $\pi_1 e$ and $\pi_1 e'$ are defined with $\pi_1 y \xrightarrow{\pi_1(e)}\!\!\subset \pi_1 y_1 \xrightarrow{\pi_1(e')}\!\!\subset$ and $\neg\, \pi_1 y \xrightarrow{\pi_1 e'}\!\!\subset$. So $\pi_1(e) \mathrel{-\!\!\triangleright}_{\pi_1 x} \pi_1(e')$ in $\mathcal{A}$, by Proposition 3.15(iii). Meanwhile from (b), this time by Proposition 3.15(i), if $\pi_2(e)$, $\pi_2(e')$ are defined then $\pi_2(e) \mathrel{-\!\!\triangleright}_{\pi_2 x} \pi_2(e')$ or $\pi_2(e)\ co_{\pi_2 x}\ \pi_2(e')$ in $\mathcal{B}$. Hence (i), above.

Similarly, the case $\neg\, \pi_2 y \xrightarrow{\pi_2 e'}\!\!\subset$, with $\pi_2 e'$ defined, yields (ii).

□

Corollary 3.26. *Let $A \times B$, Π_1, Π_2 be a product of event structures. If $p \mathrel{-\!\!\triangleright} p'$ in $A \times B$, then*

either

(i) $\Pi_1(p)$ and $\Pi_1(p')$ are both defined with $\Pi_1(p) \mathrel{-\!\!\triangleright} \Pi_1(p')$ in A and if $\Pi_2(p)$, $\Pi_2(p')$ are defined then $\Pi_2(p) \mathrel{-\!\!\triangleright} \Pi_2(p')$ or $\Pi_2(p)\ co\ \Pi_2(p')$ in B,

or

(ii) $\Pi_2(p)$ and $\Pi_2(p')$ are both defined with $\Pi_2(p) \mathrel{-\!\!\triangleright} \Pi_2(p')$ in B and if $\Pi_1(p)$, $\Pi_1(p')$ are defined then $\Pi_1(p) \mathrel{-\!\!\triangleright} \Pi_1(p')$ or $\Pi_1(p)\ co\ \Pi_1(p')$ in A.

Proof. Directly by Lemma 3.25, because $p \mathrel{-\!\!\triangleright} p'$ in $A \times B$ implies $top(p) \mathrel{-\!\!\triangleright}_{p'} top(p')$ in $\mathcal{C}(A) \times \mathcal{C}(B)$. □

The converse to Lemma 3.25, above, is false. A more explicit, case-by-case, form of the above Lemma 3.25 is helpful:

Lemma 3.27. *Suppose $e \mathrel{-\!\!\triangleright}_x e'$ in a product of stable families $\mathcal{A} \times \mathcal{B}, \pi_1, \pi_2$.*
(i) If $e = (a, *)$ then $e' = (a', b)$ or $e' = (a', *)$ with $a \mathrel{-\!\!\triangleright}_{\pi_1 x} a'$ in $\mathcal{A}$.
(ii) If $e' = (a', *)$ then $e = (a, b)$ or $e = (a, *)$ with $a \mathrel{-\!\!\triangleright}_{\pi_1 x} a'$ in $\mathcal{A}$.
(iii) If $e = (a, b)$ and $e' = (a', b')$ then $a \mathrel{-\!\!\triangleright}_{\pi_1 x} a'$ in $\mathcal{A}$ or $b \mathrel{-\!\!\triangleright}_{\pi_2 x} b'$ in $\mathcal{B}$. Furthermore both ($a \mathrel{-\!\!\triangleright}_{\pi_1 x} a'$ or $a\ co_{\pi_1 x}\ a'$) and ($b \mathrel{-\!\!\triangleright}_{\pi_2 x} b'$ or $b\ co_{\pi_2 x}\ b'$).

The obvious analogues of (i) and (ii) hold for $e = (*, b)$ and $e' = (*, b')$.

Proof. A restatement of Lemma 3.25, writing $a = \pi_1(e)$, $b = \pi_2(e)$, $a' = \pi_1(e')$ and $b = \pi_2(e')$ when these results of projections are defined. □

Exercise 3.28. Let $z \in \mathcal{A} \times \mathcal{B}$, the product of stable families. For any chain

$$(a, *) \mathrel{-\!\!\triangleright}_z e_1 \mathrel{-\!\!\triangleright}_z \cdots \mathrel{-\!\!\triangleright}_z e_m = (*, b)$$

show there is $e_i = (a_i, b_i)$ for some events a_i of $\mathcal{A}$ and b_i of $\mathcal{B}$.

Corollary 3.29. *Let $f : A \to A'$ and $g : B \to B'$ be rigid maps of event structures. Then the map $\langle f, g\rangle : A \times B \to A' \times B'$ is rigid.*

Proof. Write Π_1, Π_2 and Π'_1, Π'_2 for the projections of $A \times B$ and $A' \times B'$ respectively. It is easy to check that the totality of f and g above implies $\langle f, g\rangle$ is total. To show that their rigidity implies $\langle f, g\rangle$ is rigid we use Corollary 3.26 above. Assuming $p \rightarrowtriangle p'$ in $A \times B$ the corollary implies $\Pi_1(p) \rightarrowtriangle \Pi_1(p')$ or $\Pi_2(p) \rightarrowtriangle \Pi_2(p')$. From the rigidity of f and g, we obtain $f\Pi_1(p) \rightarrowtriangle f\Pi_1(p')$ or $g\Pi_2(p) \rightarrowtriangle g\Pi_2(p')$. But $\Pi'_1\langle f, g\rangle(p') = f\Pi_1(p')$ and $\Pi'_2\langle f, g\rangle(p') = f\Pi_2(p')$ whence as $\langle f, g\rangle$ is a map so reflects causal dependency locally we deduce $\langle f, g\rangle(p) \le \langle f, g\rangle(p')$ (or in fact $\langle f, g\rangle(p) \rightarrowtriangle \langle f, g\rangle(p')$), showing $\langle f, g\rangle$ is rigid. □

3.3.2 Restriction

The *restriction* of $\mathcal{F}$ to a subset of events R is the stable family $\mathcal{F} \upharpoonright R =_{\text{def}} \{x \in \mathcal{F} \mid x \subseteq R\}$. Defining $E \upharpoonright R$, the restriction of an event structure E to a subset of events R, to have events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency induced by E, we obtain $\mathcal{C}(E \upharpoonright R) = \mathcal{C}(E) \upharpoonright R$.

Proposition 3.30. *Let $\mathcal{F}$ be a stable family and R a subset of its events. Then,* $\mathrm{Pr}(\mathcal{F} \upharpoonright R) = \mathrm{Pr}(\mathcal{F}) \upharpoonright top^{-1} R$.

We remark that we can regard restriction as arising as an equaliser. *E.g.* for an event structure E and a subset R of events, the inclusion map $E \upharpoonright R \hookrightarrow E$ is the equaliser of the two maps id_E, the identity map on E, and $r : E \to E$, which acts as identity on events with down-closure in R and is undefined elsewhere.

3.3.3 Synchronized compositions

Synchronized parallel compositions are obtained as restrictions of products to those events which are allowed to synchronize or occur asynchronously. For example, the synchronized composition of Milner's CCS on stable families $\mathcal{A}$ and $\mathcal{B}$ (with labelled events) is defined as $\mathcal{A} \times \mathcal{B} \upharpoonright R$ where R comprises events which are pairs $(a, *)$, $(*, b)$ and (a, b), where in the latter case the events a of $\mathcal{A}$ and b of $\mathcal{B}$ carry complementary labels. Similarly, synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. By Proposition 3.30, we can equivalently form a synchronized composition of event structures by forming the synchronized composition of their stable families of configurations, and then obtaining the resulting event structure—this has the advantage of eliminating superfluous events earlier.

Products of stable families within the subcategory of total maps can be obtained by restricting the product (w.r.t. partial maps). Construct

$$\mathcal{A} \times_t \mathcal{B} = \mathcal{A} \times \mathcal{B} \upharpoonright A \times B$$

where we restrict to the cartesian product of the sets of events of $\mathcal{A}$ and $\mathcal{B}$, called A and B respectively; projection maps are obtained from the projection functions from the cartesian product. Products of stable families within the subcategory of total maps have a particularly simple characterisation:

Proposition 3.31. *Finite configurations of a product $\mathcal{A} \times_t \mathcal{B}$ of stable families with total maps are* secured bijections *$\theta : x \cong y$ between configurations $x \in \mathcal{A}$ and $y \in \mathcal{B}$, such that the transitive relation generated on θ by taking $(a,b) \le (a',b')$ if $a \le_x a'$ or $b \le_y b'$ is a partial order.*

Proof. Let $z \in \mathcal{A} \times_t \mathcal{B}$. By Proposition3.14 the projections π_1 and π_2 locally reflect causal dependency. Hence the partial order $\le_z$ satisfies: $(a,b) \le_z (a',b')$ if $a \le_x a$ or $b \le_y b'$, for all $(a,b),(a',b') \in z$. Thus the transitive relation on z generated by taking $(a,b) \le (a',b')$ if $a \le_x a'$ or $b \le_y b'$ is certainly a partial order; failure of antisymmetry for the relation generated would imply its failure for $\le_z$, a contradiction. To see that $\le_z$ is precisely the transitive relation generated in this way, let θ be the elementary event structure comprising events the set z with causal dependency the least transitive relation $\le$ for which $(a,b) \le (a',b')$ if $a \le_x a'$ or $b \le_y b'$. Let Θ be its stable family of configurations with $r_1 : \Theta \to \mathcal{A}$ and $r_2 : \Theta \to \mathcal{B}$ the obvious projection maps. By the universal properties of the product $\mathcal{A} \times_t \mathcal{B}$, π_1, π_2 there is a unique map $h : \Theta \to \mathcal{A} \times_t \mathcal{B}$ s.t. $r_1 = \pi_1 h$ and $r_2 = \pi_2 h$. As a function on the underlying sets of events $h : \theta \to z$ acts as the identity on events and reflects causal dependency. Hence $\le_z \subseteq \le_p$. It follows that $\le_z$ and $\le_p$ coincide, so that $\le_z$ is a secured bijection.

Conversely, suppose θ is a secured bijection between $x \in \mathcal{A}$ and $y \in \mathcal{B}$ with generated partial order $\le$. Regard $\theta, \le$ as an elementary event structure with stable family of configurations Θ. From the way $\le$ is generated, there are projection maps $r_1 : \Theta \to \mathcal{A}$ and $r_2 : \Theta \to \mathcal{B}$. Hence by universality, there is a unique map $h : \Theta \to \mathcal{A} \times_t \mathcal{B}$ s.t. $r_1 = \pi_1 h$ and $r_2 = \pi_2 h$. But then h must act as the identity function, ensuring $\theta \in \mathcal{A} \times_t \mathcal{B}$. □

3.3.4 Pullbacks

The construction of pullbacks can be viewed as a special case of synchronized composition. Once we have products of event structures pullbacks are obtained by restricting products to the appropriate equalizing set. Pullbacks of event structures can also be constructed via pullbacks of stable families, in a similar manner to the way we have constructed products of event structures. We obtain pullbacks of stable families as restrictions of products. Suppose $f_1 : \mathcal{F}_1 \to \mathcal{G}$ and $f_2 : \mathcal{F}_2 \to \mathcal{G}$ are maps of stable families. Let E_1, E_2 and C be the sets of events of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$, respectively. The set $P =_{\text{def}} \{(e_1, e_2) \mid f(e_1) = f(e_2)\}$ with projections π_1, π_2 to the left and right, forms the pullback, in the category of sets, of the functions $f_1 : E_1 \to C$, $f_2 : E_2 \to C$. We obtain the pullback in stable families of f_1, f_2 as the stable family $\mathcal{P}$, consisting of those subsets of P which are also configurations of the product $\mathcal{F}_1 \times \mathcal{F}_2$—its associated maps are the projections π_1, π_2 from the events of $\mathcal{P}$. When f_1 and f_2 are total maps we obtain the pullback in the subcategory of stable families with total maps.

As a corollary of Proposition 3.31 we obtain a simple characterization of pullbacks of total maps within stable families:

Lemma 3.32. *Let $\mathcal{P}, \pi_1, \pi_2$ form a pullback of total maps $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ in the category of stable families. Configurations of $\mathcal{P}$ are precisely* *those composite bijections $\theta : x \cong fx = gy \cong y$ between configurations $x \in \mathcal{A}$ and $y \in \mathcal{B}$ s.t. $fx = gy$ for which the transitive relation generated on θ by taking $(a,b) \le (a',b')$ if $a \le_x a'$ or $b \le_y b'$ is a partial order.*

For future reference we give the detailed construction of pullbacks of total maps in stable families. Let $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ be total maps of stable families. Assume $\mathcal{A}$ and $\mathcal{B}$ have underlying sets A and B. Define $D =_{\mathrm{def}} \{(a,b) \in A \times B \mid f(a) = g(b)\}$ with projections π_1 and π_2 to the left and right components. Define a family of configurations of the *pullback* to consist of

$$
\begin{aligned}
&x \in \mathcal{D} \text{ iff} \\
&x \text{ is a finite subset of } D \text{ such that } \pi_1 x \in \mathcal{A} \;\&\; \pi_2 x \in \mathcal{B}, \\
&\forall e, e' \in x.\; e \ne e' \Rightarrow \exists y \subseteq x.\; \pi_1 y \in \mathcal{A} \;\&\; \pi_2 y \in \mathcal{B} \;\&\; (e \in y \iff e' \notin y)\,.
\end{aligned}
$$

The extra local injectivity property we needed in the definition of product is not necessary here; it follows from the definition of D and that f and g are locally injective.

We obtain the pullback of event structures by first forming the pullback in stable families of their families of configurations and then applying Pr.

As a corollary of Lemma 3.32 we obtain a useful way to understand configurations of the pullback of total maps on event structures.

Proposition 3.33. *When $f : A \to C$ and $g : B \to C$ are total, maps of event structures, in their pullback P, Π_1, Π_2*

$$
\begin{tikzcd}
 & P \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
 & C\,. &
\end{tikzcd}
$$

the finite configurations of P correspond to composite bijections

$$\theta : x \cong fx = gy \cong y$$

between finite configurations x of A and y of B such that $fx = gy$, for which the transitive relation generated on θ by $(a,b) \le (a',b')$ if $a \le_A a'$ or $b \le_B b'$ forms a partial order.

As a consequence the pullback of rigid maps, respectively rigid epi maps, across total maps are rigid, respectively rigid epi.

Proposition 3.34. *Let P, Π_1, Π_2 be a pullback of total maps $f : A \to C$ and $g : B \to C$ in the category of event structures. If f is rigid so is Π_2. If f is rigid and epi so is Π_2.*

Proof. Use Proposition 3.33 to construct the appropriate configurations of the pullback of event structures; the rigidity of f ensures their existence. $\square$

3.3.5 Projection

As we have seen, event structures support a simple form of hiding associated with the partial-total factorisation of a partial map. Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$.

Proposition 3.35. *Let $f : E \to E'$ be a total map of event structures. Let $V \subseteq E$ and $V' \subseteq E'$ be such that*

$$\forall e \in E.\ e \in V \iff f(e) \in V' .$$

Then f restricts to a total map $f \upharpoonright V : E \downarrow V \to E' \downarrow V'$. Moreover, if f is rigid then so is $f \upharpoonright V$.

3.3.6 Recursion

Both stable families and event structures support recursive definitions via the 'large cpo' based on the substructure relation $\trianglelefteq$ [4, ?]. For two stable families $\mathcal{F}$ and $\mathcal{G}$ with events F and G respectively,

$$\mathcal{F} \trianglelefteq \mathcal{G} \text{ iff } F \subseteq G \ \&\ \forall x \subseteq_{\mathrm{fin}} F.\ x \in \mathcal{F} \iff x \in \mathcal{G} .$$

Chapter 4

Games and strategies

Very general nondeterministic concurrent games and strategies are presented. The intention is to formalize distributed games in which both Player (or a team of players) and Opponent (or a team of opponents) can interact in highly distributed fashion, without, for instance, enforcing that their moves alternate. Strategies, those nondeterministic plays which compose well with copy-cat strategies, are characterized.[1]

4.1 Event structures with polarities

We shall represent both a game and a strategy in a game as an event structure with polarity, comprising an event structure together with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ or $-$ to its events E. The events correspond to (occurrences of) moves. The two polarities $+/-$ express the dichotomy: Player/Opponent; Process/Environment; Prover/Disprover; or Ally/Enemy. Maps of event structures with polarity are maps of event structures which preserve polarity.

4.2 Operations

4.2.1 Dual

The *dual*, $E^\perp$, of an event structure with polarity E comprises a copy of the event structure E but with a reversal of polarities. It obviously extends to a functor. Write $\overline{e} \in E^\perp$ for the event complementary to $e \in E$ and *vice versa*.

4.2.2 Simple parallel composition

This operation simply juxtaposes two event structures with polarity. Let $(A, \leq_A, \mathrm{Con}_A, pol_A)$ and $(B, \leq_B, \mathrm{Con}_B, pol_B)$ be event structures with polarity. The

[1]This key chapter is the result of joint work with Silvain Rideau [5].

events of $A\|B$ are $(\{1\}\times A)\cup(\{2\}\times B)$, their polarities unchanged, with: the only relations of causal dependency given by $(1,a) \le (1,a')$ iff $a \le_A a'$ and $(2,b) \le (2,b')$ iff $b \le_B b'$; a subset of events C is consistent in $A\|B$ iff $\{a \mid (1,a) \in C\} \in \mathrm{Con}_A$ and $\{b \mid (2,b) \in C\} \in \mathrm{Con}_B$. The operation extends to a functor—put the two maps in parallel. The empty event structure with polarity $\varnothing$ is the unit w.r.t. $\|$.

4.3 Pre-strategies

Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A *pre-strategy* in A is a total map $\sigma : S \to A$ from an event structure with polarity S. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy* (and *winning strategy* in Section 9.1).

A map from a pre-strategy $\sigma : S \to A$ to a pre-strategy $\sigma' : S' \to A$ is a map $f : S \to S'$ such that

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "\sigma"'] & S' \arrow[d, "\sigma'"] \\ & A \end{tikzcd}$$

commutes. Accordingly, we regard two pre-strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ as essentially the same when they are isomorphic, and write $\sigma \cong \sigma'$, *i.e.* when there is an isomorphism of event structures $\theta : S \cong S'$ such that

$$\begin{tikzcd} S \arrow[r, phantom, "\overset{\theta}{\cong}"] \arrow[dr, "\sigma"'] & S' \arrow[d, "\sigma'"] \\ & A \end{tikzcd}$$

commutes.

Let A and B be event structures with polarity. Following Joyal [6], a pre-strategy from A to B is a pre-strategy in $A^\perp\|B$, so a total map $\sigma : S \to A^\perp\|B$. It thus determines a span

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ A^\perp & & B, \end{tikzcd}$$

of event structures with polarity where σ_1, σ_2 are *partial* maps. In fact, a pre-strategy from A to B corresponds to such spans where for all $s \in S$ either, but not both, $\sigma_1(s)$ or $\sigma_2(s)$ is defined. Two pre-strategies σ and τ from A to B are isomorphic, $\sigma \cong \tau$, when their spans are isomorphic, *i.e.*

$$\begin{tikzcd}
& S \arrow[d, "\cong"] \arrow[ddl, "\sigma_1"'] \arrow[ddr, "\sigma_2"] & \\
& T \arrow[dl, "\tau_1"] \arrow[dr, "\tau_2"'] & \\
A^\perp & & B
\end{tikzcd}$$

commutes. We write $\sigma : A \mathrel{-\!\!\!+\!\!\!\!\!\to} B$ to express that σ is a pre-strategy from A to B. Note a pre-strategy in a game A coincides with a pre-strategy from the empty game $\sigma : \varnothing \mathrel{-\!\!\!+\!\!\!\!\!\to} A$.

4.3.1 Concurrent copy-cat

Identities on games are given by copy-cat strategies—strategies for Player based on copying the latest moves made by Opponent.

Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$. It describes a concurrent, or distributed, strategy based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity.

For $c \in A^\perp \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component, *i.e.*

$$\overline{(1,a)} = (2,\overline{a}) \text{ and } \overline{(2,a)} = (1,\overline{a})\,.$$

Proposition 4.1. *Let A be an event structure with polarity. There is an event structure with polarity $\mathbb{C}\mathbb{C}_A$ having the same events and polarity as $A^\perp \| A$ but with causal dependency $\le_{\mathbb{C}\mathbb{C}_A}$ given as the transitive closure of the relation*

$$\le_{A^\perp \| A} \cup \{(\overline{c}, c) \mid c \in A^\perp \| A \;\&\; pol_{A^\perp \| A}(c) = +\}$$

and finite subsets of $\mathbb{C}\mathbb{C}_A$ consistent if their down-closure w.r.t. $\le_{\mathbb{C}\mathbb{C}_A}$ are consistent in $A^\perp \| A$. Moreover,
(i) $c \mathrel{-\!\!\!\triangleright} c'$ in $\mathbb{C}\mathbb{C}_A$ iff

$$c \mathrel{-\!\!\!\triangleright} c' \textit{ in } A^\perp \| A \textit{ or } pol_{A^\perp \| A}(c') = + \;\&\; \overline{c} = c'\,;$$

(ii) $x \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ iff

$$x \in \mathcal{C}(A^\perp \| A) \;\&\; \forall c \in x.\; pol_{A^\perp \| A}(c) = + \implies \overline{c} \in x\,.$$

Proof. It can first be checked that defining

$$\begin{aligned} c \le_{\mathbb{C}\mathbb{C}_A} c' \text{ iff } & (i)\; c \le_{A^\perp \| A} c' \text{ or} \\ & (ii)\; \exists c_0 \in A^\perp \| A.\; pol_{A^\perp \| A}(c_0) = + \;\& \\ & \qquad c \le_{A^\perp \| A} \overline{c_0} \;\&\; c_0 \le_{A^\perp \| A} c'\,, \end{aligned}$$

yields a partial order. Note that

$$c \le_{A^\perp \| A} d \text{ iff } \overline{c} \le_{A^\perp \| A} \overline{d}\,,$$

used in verifying transitivity and antisymmetry. The relation $\le_{\mathbb{C}_A}$ is clearly the transitive closure of $\le_{A^\perp \| A}$ together with all extra causal dependencies $(\overline{c}, c)$ where $pol_{A^\perp \| A}(c) = +$. The remaining properties required for $\mathbb{C}_A$ to be an event structure follow routinely.
(i) From the above characterization of $\le_{\mathbb{C}_A}$.
(ii) From $\mathbb{C}_A$ and $A^\perp \| A$ sharing the same consistency relation on sets down-closed in $A^\perp \| A$ and w.r.t. the extra causal dependency adjoined to $\mathbb{C}_A$. $\square$

Based on Proposition 4.1, define the *copy-cat* pre-strategy from A to A to be the pre-strategy $\gamma_A : \mathbb{C}_A \to A^\perp \| A$ where $\mathbb{C}_A$ comprises the event structure with polarity $A^\perp \| A$ together with extra causal dependencies $\overline{c} \le_{\mathbb{C}_A} c$ for all events c with $pol_{A^\perp \| A}(c) = +$, and γ_A is the identity on the set of events common to both $\mathbb{C}_A$ and $A^\perp \| A$.

4.3.2 Composing pre-strategies

Consider two pre-strategies $\sigma : A \mathrel{\rightarrow\!\!\!\!\!\!+\!\!\!\!\!\rightarrow} B$ and $\tau : B \mathrel{\rightarrow\!\!\!\!\!\!+\!\!\!\!\!\rightarrow} C$ as spans:

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & & T \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ A^\perp & & B & B^\perp & & C\,. \end{tikzcd}$$

We show how to define their composition $\tau \odot \sigma : A \mathrel{\rightarrow\!\!\!\!\!\!+\!\!\!\!\!\rightarrow} C$. If we ignore polarities the partial maps of event structures σ_2 and τ_1 have a common codomain, the underlying event structure of B and $B^\perp$. The composition $\tau \odot \sigma$ will be constructed as a synchronized composition of S and T, in which output events of S synchronize with input events of T, followed by an operation of hiding 'internal' synchronization events. Only those events s from S and t from T for which $\sigma_2(s) = \overline{\tau_1(t)}$ synchronize; note that then s and t must have opposite polarities as this is so for their images $\sigma_2(s)$ in B and $\tau_1(t)$ in $B^\perp$. The event resulting from the synchronization of s and t has indeterminate polarity and will be hidden in the composition $\tau \odot \sigma$.

Formally, we use the construction of synchronized composition and projection of Section 3.3.3. Via projection we hide all those events with undefined polarity.

We first define the composition of the families of configurations of S and T as a synchronized composition of stable families. We form the product of stable families $\mathcal{C}(S) \times \mathcal{C}(T)$ with projections π_1 and π_2, and then form a restriction:

$$\mathcal{C}(T) \circledast \mathcal{C}(S) =_{\mathrm{def}} \mathcal{C}(S) \times \mathcal{C}(T) \restriction R$$

where

$$\begin{aligned} R \;=\; & \{(s,*) \mid s \in S \;\&\; \sigma_1(s) \text{ is defined}\} \cup \\ & \{(s,t) \mid s \in S \;\&\; t \in T \;\&\; \sigma_2(s) = \overline{\tau_1(t)} \text{ with both defined}\} \cup \\ & \{(*,t) \mid t \in T \;\&\; \tau_2(t) \text{ is defined}\}\,. \end{aligned}$$

The stable family $\mathcal{C}(T) \circledast \mathcal{C}(S)$ is the synchronized composition of the stable families $\mathcal{C}(S)$ and $\mathcal{C}(T)$ in which synchronizations are between events of S and T which project, under σ_2 and τ_1 respectively, to complementary events in B and $B^\perp$. The stable family $\mathcal{C}(T) \circledast \mathcal{C}(S)$ represents all the configurations of the composition of pre-strategies, including internal events arising from synchronizations. We obtain the synchronized composition as an event structure by forming $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, in which events are the primes of $\mathcal{C}(T) \circledast \mathcal{C}(S)$. This synchronized composition still has internal events.

To obtain the composition of pre-strategies we hide the internal events due to synchronizations. The event structure of the composition of pre-strategies is defined to be

$$T \odot S =_{\text{def}} \Pr(\mathcal{C}(T) \circledast \mathcal{C}(S)) \downarrow V\,,$$

the projection onto "visible" events,

$$\begin{aligned} V = & \{p \in \Pr(\mathcal{C}(T) \circledast \mathcal{C}(S)) \mid \exists s \in S.\; top(p) = (s,*)\} \cup \\ & \{p \in \Pr(\mathcal{C}(T) \circledast \mathcal{C}(S)) \mid \exists t \in T.\; top(p) = (*,t)\}\,. \end{aligned}$$

Finally, the composition $\tau \odot \sigma$ is defined by the span

$$\begin{tikzcd} & T \odot S \arrow[dl, "\upsilon_1"'] \arrow[dr, "\upsilon_2"] & \\ A^\perp & & C \end{tikzcd}$$

where υ_1 and υ_2 are maps of event structures, which on events p of $T \odot S$ act so $\upsilon_1(p) = \sigma_1(s)$ when $top(p) = (s,*)$ and $\upsilon_2(p) = \tau_2(t)$ when $top(p) = (*,t)$, and are undefined elsewhere.

Proposition 4.2. *Above, υ_1 and υ_2 are partial maps of event structures with polarity, which together define a pre-strategy $\upsilon : A \twoheadrightarrow C$. For $x \in \mathcal{C}(T \odot S)$,*

$$\upsilon_1 x = \sigma_1 \pi_1 \bigcup x \;\; \textit{and} \;\; \upsilon_2 x = \tau_2 \pi_2 \bigcup x\,.$$

Proof. Consider the two maps of event structures

$$\begin{aligned} u_1 &: \Pr(\mathcal{C}(T) \circledast \mathcal{C}(S)) \xrightarrow{\Pi_1} S \xrightarrow{\sigma_1} A^\perp\,, \\ u_2 &: \Pr(\mathcal{C}(T) \circledast \mathcal{C}(S)) \xrightarrow{\Pi_2} T \xrightarrow{\tau_2} C\,, \end{aligned}$$

where Π_1, Π_2 are (restrictions of) projections of the product of event structures. *E.g.* for $p \in \Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, $\Pi_1(p) = s$ precisely when $top(p) = (s,*)$, so $\sigma_1(s)$

is defined, or when $top(p) = (s,t)$, so $\sigma_1(s)$ is undefined. The partial functions υ_1 and υ_2 are restrictions of the two maps u_1 and u_2 to the projection set V. But V consists exactly of those events in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$ where u_1 or u_2 is defined. It follows that υ_1 and υ_2 are maps of event structures.

Clearly one and only one of υ_1, υ_2 are defined on any event in $T \odot S$ so they form a pre-strategy. Their effect on $x \in \mathcal{C}(T \odot S)$ follows directly from their definition. □

Proposition 4.3. *Let $\sigma : A \twoheadrightarrow B$, $\tau : B \twoheadrightarrow C$ and $\upsilon : C \twoheadrightarrow D$ be pre-strategies. The two compositions $\upsilon \odot (\tau \odot \sigma)$ and $(\upsilon \odot \tau) \odot \sigma$ are isomorphic.*

Proof. The natural isomorphism $S \times (T \times U) \cong (S \times T) \times U$, associated with the product of event structures S, T, U, restricts to the required isomorphism of spans as the synchronizations involved in successive compositions are disjoint. □

4.3.3 Composition via pullback

We can alternatively present the composition of pre-strategies via pullbacks.[2] For this section assume that the correspondence $a \leftrightarrow \overline{a}$ between the events of A and its dual $A^\perp$ is the identity, so A and $A^\perp$ share the same events, though assign opposite polarities to them. Given two pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in $\mathcal{E}$ (or $\mathcal{E}_t$, as the maps along which we are pulling back are total)

$$\begin{tikzcd}[column sep=small]
 & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
S \| C \arrow[dr, "\sigma \| C"'] & & A \| T \arrow[dl, "A \| \tau"] \\
 & A \| B \| C\,. &
\end{tikzcd}$$

There is an obvious partial map of event structures $A \| B \| C \to A \| C$ undefined on B and acting as identity on A and C. The partial map from P to $A \| C$ given

[2]I'm grateful to Nathan Bowler for the observations of this section.

by following the diagram (either way round the pullback square)

$$\begin{tikzcd}
 & P \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
S\|C \arrow[dr, "\sigma\|C"'] & & A\|T \arrow[dl, "A\|\tau"] \\
 & A\|B\|C \arrow[d] & \\
 & A\|C &
\end{tikzcd}$$

factors through the projection of P to V, those events at which the partial map is defined:

$$P \to P \downarrow V \to A\|C\,.$$

The resulting total map $\upsilon : P \downarrow V \to A\|C$ gives us the composition $\tau\odot\sigma : P \downarrow V \to A^{\perp}\|C$ once we reinstate polarities.

4.3.4 Duality

A pre-strategy $\sigma : A \longrightarrow\!\!\!\!\!\!+\;\; B$ corresponds to a dual pre-strategy $\sigma^{\perp} : B^{\perp} \longrightarrow\!\!\!\!\!\!+\;\; A^{\perp}$. This duality arises from the correspondence

$$\begin{tikzcd}
 & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & \longleftrightarrow & & S \arrow[dl, "\sigma_2"'] \arrow[dr, "\sigma_1"] & \\
A^{\perp} & & B & & (B^{\perp})^{\perp} & & A^{\perp}\,.
\end{tikzcd}$$

It is easy to check that the dual of copy-cat, $\gamma_A^{\perp}$, is isomorphic, as a span, to the copy-cat of the dual, $\gamma_{A^{\perp}}$, for A an event structure with polarity. It is also straightforward, though more involved, to show that the dual of a composition of pre-strategies $(\tau\odot\sigma)^{\perp}$ is isomorphic as a span to the composition $\sigma^{\perp}\odot\tau^{\perp}$. Duality, as usual, will save us work.

4.4 Strategies

This section is devoted to the main result of this chapter: that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient in order for copy-cat to behave as identity w.r.t. the composition of pre-strategies. It becomes compelling to define a *(nondeterministic) concurrent strategy*, in general, as a pre-strategy which is receptive and innocent.

4.4.1 Necessity of receptivity and innocence

The properties of *receptivity* and *innocence* of a pre-strategy, described below, will play a central role.

Receptivity. Say a pre-strategy $\sigma : S \to A$ is *receptive* when $\sigma x \xrightarrow{a}\!\!\subset$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \xrightarrow{s}\!\!\subset$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$. Receptivity ensures that no Opponent move which is possible is disallowed.

Innocence. Say a pre-strategy σ is *innocent* when it is both +-innocent and −-innocent:

+-Innocence: If $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

−-Innocence: If $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

The definition of a pre-strategy $\sigma : S \to A$ ensures that the moves of Player and Opponent respect the causal constraints of the game A. Innocence restricts Player further. Locally, within a configuration, Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$. Thus innocence gives Player the freedom to await Opponent moves before making their move, but prevents Player having any influence on the moves of Opponent beyond those stipulated in the game A; more surprisingly, innocence also disallows any immediate causality of the form $\oplus \rightarrowtriangle \oplus$, purely between Player moves, not already stipulated in the game A.

Two important consequences of −-innocence:

Lemma 4.4. *Let $\sigma : S \to A$ be a pre-strategy. Suppose, for $s, s' \in S$, that*

$$[s) \uparrow [s') \ \&\ pol_S(s) = pol_S(s') = - \ \&\ \sigma(s) = \sigma(s')\,.$$

(i) If σ is −-innocent, then $[s) = [s')$.

(ii) If σ is receptive and −-innocent, then $s = s'$.

[$x \uparrow y$ expresses the compatibility of $x, y \in \mathcal{C}(S)$.]

Proof. (i) Assume the property above holds of $s, s' \in S$. Assume σ is −-innocent. Suppose $s_1 \rightarrowtriangle s$. Then by −-innocence, $\sigma(s_1) \rightarrowtriangle \sigma(s)$. As $\sigma(s') = \sigma(s)$ and σ is a map of event structures there is $s_2 < s'$ such that $\sigma(s_2) = \sigma(s_1)$. But s_1, s_2 both belong to the configuration $[s) \cup [s')$ so $s_1 = s_2$, as σ is a map, and $s_1 < s'$. Symmetrically, if $s_1 \rightarrowtriangle s'$ then $s_1 < s$. It follows that $[s) = [s')$. (ii) Now both $[s) \xrightarrow{s}\!\!\subset$ and $[s) \xrightarrow{s'}\!\!\subset$ with $\sigma(s) = \sigma(s')$ where both s, s' have −ve polarity. If, further, σ is receptive, $s = s'$. □

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^- x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{-\}$, *i.e.* the configuration x' extends the configuration x solely by events of −ve polarity. In the presence of −-innocence, receptivity strengthens to the following useful *strong-receptivity* property:

Lemma 4.5. *Let $\sigma : S \to A$ be a −-innocent pre-strategy. The pre-strategy σ is* receptive *iff whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a* unique *$x' \in \mathcal{C}(S)$ so that*

$x \subseteq x'$ & $\sigma x' = y$. Diagrammatically,

$$\begin{tikzcd} x \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, "\sigma"'] & x' \arrow[d, dotted, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & y\,. \end{tikzcd}$$

[It will necessarily be the case that $x \subseteq^- x'$.]

Proof. *"if":* Clear. *"Only if":* Assuming $\sigma x \subseteq^- y$ we can form a covering chain

$$\sigma x \xrightarrow{a_1}\!\!\subset y_1 \cdots \xrightarrow{a_n}\!\!\subset y_n = y\,.$$

By repeated use of receptivity we obtain the existence of x' where $x \subseteq x'$ and $\sigma x' = y$. To show the uniqueness of x' suppose $x \subseteq z, z'$ and $\sigma z = \sigma z' = y$. Suppose that $z \neq z'$. Then, without loss of generality, suppose there is a $\leq_S$-minimal $s' \in z'$ with $s' \notin z$. Then $[s') \subseteq z$. Now $\sigma(s') \in y$ so there is $s \in z$ for which $\sigma(s) = \sigma(s')$. We have $[s), [s') \subseteq z$ so $[s) \uparrow [s')$. By Lemma 4.4(ii) we deduce $s = s'$ so $s' \in z$, a contradiction. Hence, $z = z'$. ◻

It is useful to define innocence and receptivity on partial maps of event structures with polarity.

Definition 4.6. Let $f : S \to A$ be a partial map of event structures with polarity. Say f is *receptive* when

$$f(x) \xrightarrow{a}\!\!\subset \ \&\ pol_A(a) = - \implies \exists! s \in S.\ x \xrightarrow{s}\!\!\subset \ \&\ f(s) = a$$

for all $x \in \mathcal{C}(S)$, $a \in A$.

Say f is *innocent* when it is both +-innocent and −-innocent, *i.e.*

$$\begin{aligned} s \rightarrowtriangle s' \ \&\ pol(s) = + \ \&\ f(s) \text{ is defined } &\implies \\ & f(s') \text{ is defined } \ \&\ f(s) \rightarrowtriangle f(s')\,, \\ s \rightarrowtriangle s' \ \&\ pol(s') = - \ \&\ f(s') \text{ is defined } &\implies \\ & f(s) \text{ is defined } \ \&\ f(s) \rightarrowtriangle f(s')\,. \end{aligned}$$

Proposition 4.7. *A pre-strategy $\sigma : A \twoheadrightarrow B$ is receptive, respectively +/−-innocent, iff both the partial maps σ_1 and σ_2 of its span are receptive, respectively +/−-innocent.*

Proposition 4.8. *For $\sigma : A \twoheadrightarrow B$ a pre-strategy, σ_1 is receptive, respectively +/−-innocent, iff $(\sigma^\perp)_2$ is receptive, respectively +/−-innocent; σ is receptive and innocent iff $\sigma^\perp$ is receptive and innocent.*

The next lemma will play a major role in importing receptivity and innocence to compositions of pre-strategies.

Lemma 4.9. *For pre-strategies $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$, if σ_1 is receptive, respectively +/−-innocent, then $(\tau \odot \sigma)_1$ is receptive, respectively +/−-innocent.*

Proof. Abbreviate $\tau\odot\sigma$ to υ.
Receptivity: We show the receptivity of υ_1 assuming that σ_1 is receptive. Let $x \in \mathcal{C}(T\odot S)$ such that $\upsilon_1 x \xrightarrow{a}\subset$ in $\mathcal{C}(A^\perp)$ with $pol_{A^\perp}(a) = -$. By Proposition 4.2, $\sigma_1\pi_1 \bigcup x \xrightarrow{a}\subset$ with $\pi_1 \bigcup x \in \mathcal{C}(S)$. As σ_1 is receptive there is a unique $s \in S$ such that $\pi_1 \bigcup x \xrightarrow{s}\subset$ in S and $\sigma_1(s) = a$. It follows that $\bigcup x \xrightarrow{(s,*)}\subset z$, for some z, in $\mathcal{C}(T) \circledast \mathcal{C}(S)$. Defining $p =_{\text{def}} [(s,*)]_z$ we obtain $x \xrightarrow{p}\subset$ and $\upsilon_1(p) = a$, with p the unique such event.
Innocence: Assume that σ_1 is innocent. To show the +-innocence of υ_1 we first establish a property of the $\multimap$-relation in the event structure $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, the synchronized composition of event structures S and T, before projection to V:

> *If $e \multimap e'$ in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$ with $e \in V$, $pol(e) = +$ and $\upsilon_1(e)$ defined, then $e' \in V$ and $\upsilon_1(e')$ is defined.*

Assume $e \multimap e'$ in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, $e \in V$, $pol(e) = +$ and $\upsilon_1(e)$ is defined. From the definition of $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, the event e is a prime configuration of $\mathcal{C}(T) \circledast \mathcal{C}(S)$ where $top(e)$ must have the form $(s,*)$, for some event s of S where $\sigma_1(s)$ is defined. By Lemma 3.27, $top(e')$ has the form $(s',*)$ or (s',t) with $s \multimap s'$ in S. Now, as $s \multimap s'$ and $pol(s) = +$, from the +-innocence of σ_1, we obtain $\sigma_1(s) \multimap \sigma_1(s')$ in $A^\perp \| A$. Whence $\sigma_1(s')$ is defined ensuring $top(e') = (s',*)$. It follows that $e' \in V$ and $\upsilon_1(e')$ is defined.

Now suppose $e \multimap e'$ in $T\odot S$. Then either
(i) $e \multimap e'$ in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, or
(ii) $e \multimap e_1 < e'$ in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$ for some 'invisible' event $e_1 \notin V$.

But the above argument shows that case (ii) cannot occur when $pol(e) = +$ and $\upsilon_1(e)$ is defined. It follows that whenever $e \multimap e'$ in $T\odot S$ with $pol(e) = +$ and $\upsilon_1(e)$ defined, then $\upsilon_1(e')$ is defined and $\upsilon_1(e) \multimap \upsilon_1(e')$, as required.

The argument showing $-$-innocence of υ_1 assuming that of σ_1 is similar. □

Corollary 4.10. *For pre-strategies $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$, if τ_2 is receptive, respectively $+/-$-innocent, then $(\tau\odot\sigma)_2$ is receptive, respectively $+/-$-innocent.*

Proof. By duality using Lemma 4.9: if τ_2 is receptive, respectively $+/-$-innocent, then $(\tau^\perp)_1$ is receptive, respectively $+/-$-innocent, and hence $(\sigma^\perp\odot\tau^\perp)_1 = ((\tau\odot\sigma)^\perp)_1 = (\tau\odot\sigma)_2$ is receptive, respectively $+/-$-innocent. □

Lemma 4.11. *For an event structure with polarity A, the pre-strategy copy-cat $\gamma_A : A \twoheadrightarrow A$ is receptive and innocent.*

Proof. *Receptive:* Suppose $x \in \mathcal{C}(\mathbb{C}_A)$ such that $\gamma_A x \xrightarrow{c}\subset$ in $\mathcal{C}(A^\perp \| A)$ where $pol_{A^\perp\|A}(c) = -$. Now $\gamma_A x = x$ and $x' =_{\text{def}} x \cup \{c\} \in \mathcal{C}(A^\perp \| A)$. Proposition 4.1(ii) characterizes those configurations of $A^\perp \| A$ which are also configurations of $\mathbb{C}_A$: the characterization applies to x and to its extension $x' = x \cup \{c\}$ because of the

−ve polarity of c. Hence $x' \in \mathcal{C}(\mathbb{C}_A)$ and $x \overset{c}{\longrightarrow\!\!\subset} x'$ in $\mathcal{C}(\mathbb{C}_A)$, and clearly c is unique so $\gamma_A(c) = c$.

−-*Innocent:* Suppose $c \twoheadrightarrow c'$ in $\mathbb{C}_A$ and $pol(c') = -$. By Proposition 4.1(i), $c \twoheadrightarrow c'$ in $A^\perp \| A$. The argument for +-innocence is similar. □

Theorem 4.12. *Let $\sigma : A \twoheadrightarrow B$ be a pre-strategy from A to B. If $\sigma \odot \gamma_A \cong \sigma$ and $\gamma_B \odot \sigma \cong \sigma$, then σ is receptive and innocent.*

Let $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$ be pre-strategies which are both receptive and innocent. Then their composition $\tau \odot \sigma : A \twoheadrightarrow C$ is receptive and innocent.

Proof. We know the copy-cat pre-strategies γ_A and γ_B are receptive and innocent—Lemma 4.11. Assume $\sigma \odot \gamma_A \cong \sigma$ and $\gamma_B \odot \sigma \cong \sigma$. By Lemma 4.9, $(\sigma \odot \gamma_A)_1$ is receptive and innocent so σ_1 is receptive and innocent. From its dual, Corollary 4.10, $(\gamma_B \odot \sigma)_2$ so σ_2 is receptive and innocent. Hence σ is receptive and innocent.

Assume that $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$ are receptive and innocent. The fact that σ is receptive and innocent ensures that $(\tau \odot \sigma)_1$ is receptive and innocent, that τ is receptive and innocent that $(\tau \odot \sigma)_2$ is too. Combining, we obtain that $\tau \odot \sigma$ is receptive and innocent. □

In other words, if a pre-strategy is to compose well with copy-cat, in the sense that copy-cat behaves as an identity w.r.t. composition, the pre-strategy must be receptive and innocent. Copy-cat behaving as identity is a hallmark of game-based semantics, so any sensible definition of concurrent strategy will have to ensure receptivity and innocence.

4.4.2 Sufficiency of receptivity and innocence

In fact, as we will now see, not only are the conditions of receptivity and innocence on pre-strategies necessary to ensure that copy-cat acts as identity. They are also sufficient.

Technically, this section establishes that for a pre-strategy $\sigma : A \twoheadrightarrow B$ which is receptive and innocent both the compositions $\sigma \odot \gamma_A$ and $\gamma_B \odot \sigma$ are isomorphic to σ. We shall concentrate on the isomorphism from $\sigma \odot \gamma_A$ to σ. The isomorphism from $\gamma_B \odot \sigma$ to σ follows by duality.

Recall, from Section 4.3.2, the construction of the pre-strategy $\sigma \odot \gamma_A$ as a total map $S \odot \mathbb{C}_A \to A^\perp \| B$. The event structure $S \odot \mathbb{C}_A$ is built from the synchronized composition of stable families $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$, a restriction of the product of stable families to events

$$
\begin{aligned}
&\{(c,*) \mid c \in \mathbb{C}_A \ \&\ \gamma_{A1}(c) \text{ is defined}\} \cup \\
&\{(c,s) \mid c \in \mathbb{C}_A \ \&\ s \in S \ \&\ \gamma_{A2}(c) = \overline{\sigma_1(s)}\} \cup \\
&\{(*,s) \mid s \in S \ \&\ \sigma_2(t) \text{ is defined}\} \ :
\end{aligned}
$$

$$
\begin{tikzcd}[column sep=small]
 & & & \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}\!\!\mathbb{C}_A) \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & & & \\
 & \mathcal{C}(\mathbb{C}\!\!\mathbb{C}_A) \arrow[dl, "\gamma_{A1}"'] \arrow[drr, "\gamma_{A2}"] & & & & \mathcal{C}(S) \arrow[dll, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\
\mathcal{C}(A^{\perp}) & & & \mathcal{C}(A)\ \mathcal{C}(A^{\perp}) & & & \mathcal{C}(B)
\end{tikzcd}
$$

Finally $S \odot \mathbb{C}\!\!\mathbb{C}_A$ is obtained from the prime configurations of $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}\!\!\mathbb{C}_A)$ whose maximum events are defined under $\gamma_{A1}\pi_1$ or $\sigma_2\pi_2$.

We will first present the putative isomorphism from $\sigma \odot \gamma_A$ to σ as a total map of event structures $\theta : S \odot \mathbb{C}\!\!\mathbb{C}_A \to S$. The definition of θ depends crucially on the lemmas below. They involve special configurations of $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}\!\!\mathbb{C}_A)$, *viz.* those of the form $\bigcup x$, where x is a configuration of $S \odot \mathbb{C}\!\!\mathbb{C}_A$.

Lemma 4.13. *For $x \in \mathcal{C}(S \odot \mathbb{C}\!\!\mathbb{C}_A)$,*

$$(c, s) \in \bigcup x \implies (\overline{c}, *) \in \bigcup x \,.$$

Proof. The case when $pol(c) = +$ follows directly because then $\overline{c} \rightarrowtriangle c$ in $\mathbb{C}\!\!\mathbb{C}_A$ so $(\overline{c}, *) \rightarrowtriangle_{\bigcup x} (c, s)$.
Suppose the lemma fails in the case when $pol(c) = -$, so there is a $\le_{\bigcup x}$-maximal $(c, s) \in \bigcup x$ such that

$$pol(c) = - \ \&\ (\overline{c}, *) \notin \bigcup x \,. \tag{$\dagger$}$$

The event (c, s) cannot be maximal in $\bigcup x$ as its maximal events take the form $(c', *)$ or $(*, s')$. There must be $e \in \bigcup x$ for which

$$(c, s) \rightarrowtriangle_{\bigcup x} e \,.$$

Consider the possible forms of e:
Case $e = (c', s')$: Then, by Lemma 3.27, either $c \rightarrowtriangle c'$ in $\mathbb{C}\!\!\mathbb{C}_A$ or $s \rightarrowtriangle s'$ in S. However if $s \rightarrowtriangle s'$ then, as $pol(s) = +$ by innocence, $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^{\perp}$, so $\gamma_{A2}(c) \rightarrowtriangle \gamma_{A2}(c')$ in A; but then $c \rightarrowtriangle c'$ in $\mathbb{C}\!\!\mathbb{C}_A$. Either way, $c \rightarrowtriangle c'$ in $\mathbb{C}\!\!\mathbb{C}_A$.

Suppose $pol(c') = +$. Then,

$$(c, s) \rightarrowtriangle_{\bigcup x} (\overline{c}, *) \rightarrowtriangle_{\bigcup x} (\overline{c'}, *) \rightarrowtriangle_{\bigcup x} (c', s') \,.$$

But this contradicts $(c, s) \rightarrowtriangle_{\bigcup x} (c', s')$.

Suppose $pol(c') = -$. Because (c, s) is maximal such that ($\dagger$), $(\overline{c'}, *) \in \bigcup x$. But $(\overline{c}, *) \rightarrowtriangle_{\bigcup x} (\overline{c'}, *)$ whence $(\overline{c}, *) \in \bigcup x$, contradicting ($\dagger$).
Case $e = (*, s')$: Now $(c, s) \rightarrowtriangle_{\bigcup x} (*, s')$. By Lemma 3.27, $s \rightarrowtriangle s'$ in S with $pol(s) = +$. By innocence, $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ and in particular $\sigma_1(s')$ is defined, which forbids $(*, s')$ as an event of $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}\!\!\mathbb{C}_A)$.
Case $e = (c', *)$: Now $(c, s) \rightarrowtriangle_{\bigcup x} (c', *)$. By Lemma 3.27, $c \rightarrowtriangle c'$ in $\mathbb{C}\!\!\mathbb{C}_A$. Because (c, s) and $(c', *)$ are events of $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}\!\!\mathbb{C}_A)$ we must have $\gamma_2(c)$ and $\gamma_1(c')$ are defined—they are in different components of $\mathbb{C}\!\!\mathbb{C}_A$. By Proposition 4.1, $c' = \overline{c}$, contradicting ($\dagger$).

In all cases we obtain a contradiction—hence the lemma. $\square$

Lemma 4.14. *For $x \in \mathcal{C}(S \odot \mathbb{C}_A)$,*

$$\sigma_1 \pi_2 \bigcup x \subseteq^- \gamma_{A\,1} \pi_1 \bigcup x\,.$$

Proof. As a direct corollary of Lemma 4.13, we obtain:

$$\sigma_1 \pi_2 \bigcup x \subseteq \gamma_{A\,1} \pi_1 \bigcup x\,.$$

The current lemma will follow provided all events of +ve polarity in $\gamma_{A\,1}\pi_1 \bigcup x$ are in $\sigma_1\pi_2 \bigcup x$. However, $(\overline{c}, s) \rightarrowtriangle_{\bigcup x} (c, *)$, for some $s \in S$, when $pol(c) = +$. □

Lemma 4.15. *For $x \in \mathcal{C}(S \odot \mathbb{C}_A)$,*

$$\sigma \pi_2 \bigcup x \subseteq^- \sigma \odot \gamma_A\, x\,.$$

Proof.

$$\begin{aligned} \sigma\pi_2 \bigcup x &= \{1\} \times \sigma_1 \pi_2 \bigcup x \;\cup\; \{2\} \times \sigma_2 \pi_2 \bigcup x \\ &\subseteq^- \{1\} \times \gamma_{A\,1} \pi_1 \bigcup x \;\cup\; \{2\} \times \sigma_2 \pi_2 \bigcup x\,, \quad \text{by Lemma 4.14} \\ &= \sigma \odot \gamma_A\, x\,, \quad \text{by Proposition 4.2.} \end{aligned}$$

□

Lemma 4.15 is the key to defining a map $\theta : S \odot \mathbb{C}_A \to S$ via the following map-lifting property of receptive maps:

Lemma 4.16. *Let $\sigma : S \to C$ be a total map of event structures with polarity which is receptive and $-$-innocent. Let $p : \mathcal{C}(V) \to \mathcal{C}(S)$ be a monotonic function, i.e. such that $p(x) \subseteq p(y)$ whenever $x \subseteq y$ in $\mathcal{C}(V)$. Let $\upsilon : V \to C$ be a total map of event structures with polarity such that*

$$\forall x \in \mathcal{C}(V).\ \sigma p(x) \subseteq^- \upsilon\, x\,.$$

Then, there is a unique total map of event structures with polarity $\theta : V \to S$ such that $\forall x \in \mathcal{C}(V).\ p(x) \subseteq^- \theta\, x$ and $\upsilon = \sigma\theta$:

$$\begin{tikzcd} V \arrow[r, dashed, "p", "\subseteq^-"'] \arrow[r, dotted, bend left=60, "\theta"] \arrow[dr, "\upsilon"'] & S \arrow[d, "\sigma"] \\ & C\,. \end{tikzcd}$$

[We use a broken arrow to signify that p is not a map of event structures.]

Proof. Let $x \in \mathcal{C}(V)$. Then $\sigma p(x) \subseteq^- \upsilon\, x$. Define $\Theta(x)$ to be the unique configuration of $\mathcal{C}(S)$, determined by the receptivity of σ, such that

$$\begin{tikzcd} p(x) \arrow[r, dotted, no head, "\subseteq^-"] \arrow[d, mapsto, "\sigma"'] & \Theta(x) \arrow[d, dotted, mapsto, "\sigma"] \\ \sigma p(x) \arrow[r, phantom, "\subseteq^-"] & \upsilon\, x\,. \end{tikzcd}$$

Define θ_x to be the composite bijection

$$\theta_x : x \;\cong\; \upsilon x \;\cong\; \Theta(x)$$

where the bijection $x \cong \upsilon x$ is that determined locally by the total map of event structures υ, and the bijection $\upsilon x \cong \Theta(x)$ is the inverse of the bijection $\sigma \restriction \Theta(x) : \Theta(x) \cong \upsilon\, x$ determined locally by the total map σ.

Now, let $y \in \mathcal{C}(V)$ with $x \subseteq y$. We claim that θ_x is the restriction of θ_y. This will follow once we have shown that $\Theta(x) \subseteq \Theta(y)$. Then, treating the inclusions as inclusion maps, both squares in the diagram below will commute:

$$\begin{array}{ccccc}
\theta_y : y & \cong & \upsilon\, y & \cong & \Theta(y) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\theta_x : x & \cong & \upsilon\, x & \cong & \Theta(x)
\end{array}$$

This will make the composite rectangle commute, *i.e.* make θ_x the restriction of θ_y.

To show $\Theta(x) \subseteq \Theta(y)$ we suppose otherwise. Then there is an event $s \in \Theta(x)$ of minimum depth w.r.t. $\leq_S$ such that $s \notin \Theta(y)$. Note that $pol(s) = -$, as otherwise $s \in p(x) \subseteq p(y) \subseteq \Theta(y)$. As $\sigma(s) \in \upsilon\, x \subseteq \upsilon\, y$ there is $s' \in \Theta(y)$ such that $\sigma(s') = \sigma(s)$. From the minimality of s, both $[s), [s') \subseteq \Theta(y)$ ensuring the compatibility of $[s)$ and $[s')$. By Lemma 4.4(ii), $s = s'$ and $s \in \Theta(y)$—a contradiction.

By Proposition 2.7, the family θ_x, $x \in \mathcal{C}(V)$, determines the unique total map $\theta : V \to S$ such that $\theta\, x = \Theta(x)$. By construction, $p(x) \subseteq^- \theta\, x$, for all $x \in \mathcal{C}(V)$, and $\upsilon = \sigma\theta$. This property in itself ensures that $\theta\, x = \Theta(x)$ so determines θ uniquely. $\square$

In Lemma 4.16, instantiate $p : \mathcal{C}(S \odot \mathbb{C}_A) \to \mathcal{C}(S)$ to the function $p(x) = \pi_2 \bigcup x$ for $x \in \mathcal{C}(S \odot \mathbb{C}_A)$, the map σ to the pre-strategy $\sigma : S \to A^\perp \| B$ and υ to the pre-strategy $\sigma \odot \gamma_A$. By Lemma 4.15, $\sigma \pi_2 \bigcup x \subseteq^- \sigma \odot \gamma_A\, x$, so the conditions of Lemma 4.16 are met and we obtain a total map $\theta : S \odot \mathbb{C}_A \to S$ such that $\pi_2 \bigcup x \subseteq^- \theta\, x$, for all $x \in \mathcal{C}(S \odot \mathbb{C}_A)$, and $\sigma\theta = \sigma \odot \gamma_A$:

$$\begin{tikzcd}
S \odot \mathbb{C}_A \arrow[r, dashed, "p"'] \arrow[r, dotted, bend left=60, "\theta"] \arrow[r, phantom, bend left=30, "\rotatebox{90}{$\subseteq^-$}"] \arrow[dr, "\sigma \odot \gamma_A"'] & S \arrow[d, "\sigma"] \arrow[dl, phantom, near end, "\supseteq^-"] \\
& A^\perp \| B\,.
\end{tikzcd}$$

The next lemma is used in showing θ is an isomorphism.

Lemma 4.17. *(i) Let $z \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$. If $e \leq_z e'$ and $\pi_2(e)$ and $\pi_2(e')$ are defined, then $\pi_2(e) \leq_S \pi_2(e')$. (ii) The map π_2 is surjective on configurations.*

Proof. (i) It suffices to show when

$$e \multimap_z e_1 \multimap_z \cdots \multimap_z e_{n-1} \multimap_z e'$$

with $\pi_2(e)$ and $\pi_2(e')$ defined and all $\pi_2(e_i)$, $1 \le i \le n-1$, undefined, that $\pi_2(e) \le_S \pi_2(e')$.
Case $n = 1$, so $e \multimap_z e'$: Use Lemma 3.27. If either e or e' has the form $(*, s)$ then the other event must have the form $(*, s')$ or (c', s') with $s \multimap s'$ in S. In the remaining case $e = (c, s)$ and $e' = (c', s')$ with either (1) $c \multimap c'$ in $\mathbb{C}_A$, and $\gamma_{A2}(c) \multimap \gamma_{A2}(c')$ in A, or (2) $s \multimap s'$ in S. If (1), $\sigma_1(s) \multimap \sigma_1(s')$ in $A^\perp$ where $s, s' \in \pi_2 z$. By Proposition 3.14, $s \le_S s'$. In either case (1) or (2), $\pi_2(e) \le_S \pi_2(e')$.
Case $n > 1$: Each e_i has the form $(c_i, *)$, for $1 \le i \le n-1$. By Lemma 3.27, events e and e' must have the form (c, s) and (c', s') with $c \multimap c_1$ and $c_{n-1} \multimap c'$ in $\mathbb{C}_A$. As $\gamma_{A1}(c)$ and $\gamma_{A2}(c_1)$ are defined, $c_1 = \overline{c}$ and similarly $c_{n-1} = \overline{c'}$. Again by Lemma 3.27, $c_i \multimap c_{i+1}$ in $\mathbb{C}_A$ for $1 \le i \le i-2$. Consequently $\gamma_{A2}(c) \le_A \gamma_{A2}(c')$. Now, $s, s' \in \pi_2 z$ with $\sigma_1(s) \le_{A^\perp} \sigma_1(s')$. By Proposition 3.14, $s \le_S s'$, as required.
(ii) Let $y \in \mathcal{C}(S)$. Then $\sigma_1 y \in \mathcal{C}(A^\perp)$ and by the clear surjectivity of γ_{A2} on configurations there exists $w \in \mathcal{C}(\mathbb{C}_A)$ such that $\gamma_{A2} w = \sigma_1 y$. Now let

$$\begin{aligned} z = &\{(c, *) \mid c \in w \ \&\ \gamma_{A1}(c) \text{ is defined}\} \\ &\cup \{(c, s) \mid c \in w \ \&\ s \in y \ \&\ \gamma_{A2}(c) = \sigma_1(s)\} \\ &\cup \{(*, s) \mid s \in y \ \&\ \sigma_2(s) \text{ is defined}\}. \end{aligned}$$

Then, from the definition of the product of stable families—3.3.1, it can be checked that $z \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$. By construction, $\pi_2 z = y$. Hence π_2 is surjective on configurations. □

Theorem 4.18. *$\theta : \sigma \odot \gamma_A \cong \sigma$, an isomorphism of pre-strategies.*

Proof. We show θ is an isomorphism of event structures by showing θ is rigid and both surjective and injective on configurations (Lemma 3.3 of [7]). The rest is routine.
Rigid: It suffices to show $p \multimap p'$ in $S \odot \mathbb{C}_A$ implies $\theta(p) \le_S \theta(p')$. Suppose $p \multimap p'$ in $S \odot \mathbb{C}_A$ with $top(p) = e$ and $top(p') = e'$. Take $x \in \mathcal{C}(S \odot \mathbb{C}_A)$ containing p' so p too. Then

$$e \multimap_{\bigcup x} e_1 \multimap_{\bigcup x} \cdots \multimap_{\bigcup x} e_{n-1} \multimap_{\bigcup x} e'$$

where $e, e' \in V_0$ and $e_i \notin V_0$ for $1 \le i \le n-1$. (V_0 consists of 'visible' events of the form $(c, *)$ with $\gamma_{A1}(c)$ defined, or $(*, s)$, with $\sigma_2(s)$ defined.)
Case $n = 1$, so $e \multimap_{\bigcup x} e'$: By Lemma 3.27, either (i) $e = (*, s)$ and $e' = (*, s')$ with $s \multimap s'$ in S, or (ii) $e = (c, *)$ and $e' = (c', *)$ with $c \multimap c'$ in $\mathbb{C}_A$.
If (i), we observe, via $\sigma\theta = \sigma \odot \gamma_A$, that $s \in \pi_2 \bigcup x \subseteq \theta x$ and $\theta(p) \in \theta x$ with $\sigma(\theta(p)) = \sigma(s)$, so $\theta(p) = s$ by the local injectivity of σ. Similarly, $\theta(p') = s'$, so $\theta(p) \le_S \theta(p')$.
If (ii), we obtain $\theta(p), \theta(p') \in \theta x$ with $\sigma_1\theta(p) = \gamma_{A1}(c)$, $\sigma_1\theta(p') = \gamma_{A1}(c')$ and $\gamma_{A1}(c) \multimap \gamma_{A1}(c')$ in $A^\perp$. By Proposition 3.14, $\theta(p) \le_S \theta(p')$.

Case $n > 1$: Note $e_i = (c_i, s_i)$ for $1 \le i \le n-1$, and that $s_1 \le_S s_{n-1}$ by Lemma 4.17(i). Consider the case in which $e = (c, *)$ and $e' = (c', *)$—the other cases are similar. By Lemma 3.27, $c \multimap c_1$ and $c_{n-1} \multimap c'$ in $\mathbb{C}_A$. But $\gamma_{A1}(c)$ and $\gamma_{A2}(c_1)$ are defined, so $c_1 = \overline{c}$, and similarly $c_{n-1} = \overline{c'}$. We remark that $\theta(p) = s_1$, by the local injectivity of σ, as both $s_1 \in \pi_2 \bigcup x \subseteq \theta x$ and $\theta(p) \in \theta x$ with $\sigma(\theta(p)) = \sigma(s_1)$. Similarly $\theta(p') = s_{n-1}$, whence $\theta(p) \le_S \theta(p')$.
Surjective: Let $y \in \mathcal{C}(S)$. By Lemma 4.17(ii), there is $z \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$ such that $\pi_2 z = y$. Let

$$z' = z \cup \{(c, *) \mid pol(c) = + \ \&\ \exists s \in S.\ (\overline{c}, s) \in z\}\,.$$

It is straightforward to check $z' \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$. Now let

$$z'' = z' \setminus \{(c, *) \mid pol(c) = - \ \&\ \forall s \in S.\ (\overline{c}, s) \notin z'\}\,.$$

Then $z'' \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$ by the following argument. The set z'' is certainly consistent, so it suffices to show

$$pol(c) = - \ \&\ (c, *) \le_{z'} e \in z'' \implies \exists s \in S.\ (\overline{c}, s) \in z'\,,$$

for all $c \in \mathbb{C}_A$ and $e \in z''$. This we do by induction on the number of events between $(c, *)$ and e. Suppose

$$pol(c) = - \ \&\ (c, *) \multimap_{z'} e_1 \le_{z'} e \in z'\,.$$

In the case where $e_1 = (c_1, s_1)$, we deduce $c \multimap c_1$ in $\mathbb{C}_A$ and as $\gamma_{A1}(c)$ is defined while $\gamma_{A2}(c_1)$ is defined, we must have $c_1 = \overline{c}$, as required. In the case where $e_1 = (c_1, *)$ and $pol(c_1) = -$, by induction, we obtain $(\overline{c_1}, s_1) \in z'$ for some $s_1 \in S$. Also $c \multimap c_1$, so $\overline{c} \multimap \overline{c_1}$ in $\mathbb{C}_A$. As z' is a configuration we must have $(\overline{c}, s) \le_{z'} (\overline{c_1}, s_1)$, for some $s \in S$, so $(\overline{c}, s) \in z'$. In the case where $e_1 = (c_1, *)$ and $pol(c_1) = +$, we have $c \multimap c_1$ in $\mathbb{C}_A$. Moreover, $(\overline{c}_1, s) \in z'$, for some $s \in S$, as z' is a configuration and $\overline{c_1} \multimap c_1$ in $\mathbb{C}_A$. Again, from the fact that z' is a configuration, there must be $(\overline{c}, s) \in z'$ for some $s \in S$. We have exhausted all cases and conclude $z'' \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$ with $\theta z'' = \pi_2 z = y$, as required to show θ is surjective on configurations.
Injective: Abbreviate $\sigma \odot \gamma_A$ to υ. Assume $\theta x = \theta y$, where $x, y \in \mathcal{C}(S \odot \mathbb{C}_A)$. Via the commutativity $\upsilon = \sigma\theta$, we observe

$$\upsilon x = \sigma\theta\, x = \sigma\theta\, y = \upsilon y\,.$$

Recall by Proposition 4.2, that $\upsilon_1 x = \gamma_{A1}\pi_1 \bigcup x = \pi_1 \bigcup x$. It follows that

$$(c, *) \in \bigcup x \iff c \in \upsilon_1 x \iff c \in \upsilon_1 y \iff (c, *) \in \bigcup y\,.$$

Observe

$$(*, s) \in \bigcup x \iff \sigma_2(s) \text{ is defined } \&\ s \in \theta x\ :$$

"$\Rightarrow$" by the local injectivity of σ_2, as $p =_{\mathrm{def}} [(*, s)]_{\bigcup x}$ yields $\theta(p) \in \theta x$ and $s \in \pi_2 \bigcup x \subseteq \theta x$ with $\sigma_2(\theta(p)) = \sigma_2(s)$, so $\theta(p) = s$; "$\Leftarrow$" as $\sigma_2(s)$ defined and

$s \in \theta x$ entails $s = \theta(p)$ for some $p \in x$, necessarily with $top(p) = (*, s)$. Hence

$$\begin{aligned}(*, s) \in \bigcup x &\iff \sigma_2(s) \text{ is defined } \& \ s \in \theta x\\ &\iff \sigma_2(s) \text{ is defined } \& \ s \in \theta y\\ &\iff (*, s) \in \bigcup y\,.\end{aligned}$$

Assuming $(c, s) \in \bigcup x$ we now show $(c, s) \in \bigcup y$. (The converse holds by symmetry.) There is $p \in x$, such that $(c, s) \in p$. If $top(p) = (*, s')$ (also in $\bigcup y$ as it is visible) then as π_2 is rigid, $s \le s'$ and we must have $(c', s) \in \bigcup y$. Otherwise, $top(p) = (d, *)$ and we can suppose (by taking p minimal) that $(c, s) \le_{\bigcup x} (d', s') \rightarrowtriangle_{\bigcup x} (d, *)$. But then $\theta(p) = s' \in \theta x = \theta y$. Also $s \le_S s'$, by the rigidity of π_2, and, as we have seen before, $d' = \overline{d}$ with d' −ve. Hence s' is +ve and as θy is a −ve extension of $\pi_2 \bigcup y$ we must have $s' \in \pi_2 \bigcup y$. Hence there is $(*, s')$ or (c'', s') in $\bigcup y$, and as $s \le_S s'$ there is some $(c', s) \in \bigcup y$. In both cases, $\gamma_{A2}(c') = \overline{\sigma_1(s)} = \gamma_{A2}(c)$, so $c' = c$, and thus $(c, s) \in \bigcup y$.

We conclude $\bigcup x = \bigcup y$, so $x = y$, as required for injectivity. □

4.5 Concurrent strategies

Define a *strategy* to be a pre-strategy which is receptive and innocent. We obtain a bicategory, **Strat**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \rightarrow\!\!\!\!\!\!\rightarrow B$ and the 2-cells are maps of pre-strategies. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the functoriality of synchronized composition). The isomorphisms expressing associativity and the identity of copy-cat are those of Proposition 4.3 and Theorem 4.18 with its dual.

We remark for future use that composition of strategies respects less general notions of 2-cell. The horizontal composition of rigid 2-cells is rigid. The essential ingredients in showing this are that the product and pullback of event structures preserve rigid maps when regarded as functor (from Corollary 3.29) and that under appropriate conditions hiding as formalized through projection preserves rigid maps (Proposition 3.35).

Proposition 4.19. *Let $\sigma : S \to A$ be a strategy in A and $\sigma' : S' \to A$ a receptive total map of event structures with polarity. Let $f : S \to S'$ be a total map of event structures with polarity s.t. $\sigma' f = \sigma$. Then, f is receptive and innocent. A fortiori if f is 2-cell from strategy σ to strategy σ' in the bicategory of games and strategies, then f is receptive and innocent.*

Proof. We first show f is receptive. Assume $x \in \mathcal{C}(S)$ and $fx \subseteq^- x'$. Then $\sigma' fx \subseteq^- \sigma' x'$, *i.e.* $\sigma x \subseteq^- \sigma' x'$ in A. Hence as σ is receptive (existence part), there is $z \in \mathcal{C}(S)$ such that $\sigma z = \sigma' x'$. Now both $fx \subseteq fz$ and $fx \subseteq x'$ with $\sigma' fz = \sigma' x'$. From the receptivity of σ' (uniqueness part) we obtain $fz = x'$, as required.

It remains to show f is innocent. Suppose $s' \rightarrowtriangle s$ and $pol(s') = +$ or $pol(s) = -$ in S. We require $f(s') \rightarrowtriangle f(s)$ in S'. As σ is innocent, $\sigma(s') \rightarrowtriangle \sigma(s)$ in A. Being a map σ' locally reflects causal dependency. So given that $f(s')$ and $f(s)$ both belong to the configuration $f[s]_S$ and $\sigma'(f(s')) \rightarrowtriangle \sigma'(f(s))$ we obtain $f(s') \leq f(s)$. The depenency $f(s') \leq f(s)$ must be realised by a chain of immediate causal dependencies

$$f(s') \rightarrowtriangle \cdots \rightarrowtriangle f(s)$$

in S'. Suppose to obtain a contradiction, that the chain were of length greater than one. Then, as f is total and reflects causal dependency locally w.r.t. $[s]$, we would obtain a chain

$$s' \rightarrowtriangle \cdots \rightarrowtriangle s$$

of length greater than one in S—contradicting $s' \rightarrowtriangle s$. Consequently, $f(s') \rightarrowtriangle f(s)$, as required. $\square$

4.5.1 Alternative characterizations

Via saturation conditions

An alternative description of concurrent strategies exhibits the correspondence between innocence and earlier "saturation conditions," *reflecting* specific independence, in [8, 9, 10]:

Proposition 4.20. *A strategy in a game A exactly comprises a total map of event structures with polarity $\sigma : S \to A$ such that*
(i) $\sigma x \xrightarrow{a}\subset$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \xrightarrow{s}\subset$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$;
(ii)(+) If $x \xrightarrow{e}\subset x_1 \xrightarrow{e'}\subset$ & $pol_S(e) = +$ in $\mathcal{C}(S)$ and $\sigma x \xrightarrow{\sigma(e')}\subset$ in $\mathcal{C}(A)$, then $x \xrightarrow{e'}\subset$ in $\mathcal{C}(S)$; and
(ii)(−) If $x \xrightarrow{e}\subset x_1 \xrightarrow{e'}\subset$ & $pol_S(e') = -$ in $\mathcal{C}(S)$ and $\sigma x \xrightarrow{\sigma(e')}\subset$ in $\mathcal{C}(A)$, then $x \xrightarrow{e'}\subset$ in $\mathcal{C}(S)$.

Proof. Note that if $x \xrightarrow{e}\subset x_1 \xrightarrow{e'}\subset$ then either $e\ co\ e'$ or $e \rightarrowtriangle e'$. Condition (ii) is a contrapositive reformulation of innocence. $\square$

Via lifting conditions

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^+ x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{+\}$, *i.e.* the configuration x' extends the configuration x solely by events of +ve polarity. With this notation in place we can give an attractive characterization of concurrent strategies:

Lemma 4.21. *A strategy in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that* *(i) whenever* $y \subseteq^+ \sigma x$ *in* $\mathcal{C}(A)$ *there is a (necessarily unique)* $x' \in \mathcal{C}(S)$ *so that* $x' \subseteq x$ & $\sigma x' = y$ *, i.e.*

$$\begin{tikzcd} x' \arrow[r, phantom, dotted, "\subseteq"] \arrow[d, dotted, mapsto, "\sigma"'] & x \arrow[d, mapsto, "\sigma"] \\ y \arrow[r, phantom, "\subseteq^+"] & \sigma x\,, \end{tikzcd}$$

and

(ii) whenever $\sigma x \subseteq^- y$ *in* $\mathcal{C}(A)$ *there is a unique* $x' \in \mathcal{C}(S)$ *so that* $x \subseteq x'$ & $\sigma x' = y$ *, i.e.*

$$\begin{tikzcd} x \arrow[r, phantom, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, dotted, mapsto, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & y\,. \end{tikzcd}$$

Proof. Let $\sigma : S \to A$ be a total map of event structures with polarity. It is claimed that σ is a strategy iff (i) and (ii).

"Only if": Lemma 4.5 directly implies (ii). To establish (i) it suffices to show the seemingly weaker property (i)$'$ that

$$y \overset{a}{\relbar\!\!\subset} \sigma x \;\&\; pol(a) = + \implies \exists x' \in \mathcal{C}(S).\, x' \relbar\!\!\subset x \;\&\; \sigma x' = y$$

for $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$. Then (i), with $y \subseteq^+ \sigma x$, follows by considering a covering chain $y \relbar\!\!\subset \cdots \relbar\!\!\subset \sigma x$. (The uniqueness of x is a direct consequence of σ being a total map of event structures.) To show (i)$'$, suppose $y \overset{a}{\relbar\!\!\subset} \sigma x$ with a +ve. Then $\sigma(s) = a$ for some unique $s \in x$ with s +ve. Supposing s were not $\leq$-maximal in x, then $s \rightarrowtriangle s'$ for some $s' \in x$. By +-innocence $a = \sigma(s) \rightarrowtriangle \sigma(s') \in \sigma x$ implying a is not $\leq$-maximal in σx. This contradicts $y \overset{a}{\relbar\!\!\subset} \sigma x$. Hence s is $\leq$-maximal and $x' =_{\mathrm{def}} x \setminus \{s\} \in \mathcal{C}(S)$ with $x' \relbar\!\!\subset x$ and $\sigma x' = y$.

"If": Assume σ satisfies (i) and (ii). Clearly σ is receptive by (ii). We establish innocence via Proposition 4.20.

Suppose $x \overset{s}{\relbar\!\!\subset} x_1 \overset{s'}{\relbar\!\!\subset} x'$ and $pol(s) = +$ with $\sigma x \overset{\sigma(s')}{\relbar\!\!\subset} y_2$. Then $y_2 \overset{\sigma(s)}{\relbar\!\!\subset} \sigma x'$ with $pol(\sigma(s)) = +$. From (i) we obtain a unique $x_2 \in \mathcal{C}(S)$ such that $x_2 \subseteq x'$ and $\sigma x_2 = y_2$. As σ is a total map of event structures, we obtain $x_2 \overset{s}{\relbar\!\!\subset} x'$ and subsequently $x \overset{s'}{\relbar\!\!\subset} x_2$, as required by Proposition 4.20(ii)+.

Suppose $x \overset{s}{\relbar\!\!\subset} x_1 \overset{s'}{\relbar\!\!\subset} x'$ and $pol(s') = -$ with $\sigma x \overset{\sigma(s')}{\relbar\!\!\subset} y_2$. The case where $pol(s) = +$ is covered by the previous argument: we obtain $x \overset{s'}{\relbar\!\!\subset} x_2$, as required by Proposition 4.20(ii)−. Suppose $pol(s) = -$. We have

$$\sigma x \overset{\sigma(s')}{\relbar\!\!\subset} y_2 \overset{\sigma(s)}{\relbar\!\!\subset} \sigma x'\,.$$

As σ is already known to be receptive, we obtain

$$x \overset{e'}{\relbar\!\!\subset} x_2 \overset{e}{\relbar\!\!\subset} x'' \;\&\; \sigma x_2 = y_2 \;\&\; \sigma x'' = \sigma x'\,.$$

From the uniqueness part of (ii) we deduce $x'' = x'$. As σ is a total map of event structures, $e = s$ and $e' = s'$ ensuring $x \stackrel{s'}{-\!\!\!\subset}$, as required by Proposition 4.20(ii)−.
□

As its proof makes clear, condition (i) in Lemma 4.21 can be replaced by: for all $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$,

$$y \stackrel{+}{-\!\!\!\subset} \sigma x \implies \exists x' \in \mathcal{C}(S).\, x' -\!\!\!\subset x \ \&\ \sigma x' = y\,, \quad \textit{i.e.}$$

$$\begin{tikzcd} x' \arrow[r, phantom, "\cdots -\!\!\!\subset \cdots"] \arrow[d, dotted, "\sigma"'] & x \arrow[d, "\sigma"] \\ y \arrow[r, phantom, "\stackrel{+}{-\!\!\!\subset}"] & \sigma x\,, \end{tikzcd}$$

where the relation $\stackrel{+}{-\!\!\!\subset}$ signifies the covering relation induced by an event of +ve polarity.

The proposition above generalises to the situation in which configurations may be infinite, but first a lemma extending receptivity to possibly infinite configurations.

Lemma 4.22. *Let $\sigma : S \to A$ be receptive and −-innocent. Then,*

$$\sigma x \stackrel{a}{-\!\!\!\subset} \ \&\ pol_A(a) = - \Rightarrow \exists! s \in S.\ x \stackrel{s}{-\!\!\!\subset} \ \&\ \sigma(s) = a\,,$$

for all $x \in \mathcal{C}^\infty(S)$, $a \in A$.

Proof. Suppose $\sigma x \stackrel{a}{-\!\!\!\subset}$ and $pol_A(a) = -$. Then there is $x_0 \in \mathcal{C}(S)$ with $x_0 \subseteq x$ and $\sigma x_0 \stackrel{a}{-\!\!\!\subset}$. By receptivity, there is a unique $s \in S$ such that $x_0 \stackrel{s}{-\!\!\!\subset}$ & $\sigma(s) = a$. In fact, $x \cup \{s\} \in \mathcal{C}^\infty(S)$. Suppose otherwise. Then there is $x_1 \in \mathcal{C}(S)$ with $x_0 \subseteq x_1 \subseteq x$ for which $x_1 \cup \{s\} \notin \mathcal{C}(S)$. But $\sigma x_1 \stackrel{a}{-\!\!\!\subset}$ so there is a unique $s_1 \in S$ such that $x_1 \stackrel{s_1}{-\!\!\!\subset}$ & $\sigma(s_1) = a$. Both $[s)$ and $[s_1)$ are included in x_1 so $s = s_1$ by Lemma 4.4—a contradiction. Now that $x \cup \{s\} \in \mathcal{C}^\infty(S)$ we have $x \stackrel{s}{-\!\!\!\subset}$ and $\sigma(s) = a$. Uniqueness of s follows by Lemma 4.4: if also $x \stackrel{s'}{-\!\!\!\subset}$ and $\sigma(s') = a$ then $[s) \uparrow [s')$. □

Corollary 4.23. *A strategy in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that*
(i) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}^\infty(A)$ there is a (necessarily unique) $x' \in \mathcal{C}^\infty(S)$ so that $x' \subseteq x$ & $\sigma x' = y$, i.e.

$$\begin{tikzcd} x' \arrow[r, phantom, "\cdots \subseteq \cdots"] \arrow[d, dotted, "\sigma"'] & x \arrow[d, "\sigma"] \\ y \arrow[r, phantom, "\subseteq^+"] & \sigma x\,, \end{tikzcd}$$

and

(ii) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}^\infty(A)$ there is a unique $x' \in \mathcal{C}^\infty(S)$ so that $x \subseteq x'$ & $\sigma x' = y$, i.e.

$$\begin{tikzcd} x \arrow[r, dotted, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, dotted, mapsto, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & y\,. \end{tikzcd}$$

Proof. Let $\sigma : S \to A$ be a total map of event structures with polarity. It is claimed that σ is a strategy iff (i) and (ii). The *"If"* case is obvious by Lemma 4.21. *"Only if":*
(i) Take $x' =_{\mathrm{def}} \{s \in x \mid \sigma(s) \notin (\sigma x) \setminus y\}$. Suppose $s' \rightarrowtriangle s$ in x. Then

$$\sigma(s') \in (\sigma x) \setminus y \implies \sigma(s) \in (\sigma x) \setminus y$$

by +-innocence. Hence its contrapositive, *viz.*

$$\sigma(s) \notin (\sigma x) \setminus y \implies \sigma(s') \notin (\sigma x) \setminus y\,,$$

so that $s \in x'$ implies $s' \in x'$. Thus, being down-closed and consistent, $x' \in \mathcal{C}^\infty(S)$ with $\sigma x' = y$ from the definition of x'.
(ii) Let $x' \supseteq x$ be a $\subseteq$-maximal $x' \in \mathcal{C}^\infty(S)$ for which $\sigma x' \subseteq y$—this exists by Zorn's lemma. Then, $\sigma x \subseteq^- \sigma x' \subseteq^- y$. Supposing $\sigma x' \subsetneq^- y$ there is $a \in A$ with $pol_A(a) = -$ such that $\sigma x' \xrightarrow{a}\!\subset y_1 \subsetneq^- y$. But, by Lemma 4.22, there is $s \in S$ for which $x' \xrightarrow{s}\!\subset$ and $\sigma(s) = a$, contradicting the $\subseteq$-maximality of x'. Hence $\sigma x' = y$. Uniqueness of x' follows as in the proof of Lemma 4.5. □

Via +-moves

A strategy is determined by its +-moves. More precisely, a strategy $\sigma : S \to A$ determines a monotone function $d : \mathcal{C}(S^+) \to \mathcal{C}(A)$ given by $d(x) = \sigma[x]_S$ for $x \in \mathcal{C}(S^+)$. The event structure S^+ is the projection of S to its purely +-ve moves. Intuitively, d specifies the position in the game at which Player moves occur. The function d determines the original strategy σ via the universal property described in the proposition below.

Proposition 4.24. *Let $\sigma : S \to A$ be a receptive −-innocent pre-strategy. Define $q : S \to S^+$ be the partial map of event structures with polarity mapping S to its projection S^+ comprising only the +ve events of S, so $q\,y = y^+$ for $y \in \mathcal{C}(S)$. Define the function $d : \mathcal{C}(S^+) \to \mathcal{C}(A)$ to act as $d(x) = \sigma[x]_S$ for $x \in \mathcal{C}(S^+)$. Then, $d(qy) \subseteq^- \sigma y$ for all $y \in \mathcal{C}(S)$, i.e.*

$$\begin{tikzcd} S \arrow[r, "q"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "{}^-\!\supseteq"] & S^+ \arrow[dl, dotted, "d"] \\ A\,. & \end{tikzcd} \tag{1}$$

[The dotted line indicates that d is not a map of event structures.]
Suppose $f : U \to A$ is a total map and $g : U \to S^+$ a partial map of event structures with polarity such that $d(gy) \subseteq^- fy$ for all $y \in \mathcal{C}(U)$, i.e.

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "{}^{-}\supseteq" near start] & S^+ \arrow[dl, dotted, "d"] \\ A\,. & \end{tikzcd} \tag{2}$$

Then, there is a unique total map of event structures with polarity $\theta : U \to S$ such that $f = \sigma\theta$ and $g = q\theta$,

$$\begin{tikzcd} U \arrow[r, dashed, "\theta"] \arrow[rr, bend left, "g"] \arrow[dr, bend right, "f"'] & S \arrow[r, "q"'] \arrow[d, "\sigma"'] \arrow[dr, phantom, "{}^{-}\supseteq" near start] & S^+ \arrow[dl, dotted, "d"] \\ & A\,. & \end{tikzcd} \tag{3}$$

Proof. We first check (1). Letting $y \in \mathcal{C}(S)$,

$$d(q\,y) = d(y^+) = \sigma[y^+]_S \subseteq^- y\,.$$

Suppose (2). Define $p : \mathcal{C}(U) \to \mathcal{C}(S)$ by taking

$$p(z) =_{\rm def} [g\,z]_S\,.$$

Clearly p is monotonic and

$$\sigma p(z) = \sigma[g\,z]_S = d(gz) \subseteq^- f\,z$$

for all $z \in \mathcal{C}(U)$. By Lemma 4.16, there is a unique total map of event structures with polarity $\theta : U \to S$ such that

$$f = \sigma\theta \quad \text{and} \quad \forall z \in \mathcal{C}(U).\ p(z) \subseteq^- \theta\,z\,.$$

From the latter, $[g\,z]_S \subseteq^- \theta z$ from which $g\,z = (g\,z)^+ = (\theta\,z)^+$, so $g\,z = q\theta\,z$, for all $z \in \mathcal{C}(U)$. Hence we have the commuting diagram (3). Noting

$$\forall z \in \mathcal{C}(U).\ \ g\,z = (\theta\,z)^+ \iff [g\,z]_S \subseteq^- \theta z\,,$$

we see that θ is the unique map making (3) commute. $\square$

It follows that a strategy σ is determined up to isomorphism by its ‘position function’ d specifying at what state of the game Player moves are made. The position functions d which arise from strategies have been characterized by Alex Katovsky and GW [11].

4.6 Rigid-image strategies

It can be useful to replace a strategy by its rigid image in its game. As is to be expected something can be lost in the process. Precisely what is related to notions of equivalence between strategies. For now suffice it to say, that while 'may' behaviour is preserved, 'must' behaviour need not be. What is gained is that we can replace the bicategory of games by a category; a rigid-image strategy can be identified with its rigid image, a substructure of the game so we have canonical representatives of isomorphism classes of rigid-image strategies. Rigid images are important for equivalences on strategies. For several important behavioural equivalences, a representative of an equivallence class of strategies can be found in their sharing a common rigid image and some additional structure (probability or stopping configurations, for instance).

A strategy $\sigma : S \to A$ factors through its rigid image

$$S \xrightarrow{f} S_0 \xrightarrow{\sigma_0} A$$

where f is rigid epi (*i.e.* both rigid and surjective) and $\sigma_0 : S_0 \to A$ is itself a strategy. In a *rigid-image* strategy such as $\sigma_0 : S_0 \to A$ the rigid image S_0 is bounded to be a substructure of $aug(A)$. This provides us with a characterisation of rigid-image strategies. A rigid-image strategy in a game A is an innocent, receptive substructure S_0 of $aug(A)$ in the sense that there is a rigid inclusion $i_0 : S_0 \hookrightarrow aug(A)$ for which the composition $\epsilon_A \circ i_0$ is innocent and i_0 is receptive. In other words S_0 is a down-closed subset of $aug(A)$ which is closed under possible Opponent moves and comprises only innocent augmentations of A.

The following example shows that the composition of the rigid images of two strategies is not necessarily a rigid image, both for composition of strategies with and without hiding.

Example 4.25. Let B be the game

$$\begin{tikzcd} & \oplus & \\ \ominus & \oplus & \ominus \arrow[l] \end{tikzcd} .$$

Let C be the game consisting of a single Player move $\oplus$. Let $\sigma : S \to B$ be the strategy sending S equal to

$$\begin{tikzcd} \oplus \arrow[r, no head, squiggly] & \oplus & \\ \ominus \arrow[u] \arrow[ur] & \oplus & \ominus \arrow[l] \arrow[ul] \end{tikzcd} .$$

to B in the obvious way indicated by the layout. Let $\tau : T \to B^{\perp} \| C$ be the strategy sending T equal to

$$\begin{tikzcd} & \ominus \arrow[r] & \oplus \\ \oplus & \ominus \arrow[l] & \oplus \arrow[l] \end{tikzcd}$$

to $B^{\perp}\|C$, which we can draw as

$$\begin{tikzcd} & \ominus & \oplus \\ \oplus & \ominus & \oplus \arrow[l], \end{tikzcd}$$

in the obvious way. Their composition, before hiding, is given by $T \circledast S$:

$$\begin{tikzcd} & & \oplus \arrow[d, no head, squiggly] \\ \circledcirc \arrow[urr] & \circledcirc \arrow[r] & \oplus \\ \circledcirc \arrow[u] \arrow[ur] & \circledcirc \arrow[l] & \circledcirc \arrow[l]. \end{tikzcd}$$

Both σ and τ are rigid-image strategies yet there composition both before and after hiding is not. Before hiding the two Player moves in $T \circledast S$ over the common move in C go to a common image. After hiding $T \odot S$ looks like

$$\begin{tikzcd} \oplus \arrow[d, no head, squiggly] \\ \oplus \end{tikzcd}$$

with both moves going to the common sole move in C; while distinct they clearly go to a common event in the rigid image. $\square$

So the compositions, with and without hiding, $\tau_0 \odot \sigma_0$ and $\tau_0 \circledast \sigma_0$ of the rigid images of two strategies σ and τ is not necessarily a rigid-image strategies, we are forced to take the rigid image of the result. However once we do, the operation of forming the rigid image of a strategy respects composition, both with and without hiding: letting $\sigma : S \to A^{\perp}\|B$ and $\tau : T \to B^{\perp}\|C$ be strategies, $(\tau \odot \sigma)_0 = (\tau_0 \odot \sigma_0)_0$ and $(\tau \circledast \sigma)_0 = (\tau_0 \circledast \sigma_0)_0$.

Proposition 4.26. *Let $f : A \to B$ and $g : B \to C$ be maps of event structures. Assume that f is rigid and epi. Then, the rigid image of g equals the rigid image of $g \circ f$.*

Proof. Write the rigid image of g as $\mathrm{Im}(g)$ and the rigid image of gf as $\mathrm{Im}(gf)$. From the universal property associated with the rigid image of gf there is a unique (necessarily rigid epi) map $h : \mathrm{Im}(g) \to \mathrm{Im}(gf)$ such that

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[drr] & B \arrow[r, two heads, "g_0"] & \mathrm{Im}(g) \arrow[r, "g_1"] \arrow[d, dotted, "h"] & C \\
& & \mathrm{Im}(f)g \arrow[ur] &
\end{tikzcd}$$

commutes. Write $l =_{\mathrm{def}} hg_0$. Then l is rigid epi being the composition of such. From the universal property associated with the rigid image of g there is a unique (necessarily rigid epi) map $k : \mathrm{Im}(g)f \to \mathrm{Im}(g)$ such that

$$\begin{tikzcd}
B \arrow[r, two heads, "g_0"] \arrow[dr, "l"'] & \mathrm{Im}(g) \arrow[r, "g_1"] & C \\
& \mathrm{Im}(gf) \arrow[u, dotted, "k"] \arrow[ur] &
\end{tikzcd}$$

commutes. By uniqueness of the universal property of the rigid-image of g we obtain $kh = \mathrm{id}_{\mathrm{Im}(g)}$. By uniqueness of the universal property of the rigid-image of gf we obtain $hk = \mathrm{id}_{\mathrm{Im}(gf)}$. Hence the rigid images are isomorphic. Because they are chosen to be substructures of $aug(C)$ they are equal. $\square$

Corollary 4.27. *If two strategies are connected by a 2-cell which is rigid epi, then they share the same rigid image..*

Lemma 4.28. *Let* $\sigma : S \xrightarrow{f} S_0 \xrightarrow{\sigma_0} A^{\perp} \| B$ *and* $\tau : T \xrightarrow{g} T_0 \xrightarrow{\tau_0} B^{\perp} \| C$ *be the rigid image factorisations of strategies* $\sigma : S \to A^{\perp} \| B$ *and* $\tau : T \to B^{\perp} \| C$. *Then,*

(i) $(\tau_0 \circledast \sigma_0)_0 = (\tau \circledast \sigma)_0$ *and* *(ii)* $(\tau_0 \odot \sigma_0)_0 = (\tau \odot \sigma)_0$.

Proof. (i) Consider the following compound pullback square in which all the squares are pullbacks—we are ignoring polarites.

$$\begin{tikzcd}
& & T \circledast S \arrow[dl, "g \circledast S"'] \arrow[dr, "T \circledast f"] \arrow[dd, dotted, "g \circledast f"] & & \\
& T_0 \circledast S \arrow[dl] \arrow[dr, "T_0 \circledast f"] & & T \circledast S_0 \arrow[dl, "g \circledast S_0"'] \arrow[dr] & \\
S \| C \arrow[dr, "f \| C"'] & & T_0 \circledast S_0 \arrow[dl] \arrow[dr] \arrow[d, dotted] & & A \| T \arrow[dl, "A \| g"] \\
& S_0 \| C \arrow[dr, "\sigma_0 \| C"'] & (T_0 \circledast S_0)_0 \arrow[d, dotted, "(\tau_0 \circledast \sigma_0)_0"] & A \| T_0 \arrow[dl, "A \| \tau_0"] & \\
& & A \| B \| C & &
\end{tikzcd}$$

In the diagram we have inserted the rigid-image factorisation of the map $T_0 \circledast S_0 \to A\|B\|C$. Notice that in the uppermost square all the maps are rigid epi being the pullbacks of such maps. Consequently $g \circledast f$ is rigid epi. Now applying Corollary 4.27 we deduce that the rigid image of the map $T \circledast S$ coincides with that of $T_0 \circledast S_0$ in $A\|B\|C$ and is therefore $(T_0 \circledast S_0)_0$. This ensures that

$$(\tau_0 \circledast \sigma_0)_0 = (\tau \circledast \sigma)_0 \,.$$

(ii) We can also deduce

$$(\tau_0 \odot \sigma_0)_0 = (\tau \odot \sigma)_0 \,.$$

Recall we obtain $\tau \odot \sigma$ as the defined part of the partial map

$$T \circledast S \xrightarrow{\tau \circledast \sigma} A\|B\|C \longrightarrow A\|C$$

and similarly $\tau_0 \odot \sigma_0$ as the defined part of the partial map

$$T_0 \circledast S_0 \xrightarrow{\tau_0 \circledast \sigma_0} A\|B\|C \longrightarrow A\|C$$

—in both cases the map $A\|B\|C \to A\|C$ is that eliding B. From the diagram in (i) we see

$$\tau \circledast \sigma = (\tau_0 \circledast \sigma_0) \circ (g \circledast f) \,.$$

In the commuting diagram

$$\begin{tikzcd}
T \circledast S \arrow[r, "g \circledast f"] \arrow[d] & T_0 \circledast S_0 \arrow[d] \\
T \odot S \arrow[r, dotted, "g \odot f"] \arrow[dr, "\tau \odot \sigma"'] & T_0 \odot S_0 \arrow[d, "\tau_0 \odot \sigma_0"] \\
 & A\|C
\end{tikzcd}$$

we have filled in the total map $g \odot f$ given by the universal property of partial-total factorisation. As in (i) above $g \circledast f$ is rigid epi. It follows that the map $g \odot f$ is also rigid epi: the map $g \odot f$ preserves causal dependency because $g \circledast f$ does; it is epi because the composite map $T \circledast S \xrightarrow{g \circledast f} T_0 \circledast S_0 \longrightarrow T_0 \odot S_0$ is epi—the latter projection map is epi. Now by Corollary 4.27 we deduce that $\tau_0 \odot \sigma_0$ and $\tau \odot \sigma$ share the same rigid image in $A\|C$. Consequently $(\tau_0 \odot \sigma_0)_0 = (\tau \odot \sigma)_0$. $\square$

Let $\mathbf{Strat}_0$ be the order-enriched category of rigid-image strategies defined as follows. Its objects are games. Its maps are rigid-image strategies. Its 2-cells are rigid 2-cells between strategies which are necessarily rigid inclusions as they are between rigid images. Under composition composable strategies σ and τ are taken to $(\tau \odot \sigma)_0$. The associativity law and identity laws for composition are verfied using Lemma 4.28; recall that in a copycat strategy $\gamma_A : ℂℂ_A \to A^\perp \| A$ the underlying function of the map γ_A acts as the identity on events; this ensures that copycat strategies are rigid-image.

The operation of taking the rigid image of a strategy yields a functor from $\mathbf{Strat}_r$, the bicategory of strategies with with rigid 2-cells, to $\mathbf{Strat}_0$. From the results above composition is preserved. A rigid 2-cell $f : \sigma \Rightarrow \tau$ is sent to a rigid inclusion between their rigid images: by taking its image, any rigid 2-cell between strategies factors into a 2-cell which is a rigid epi, followed by 2-cells which is a rigid inclusion; strategies connected by a rigid epi share the same rigid image, while rigid inclusions are preserved in taking the rigid image.

A concrete, relatively elementary, presentation of rigid-image strategies and probabilistic rigid-image strategies is given in [**?**].

Chapter 5

Deterministic strategies

This chapter concentrates on the important special case of deterministic concurrent strategies and their properties. They are shown to coincide with Melliès and Mimram's *receptive ingenuous strategies*.

5.1 Definition

We say an event structure with polarity S is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ \mathit{Neg}[X] \in \mathrm{Con}_S \implies X \in \mathrm{Con}_S\,,$$

where $\mathit{Neg}[X] =_{\text{def}} \{s' \in S \mid \mathit{pol}(s') = - \ \&\ \exists s \in X.\ s' \leq s\}$. In other words, S is deterministic iff any finite set of moves is consistent when it causally depends only on a consistent set of opponent moves. Say a strategy $\sigma : S \to A$ is *deterministic* if S is deterministic.

Lemma 5.1. *An event structure with polarity S is deterministic iff*

$$\forall s, s' \in S, x \in \mathcal{C}(S).\ \ x \xrightarrow{s}\!\!\subset\ \&\ x \xrightarrow{s'}\!\!\subset\ \&\ \mathit{pol}(s) = + \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. *"Only if"*: Assume S is deterministic, $x \xrightarrow{s}\!\!\subset$, $x \xrightarrow{s'}\!\!\subset$ and $\mathit{pol}(s) = +$. Take $X =_{\text{def}} x \cup \{s, s'\}$. Then $\mathit{Neg}[X] \subseteq x \cup \{s\}$ so $\mathit{Neg}[X] \in \mathrm{Con}_S$. As S is deterministic, $X \in \mathrm{Con}_S$ and being down-closed $X = x \cup \{s, s'\} \in \mathcal{C}(S)$.
"If": Assume S satisfies the property stated above in the proposition. Let $X \subseteq_{\text{fin}} S$ with $\mathit{Neg}[X] \in \mathrm{Con}_S$. Then the down-closure $[\mathit{Neg}[X]] \in \mathcal{C}(S)$. Clearly $[\mathit{Neg}[X]] \subseteq [X]$ where all events in $[X] \setminus [\mathit{Neg}[X]]$ are necessarily +ve. Suppose, to obtain a contradiction, that $X \notin \mathrm{Con}_S$. Then there is a maximal $z \in \mathcal{C}(S)$ such that

$$[\mathit{Neg}[X]] \subseteq z \subseteq [X]$$

and some $e \in [X] \setminus z$, necessarily +ve, for which $[e) \subseteq z$. Take a covering chain

$$[e) \xrightarrow{s_1}\!\!\subset z_1 \xrightarrow{s_2}\!\!\subset \cdots \xrightarrow{s_k}\!\!\subset z_k = z\,.$$

As $[e)\overset{e}{\longrightarrow\!\!\!\!\subset}[e]$ with e +ve, by repeated use of the property of the lemma—illustrated below—we obtain $z\overset{e}{\longrightarrow\!\!\!\!\subset}z'$ in $\mathcal{C}(S)$ with $[Neg[X]] \subseteq z' \subseteq [X]$, which contradicts the maximality of z.

$$
\begin{array}{ccccccccc}
[e] & \overset{s_1}{\longrightarrow\!\!\!\!\subset} & z_1' & \overset{s_2}{\longrightarrow\!\!\!\!\subset} & \cdots & \overset{s_k}{\longrightarrow\!\!\!\!\subset} & z_k' & = & z' \\
{\scriptstyle e}\uparrow & & {\scriptstyle e}\uparrow & & \cdots & & {\scriptstyle e}\uparrow & & \\
[e) & \overset{s_1}{\longrightarrow\!\!\!\!\subset} & z_1 & \overset{s_2}{\longrightarrow\!\!\!\!\subset} & \cdots & \overset{s_k}{\longrightarrow\!\!\!\!\subset} & z_k & = & z
\end{array}
$$

□

So, above, an event structure with polarity can fail to be deterministic in two ways, either with $pol(s) = pol(s') = +$ or with $pol(s) = +$ & $pol(s') = -$. In general for an event structure with polarity A the copy-cat strategy can fail to be deterministic in either way, illustrated in the examples below.

Example 5.2. (i) Take A to consist of two +ve events and one −ve event, with any two but not all three events consistent. The construction of $\mathbb{C}_A$ is pictured:

$$
\begin{array}{cccc}
 & \ominus \rightarrow \oplus & \\
A^\perp & \ominus \rightarrow \oplus & A \\
 & \oplus \leftarrow \ominus &
\end{array}
$$

Here γ_A is not deterministic: take x to be the set of all three −ve events in $\mathbb{C}_A$ and s, s' to be the two +ve events in the A component.
(ii) Take A to consist of two events, one +ve and one −ve event, inconsistent with each other. The construction $\mathbb{C}_A$:

$$
\begin{array}{cccc}
A^\perp & \ominus \rightarrow \oplus & A \\
 & \oplus \leftarrow \ominus &
\end{array}
$$

To see $\mathbb{C}_A$ is not deterministic, take x to be the singleton set consisting *e.g.* of the −ve event on the left and s, s' to be the +ve and −ve events on the right.

5.2 The bicategory of deterministic strategies

We first characterize those games for which copy-cat is deterministic; they only allow immediate conflict between events of the same polarity; there can be no races between Player and Opponent moves.

Lemma 5.3. *Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff A satisfies*

$$\forall x \in \mathcal{C}(A).\ x\overset{a}{\longrightarrow\!\!\!\!\subset}\ \&\ x\overset{a'}{\longrightarrow\!\!\!\!\subset}\ \&\ pol(a) = +\ \&\ pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(race-free)}$$

Proof. *"Only if":* Suppose $x \in \mathcal{C}(A)$ with $x\overset{a}{\longrightarrow\!\!\!\!\subset}$ and $x\overset{a'}{\longrightarrow\!\!\!\!\subset}$ where $pol(a) = +$ and $pol(a') = -$. Construct $y =_{\mathrm{def}} \{(1, \overline{b}) \mid b \in x\} \cup \{(1, \overline{a})\} \cup \{(2, b) \mid b \in x\}$. Then

$y \in \mathcal{C}(\mathbb{C}_A)$ with $y \stackrel{(2,a)}{-\!\!\!\subset}$ and $y \stackrel{(2,a')}{-\!\!\!\subset}$, by Proposition 4.1(ii). Assuming $\mathbb{C}_A$ is deterministic, we obtain $y \cup \{(2,a),(2,a')\} \in \mathcal{C}(\mathbb{C}_A)$, so $y \cup \{(2,a),(2,a')\} \in \mathcal{C}(A^\perp \| A)$. This entails $x \cup \{a, a'\} \in \mathcal{C}(A)$, as required to show (**race-free**).
"If": Assume A satisfies (**race-free**). It suffices to show for $X \subseteq_{\text{fin}} \mathbb{C}_A$, with X down-closed, that $Neg[X] \in \text{Con}_{\mathbb{C}_A}$ implies $X \in \text{Con}_{\mathbb{C}_A}$. Recall for Z down-closed, $Z \in \text{Con}_{\mathbb{C}_A}$ iff $Z \in \text{Con}_{A^\perp \| A}$.

Let $X \subseteq_{\text{fin}} \mathbb{C}_A$ with X down-closed. Assume $Neg[X] \in \text{Con}_{\mathbb{C}_A}$. Observe

(i) $\{c \mid c \in X \ \&\ pol(c) = -\} \subseteq Neg[X]$ and

(ii) $\{\bar{c} \mid c \in X \ \&\ pol(c) = +\} \subseteq Neg[X]$ as by Proposition 4.1, X being down-closed must contain $\bar{c}$ if it contains c with $pol(c) = +$.

Consider $X_2 =_{\text{def}} \{a \mid (2,a) \in X\}$. Then X_2 is a finite down-closed subset of A. From (i),

$$X_2^- =_{\text{def}} \{a \in X_2 \mid pol(a) = -\} \in \text{Con}_A \,.$$

From (ii),

$$X_2^+ =_{\text{def}} \{a \in X_2 \mid pol(a) = +\} \in \text{Con}_A \,.$$

We show (**race-free**) implies $X_2 \in \text{Con}_A$.

Define $z^- =_{\text{def}} [X_2^-]$ and $z^+ =_{\text{def}} [X_2^+]$. Being down-closures of consistent sets, $z^-, z^+ \in \mathcal{C}(A)$. We show $z^- \uparrow z^+$ in $\mathcal{C}(A)$. First note $z^- \cap z^+ \in \mathcal{C}(A)$. If $a \in z^- \setminus z^- \cap z^+$ then $pol(a) = -$; otherwise, if $pol(a) = +$ then $a \in z^+$ a well as $a \in z^-$ making $a \in z^- \cap z^+$, a contradiction. Similarly, if $a \in z^+ \setminus z^- \cap z^+$ then $pol(a) = +$. We can form covering chains

$$z^- \cap z^+ \stackrel{p_1}{-\!\!\!\subset} x_1 \stackrel{p_2}{-\!\!\!\subset} \cdots \stackrel{p_k}{-\!\!\!\subset} x_k = z^- \quad \text{and} \quad z^- \cap z^+ \stackrel{n_1}{-\!\!\!\subset} y_1 \stackrel{n_2}{-\!\!\!\subset} \cdots \stackrel{n_l}{-\!\!\!\subset} y_l = z^+$$

where each p_i is +ve and each n_j is −ve.

Consequently, by repeated use of (**race-free**), we obtain $x_k \cup y_l \in \mathcal{C}(A)$, *i.e.* $z^+ \cup z^- \in \mathcal{C}(A)$, as is illustrated below. But $X_2 \subseteq z^+ \cup z^-$, so $X_2 \in \text{Con}_A$. A similar argument shows $X_1 =_{\text{def}} \{a \in A^\perp \mid (1,a) \in X\} \in \text{Con}_{A^\perp}$. It follows that $X \in \text{Con}_{A^\perp \| A}$, so $X \in \text{Con}_{\mathbb{C}_A}$ as required.

$$\begin{array}{ccccccccc}
y_l & \stackrel{p_1}{-\!\!\!\subset} & x_1 \cup y_l & \stackrel{p_2}{-\!\!\!\subset} & x_2 \cup y_l & \stackrel{p_3}{-\!\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\!\subset} & x_k \cup y_l \\
{\scriptstyle n_l}\uparrow & & {\scriptstyle n_l}\uparrow & & {\scriptstyle n_l}\uparrow & & & & {\scriptstyle n_l}\uparrow \\
\vdots & & \vdots & & \vdots & \cdots & \cdots & \cdots & \vdots \\
{\scriptstyle n_2}\uparrow & & {\scriptstyle n_2}\uparrow & & {\scriptstyle n_2}\uparrow & & & & {\scriptstyle n_2}\uparrow \\
y_1 & \stackrel{p_1}{-\!\!\!\subset} & x_1 \cup y_1 & \stackrel{p_2}{-\!\!\!\subset} & x_2 \cup y_1 & \stackrel{p_3}{-\!\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\!\subset} & x_k \cup y_1 \\
{\scriptstyle n_1}\uparrow & & {\scriptstyle n_1}\uparrow & & {\scriptstyle n_1}\uparrow & & & & {\scriptstyle n_1}\uparrow \\
z^- \cap z^+ & \stackrel{p_1}{-\!\!\!\subset} & x_1 & \stackrel{p_2}{-\!\!\!\subset} & x_2 & \stackrel{p_3}{-\!\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\!\subset} & x_k
\end{array}$$

$\square$

Exercise 5.4. Provide a direct proof of Lemma 5.3, *i.e.* show directly from the property of configurations x of copy-cat that $x \overset{c}{\longrightarrow\subset}$ and $x \overset{c'}{\longrightarrow\subset}$, with c having +ve polarity in copy-cat, implies $x \cup \{c, c'\}$ is a configuration of copy-cat. (Consider different cases of c, c', which component game the belong to and the polarity of c'.) □

Proposition 5.5. *Let A be an event structure with polarity. Then, A satisfies* (**race-free**) *iff*

$$\forall x, x_1, x_2 \in \mathcal{C}(A).\ x \subseteq^+ x_1\ \&\ x \subseteq^- x_2 \implies x_1 \cup x_2 \in \mathcal{C}(A)\,.$$

Proof. *"If"* is obvious. *"Only if"*: by repeated use of (**race-free**) as in the proof of Lemma 5.3. □

Via the next lemma, when games satisfy (**race-free**) we can simplify the condition for a strategy to be deterministic.

Lemma 5.6. *Let $\sigma : S \to A$ be a strategy. Suppose $x \overset{s}{\longrightarrow\subset} y$ & $x \overset{s'}{\longrightarrow\subset} y'$ & $pol_S(s) = -$. Then, $\sigma y \uparrow \sigma y'$ in $\mathcal{C}(A) \implies y \uparrow y'$ in $\mathcal{C}(S)$. A fortiori, if A satisfies* (**race-free**) *then so does S.*

Proof. Assume $\sigma y \uparrow \sigma y'$ in $\mathcal{C}(A)$, so $\sigma y' \overset{\sigma(s)}{\longrightarrow\subset} \sigma y \cup \sigma y'$ in $\mathcal{C}(A)$. As $\sigma(s)$ is −ve, by receptivity, there is a unique $s'' \in S$, necessarily −ve, such that $\sigma(s'') = \sigma(s)$ and $y' \overset{s''}{\longrightarrow\subset} x \cup \{s', s''\}$ in $\mathcal{C}(S)$. In particular, $x \cup \{s', s''\} \in \mathcal{C}(S)$. By −-innocence, we cannot have $s' \rightarrowtriangle s''$, so $x \cup \{s''\} \in \mathcal{C}(S)$. But now $x \overset{s}{\longrightarrow\subset}$ and $x \overset{s''}{\longrightarrow\subset}$ with $\sigma(s) = \sigma(s'')$ and both s, s'' −ve and hence $s'' = s$ by the uniqueness part of receptivity. We conclude that $x \cup \{s', s\} \in \mathcal{C}(S)$ so $y \uparrow y'$. □

Corollary 5.7. *Assume A satisfies* (**race-free**) *of Lemma 5.3. A strategy $\sigma : S \to A$ is deterministic iff it is* weakly-deterministic*, i.e. for all +ve events $s, s' \in S$ and configurations $x \in \mathcal{C}(S)$,*

$$x \overset{s}{\longrightarrow\subset}\ \&\ x \overset{s'}{\longrightarrow\subset} \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. *"Only if"*: clear. *"If"*: Let $x \overset{s}{\longrightarrow\subset}$ and $x \overset{s'}{\longrightarrow\subset}$ where $pol_S(s) = +$. For S to be deterministic we require $x \cup \{s, s'\} \in \mathcal{C}(S)$. The above assumption ensures this when $pol_S(s') = +$. Otherwise $pol_S(s') = -$ with $\sigma x \overset{\sigma(s)}{\longrightarrow\subset}$ and $\sigma x \overset{\sigma(s')}{\longrightarrow\subset}$. As A satisfies (**race-free**), $\sigma x \cup \sigma(s), \sigma(s') \in \mathcal{C}(A)$. Now by Lemma 5.6, $x \cup \{s, s'\} \in \mathcal{C}(S)$. □

Lemma 5.8. *The composition $\tau \odot \sigma$ of deterministic strategies σ and τ is deterministic.*

Proof. Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be deterministic strategies. The composition $T \odot S$ is constructed as $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S)) \downarrow V$, a synchronized composition of event structures S and T projected to visible events $e \in V$ where $top(e)$ has the form $(s, *)$ or $(*, t)$.

We first note a fact about the effect of internal, or "invisible," events not in V on configurations of $\mathcal{C}(T) \circledast \mathcal{C}(S)$. If

$$z \xrightarrow{(s,t)}\!\!\subset w \;\&\; z \xrightarrow{(s',t')}\!\!\subset w' \;\&\; w \not\uparrow w' \tag{1}$$

within $\mathcal{C}(T) \circledast \mathcal{C}(S)$, then either

$$\pi_1 z \xrightarrow{s}\!\!\subset \pi_1 w \;\&\; \pi_1 z \xrightarrow{s'}\!\!\subset \pi_1 w' \;\&\; \pi_1 w \not\uparrow \pi_1 w' \,, \tag{2}$$

within $\mathcal{C}(S)$, or

$$\pi_2 z \xrightarrow{t}\!\!\subset \pi_2 w \;\&\; \pi_2 z \xrightarrow{t'}\!\!\subset \pi_2 w' \;\&\; \pi_2 w \not\uparrow \pi_2 w' \,, \tag{3}$$

within $\mathcal{C}(T)$. Assume (1). If $t = t'$ then $\sigma(s) = \overline{\tau(t)} = \overline{\tau(t')} = \sigma(s')$ and we obtain (2) as σ is a map of event structures. Similarly if $s = s'$ then (3). Supposing $s \neq s'$ and $t \neq t'$ then if both (2) and (3) failed we could construct a configuration $z' =_{\mathrm{def}} z \cup \{(s,t),(s',t)\}$ of $\mathcal{C}(T) \circledast \mathcal{C}(S)$, contradicting (1); it is easy to check that z' is a configuration of the product $\mathcal{C}(S) \times \mathcal{C}(T)$ and its events are clearly within the restriction used in defining the synchronized composition.

We now show the impossibility of (2) and (3), and so (1). Assume (2) (case (3) is similar). One of s or s' being +ve would contradict S being deterministic. Suppose otherwise, that both s and s' are −ve. Then, because σ is a strategy, by Lemma 5.6, we have

$$\sigma_2 \pi_1 w \not\uparrow \sigma_2 \pi_1 w'$$

in $\mathcal{C}(B)$. Also, then both t and t' are +ve ensuring $\pi_2 w \uparrow \pi_2 w'$ in $\mathcal{C}(T)$, as T is deterministic. This entails

$$\tau_1 \pi_2 w \uparrow \tau_1 \pi_2 w'$$

in $\mathcal{C}(B^\perp)$. But $\sigma_2 \pi_1 w$ and $\tau_1 \pi_2 w$, respectively $\sigma_2 \pi_1 w'$ and $\tau_1 \pi_2 w'$, are the same configurations on the common event structure underlying B and $B^\perp$, of which we have obtained contradictory statements of compatibility.

As (1) is impossible, it follows that

$$z \xrightarrow{(s,t)}\!\!\subset w \;\&\; z \xrightarrow{(s',t')}\!\!\subset w' \implies w \uparrow w' \tag{4}$$

within $\mathcal{C}(T) \circledast \mathcal{C}(S)$.

Finally, we can show that $\tau \odot \sigma$ is deterministic. Suppose $x \xrightarrow{p}\!\!\subset y$ and $x \xrightarrow{p'}\!\!\subset y'$ in $\mathcal{C}(T \odot S)$ with $pol(p) = +$. Then,

$$\bigcup x \xrightarrow{e_1}\!\!\subset z_1 \xrightarrow{e_2}\!\!\subset \cdots \xrightarrow{e_k}\!\!\subset z_k = \bigcup y \quad \text{and} \quad \bigcup x \xrightarrow{e'_1}\!\!\subset z'_1 \xrightarrow{e'_2}\!\!\subset \cdots \xrightarrow{e'_l}\!\!\subset z'_l = \bigcup y'$$

in $\mathcal{C}(T) \circledast\ \mathcal{C}(S)$, where $e_k = top(p)$ and $e'_l = top(p')$, and the events e_i and e'_j otherwise have the form $e_i = (s_i, t_i)$, when $1 \le i < k$, and $e'_j = (s'_j, t'_j)$, when $1 \le j < l$. By repeated use of (4) we obtain $z_{k-1} \uparrow z'_{l-1}$. (The argument is like that ending the proof of Lemma 5.3, though with the minor difference that now we may have $e_i = e'_j$.) We obtain $w =_{\text{def}} z_{k-1} \cup z'_{l-1} \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ with $w \xrightarrow{e_k}\subset$ and $w \xrightarrow{e'_l}\subset$ and $pol(e_k) = +$.

Now, $w \cup \{e_k, e'_l\} \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ provided $w \cup \{e_k, e'_l\} \in \mathcal{C}(S) \times \mathcal{C}(T)$. Inspect the definition of configurations of the product of stable families in Section 3.3.1. If e_k and e'_l have the form $(s, *)$ and $(s', *)$ respectively, then determinacy of S ensures that the projection $\pi_1 w \cup \{s, s'\} \in \mathcal{C}(S)$ whence $w \cup \{e_k, e'_l\}$ meets the conditions needed to be in $\mathcal{C}(S) \times \mathcal{C}(T)$. Similarly, $w \cup \{e_k, e'_l\} \in \mathcal{C}(S) \times \mathcal{C}(T)$ if e_k and e'_l have the form $(*, t)$ and $(*, t')$. Otherwise one of e_k and e'_l has the form $(s, *)$ and the other $(*, t)$. In this case again an inspection of the definition of configurations of the product yields $w \cup \{e_k, e'_l\} \in \mathcal{C}(S) \times \mathcal{C}(T)$. Forming the set of primes of $w \cup \{e_k, e'_l\}$ in V we obtain $x \cup \{p, p'\} \in \mathcal{C}(T \odot S)$.

This establishes that $T \odot S$ is deterministic. □

We thus obtain a sub-bicategory **DGames** of **Strat**; its objects satisfy (**race-free**) of Lemma 5.3 and its maps are deterministic strategies.

5.3 A category of deterministic strategies

In fact, **DGames** is equivalent to an order-enriched category via the following lemma. It says weakly-deterministic strategies in a game A are essentially certain subfamilies of configurations $\mathcal{C}(A)$, for which we give a characterization in the case of deterministic strategies. Recall, from Corollary 5.7, a weakly-deterministic strategy $\sigma : S \to A$ is a a strategy in which for all +ve events $s, s' \in S$ and configurations $x \in \mathcal{C}(S)$,

$$x \xrightarrow{s}\subset \ \&\ x \xrightarrow{s'}\subset \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Lemma 5.9. *Let $\sigma : S \to A$ be a weakly-deterministic strategy. Then,*

$$\sigma y \subseteq \sigma x \implies y \subseteq x$$

for all $x, y \in \mathcal{C}(S)$. In particular, a weakly-deterministic strategy σ is injective on configurations, i.e., $\sigma x = \sigma y$ implies $x = y$, for all $x, y \in \mathcal{C}(S)$ (so is mono as a map of event structures).

Proof. Let $\sigma : S \to A$ be a weakly-deterministic strategy. We show

$$x \supseteq z \longrightarrow\subset y \ \&\ \sigma y \subseteq \sigma x \implies y \subseteq x\,,$$

for $x, y, z \in \mathcal{C}(S)$, by induction on $|x \setminus z|$.

Suppose $x \supseteq z \xrightarrow{e}\subset y$ and $\sigma y \subseteq \sigma x$. There are x_1 and event $e_1 \in S$ such that $z \xrightarrow{e_1}\subset x_1 \subseteq x$. If $\sigma(e_1) = \sigma(e)$ then e_1 and e have the same polarity; if −ve, $e_1 = e$ by receptivity; if +ve, $e_1 = e$ because σ is weakly-deterministic, using its local injectivity. Either way $y \subseteq x$. Suppose $\sigma(e_1) \neq \sigma(e)$. We show in all cases $y \cup \{e_1\} \subseteq x$, so $y \subseteq x$.

Case $pol(e_1) = pol(e) = +$: As σ is weakly-deterministic, e_1 and e are concurrent giving $x_1 \xrightarrow{e}\subset y \cup \{e_1\}$. By induction we obtain $y \cup \{e_1\} \subseteq x$.

Case $pol(e) = -$ or $pol(e_1) = -$: From Lemma 5.6, we deduce that e_1 and e are concurrent yielding $x_1 \overset{e}{\longrightarrow\!\!\subset} y \cup \{e_1\}$, and by induction $y \cup \{e_1\} \subseteq x$.

Another, simpler induction on $|y \setminus z|$ now yields

$$x \supseteq z \subseteq y \ \&\ \sigma y \subseteq \sigma x \implies y \subseteq x\,,$$

for $x, y, z \in \mathcal{C}(S)$, from which the result follows (taking z to be, for instance, $\varnothing$ or $x \cap y$). Injectivity of σ as a function on configurations is now obvious. $\square$

A deterministic strategy $\sigma : S \to A$ determines, as the image of the configurations $\mathcal{C}(S)$, a subfamily $F =_{\mathrm{def}} \sigma\mathcal{C}(S)$ of configurations of $\mathcal{C}(A)$, satisfying:

reachability: $\varnothing \in F$ and if $x \in F$ there is a covering chain $\varnothing \overset{a_1}{\longrightarrow\!\!\subset} x_1 \overset{a_2}{\longrightarrow\!\!\subset} \cdots \overset{a_k}{\longrightarrow\!\!\subset} x_k = x$ within F;

determinacy: If $x \overset{a}{\longrightarrow\!\!\subset}$ and $x \overset{a'}{\longrightarrow\!\!\subset}$ in F with $pol_A(a) = +$, then $x \cup \{a, a'\} \in F$;

receptivity: If $x \in F$ and $x \overset{a}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(A)$ and $pol_A(a) = -$, then $x \cup \{a\} \in F$;

+-innocence: If $x \overset{a}{\longrightarrow\!\!\subset} x_1 \overset{a'}{\longrightarrow\!\!\subset}$ & $pol_A(a) = +$ in F and $x \overset{a'}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{a'}{\longrightarrow\!\!\subset}$ in F (here receptivity implies $-$-innocence);

cube: In F,

$$\begin{tikzcd}
& x_1 \arrow[r, "e"] \arrow[dr, "b"] & y_1 \arrow[dr, "b"] & \\
x \arrow[ur, "a"] \arrow[dr, "b"'] & & y \arrow[r, "e"] & z \\
& x_2 \arrow[ur, "a"] \arrow[r, "e"'] & y_2 \arrow[ur, "a"'] &
\end{tikzcd}$$

implies

$$\begin{tikzcd}
& x_1 \arrow[r, "e"] & y_1 \arrow[dr, "b"] & \\
x \arrow[ur, "a"] \arrow[r, "e"] \arrow[dr, "b"'] & w \arrow[ur, "a"] \arrow[dr, "b"] & & z \\
& x_2 \arrow[r, "e"'] & y_2 \arrow[ur, "a"'] &
\end{tikzcd}$$

Theorem 5.10. *A subfamily $F \subseteq \mathcal{C}(A)$ satisfies the axioms above iff there is a deterministic strategy $\sigma : S \to A$ such that $F = \sigma\mathcal{C}(S)$, the image of $\mathcal{C}(S)$ under σ.*

Proof. (Sketch) It is routine to check that F, the image $\sigma\mathcal{C}(S)$ of a deterministic strategy, satisfies the axioms. Conversely, suppose a subfamily $F \subseteq \mathcal{C}(A)$ satisfies the axioms. We show F is a stable family. First note that from the axioms of determinacy and receptivity we can deduce:

$$\text{if } x \overset{a}{\longrightarrow\!\!\subset} \text{ and } x \overset{a'}{\longrightarrow\!\!\subset} \text{ in } F \text{ with } x \cup \{a, a'\} \in \mathcal{C}(A), \text{ then } x \cup \{a, a'\} \in F\,.$$

By repeated use of this property, using their reachability, if $x, y \in F$ and $x \uparrow y$ in $\mathcal{C}(A)$ then $x \cup y \in F$; the proof also yields a covering chain from x to $x \cup y$ and from y to $x \cup y$. (In particular, if $x \subseteq y$ in F, then there is a covering chain from x to y —a fact we shall use shortly.) Thus, if $x \uparrow y$ in F then $x \cup y \in F$. As also $\varnothing \in F$, we obtain Completeness, required of a stable family. Coincidence-freeness is a direct consequence of reachability. Repeated use of the cube axiom yields

Cube: In F,

$$\begin{tikzcd}
 & x_1 \arrow[r, "e", "\subset"'] \arrow[dr, dash, "\subseteq"] & y_1 \arrow[dr, dash, "\subseteq"] & \\
x_1 \cap x_2 \arrow[ur, dash, "\subseteq"] \arrow[dr, dash, "\subseteq"'] & & x_1 \cup x_2 \arrow[r, "e", "\subset"'] & y_1 \cup y_2 \\
 & x_2 \arrow[ur, dash, "\subseteq"'] \arrow[r, "e"', "\subset"] & y_2 \arrow[ur, dash, "\subseteq"'] &
\end{tikzcd}$$

implies

$$x_1 \cap x_2 \overset{e}{\longrightarrow\!\!\!\!\subset} \; .$$

We use *Cube* to show stability. Assume $v \uparrow w$ in F. Let $z \in F$ be maximal such that $z \subseteq v, w$. We show $z = v \cap w$. Suppose not. Then, forming covering chains in F,

$$z \overset{c_1}{\longrightarrow\!\!\!\!\subset} v_1 \overset{c_2}{\longrightarrow\!\!\!\!\subset} \cdots \overset{c_k}{\longrightarrow\!\!\!\!\subset} v_k = v \quad \text{and} \quad z \overset{d_1}{\longrightarrow\!\!\!\!\subset} w_1 \overset{d_2}{\longrightarrow\!\!\!\!\subset} \cdots \overset{d_l}{\longrightarrow\!\!\!\!\subset} w_l = w \,,$$

there are c_i and d_j such that $c_i = d_j$, where we may assume c_i is the earliest event to be repeated as some d_j. Write $e =_{\mathrm{def}} c_i = d_j$. Now, $v_{i-1} \cap w_{j-1} = z$. Also, being bounded above $v_{i-1} \cup w_{j-1} \in F$ and $v_i \cup w_j \in F$. We have an instance of *Cube*: take $x_1 = v_{i-1}$, $x_2 = w_{j-1}$, $y_1 = v_i$ and $y_2 = w_j$. Hence $z \overset{e}{\longrightarrow\!\!\!\!\subset}$ and $z \cup \{e\} \subseteq x, y$—contradicting the maximality of z. Therefore $z = v \cap w$, as required for stability.

Now we can form an event structure $S =_{\mathrm{def}} \mathrm{Pr}(F)$. The inclusion $F \subseteq \mathcal{C}(A)$ induces a total map $\sigma : S \to A$ for which $F = \sigma\mathcal{C}(S)$. Note that $-$-innocence (*viz.* if $x \overset{a}{\longrightarrow\!\!\!\!\subset} x_1 \overset{a'}{\longrightarrow\!\!\!\!\subset}$ & $pol_A(a') = -$ in F and $x \overset{a'}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{a'}{\longrightarrow\!\!\!\!\subset}$ in F) is a direct consequence of receptivity. That S is deterministic follows from determinacy, that σ is a strategy from the axioms of receptivity and $+$-innocence. $\square$

We can thus identify deterministic strategies from A to B with subfamilies of $\mathcal{C}(A^{\perp} \| B)$ satisfying the axioms above. Through this identification we obtain an order-enriched category of deterministic strategies (presented as subfamilies) equivalent to **DGames**; the order-enrichment is via the inclusion of subfamilies. As the proof of Theorem 5.10 above makes clear, in the characterization of those subfamilies F corresponding to deterministic families, the cube axiom can be replaced by

stability: if $v \uparrow w$ in F, then $v \cap w \in F$.

Chapter 6

Games people play

We briefly and incompletely examine special cases of nondeterministic concurrent games in the literature.

6.1 Categories for games

We remark that event structures with polarity appear to provide a rich environment in which to explore structural properties of games and strategies. There are adjunctions

$$
\begin{tikzcd}
\mathcal{PA}_r \arrow[r, hook, shift right] \arrow[d, shift right] & \mathcal{PF}_r \arrow[l, shift right, "\top"'] \arrow[r, hook, shift right] \arrow[d, shift right] & \mathcal{PE}_r \arrow[l, shift right, "\top"'] \arrow[r, hook, shift right] & \mathcal{PE}_t \arrow[l, shift right, "\top"'] \\
\mathcal{PA}_r^{\#} \arrow[u, hook, shift right, "\vdash"'] \arrow[r, hook, shift right] & \mathcal{PF}_r^{\#} \arrow[u, hook, shift right, "\vdash"'] \arrow[l, shift right, "\top"'] & &
\end{tikzcd}
$$

relating $\mathcal{PE}_t$, the category of event structures with polarity with total maps, to subcategories $\mathcal{PE}_r$, with rigid maps, $\mathcal{PF}_r$ of forest-like (or filiform) event structures with rigid maps, and $\mathcal{PA}_r$, its full subcategory where polarities alternate along a branch; in $\mathcal{PF}_r^{\#}$ and $\mathcal{PA}_r^{\#}$ distinct branches are inconsistent. We shall mainly be considering games in $\mathcal{PE}_t$. Lamarche games and those of sequential algorithms belong to $\mathcal{PA}_r$ [12]. Conway games inhabit $\mathcal{PF}_r^{\#}$, in fact a coreflective subcategory of $\mathcal{PE}_t$ as the inclusion is now full; Conway's 'sum' is obtained by applying the right adjoint to the $\|$-composition of Conway games in $\mathcal{PE}_t$. Further refinements are possible. The 'simple games' of [13, 14] belong to $\mathcal{PA}_r^{-\#}$, the coreflective subcategory of $\mathcal{PA}_r^{\#}$ comprising "polarized" games, starting with moves of Opponent. The 'tensor' of simple games is recovered by applying the right adjoint of $\mathcal{PA}_r^{-\#} \hookrightarrow \mathcal{PE}_t$ to their $\|$-composition in $\mathcal{PE}_t$. Generally, the right adjoints, got by composition, from $\mathcal{PE}_t$ to the other categories fail to conserve immediate causal dependency. Such facts led Melliès *et al.* to the insight that uses of pointers in game semantics can be an artifact of working with models of games which do not take account of the independence of moves [15, 10].

6.2 Related work—early results

6.2.1 Stable spans, profunctors and stable functions

The sub-bicategory of **Strat** where the events of games are purely +ve is equivalent to the bicategory of stable spans [7]. In this case, strategies correspond to *stable spans*:

$$
\begin{tikzcd}
 & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & \longleftrightarrow & & S^+ \arrow[dl, "\sigma_1^-"'] \arrow[dr, "\sigma_2^+"] & \\
A^\perp & & B & & A & & B\,,
\end{tikzcd}
$$

where S^+ is the projection of S to its +ve events; σ_2^+ is the restriction of σ_2 to S^+, necessarily a rigid map by innocence; σ_2^- is a *demand map* taking $x \in \mathcal{C}(S^+)$ to $\sigma_1^-(x) = \sigma_1[x]$; here $[x]$ is the down-closure of x in S. Composition of stable spans coincides with composition of their associated profunctors—see [16, 17, 3]. If we further restrict strategies to be deterministic (and, strictly, event structures to be countable) we obtain a bicategory equivalent to Berry's *dI-domains and stable functions* [3].

6.2.2 Ingenuous strategies

Via Theorem 5.10, deterministic concurrent strategies coincide with the *receptive ingenuous strategies* of Melliès and Mimram [10].

6.2.3 Closure operators

In [18], deterministic strategies are presented as closure operators. A deterministic strategy $\sigma : S \to A$ determines a closure operator φ on possibly infinite configurations $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \&\ Neg[\{s\}] \subseteq x\}\,.$$

Clearly φ preserves intersections of configurations and is continuous. The closure operator φ on $\mathcal{C}^\infty(S)$ induces a *partial* closure operator φ_p on $\mathcal{C}^\infty(A)$. This in turn determines a closure operator $\varphi_p^\top$ on $\mathcal{C}^\infty(A)^\top$, where configurations are extended with a top $\top$, *cf.* [18]: take $y \in \mathcal{C}^\infty(A)^\top$ to the least, fixed point of φ_p above y, if such exists, and $\top$ otherwise.

6.2.4 Simple games

"Simple games" [13, 14] arise when we restrict **Strat** to objects and deterministic strategies in $\mathcal{PA}_r^{-\#}$, described in Section 6.1. *Conway games* are tree-like, but where only strategies need alternate and begin with opponent moves.

Chapter 7

Strategies as profunctors

This chapter relates strategies to profunctors, a generalization of relations from sets to categories, and composition on strategies to composition of profunctors. Profunctors themselves provide a rich framework in which to generalize domain theory in a way that is arguably closer to that initiated by Dana Scott than game semantics [22, 23]. Early connections are made with bistructures.

7.1 The Scott order in games

Let A be an event structure with polarity. The $\subseteq$-order on its configurations is obtained as compositions of two more fundamental orders $(\subseteq^+ \cup \subseteq^-)^+$. For $x, y \in \mathcal{C}^\infty(A)$,

$$x \subseteq^- y \text{ iff } x \subseteq y \ \&\ pol_A(y \setminus x) \subseteq \{-\}\,, \text{ and}$$
$$x \subseteq^+ y \text{ iff } x \subseteq y \ \&\ pol_A(y \setminus x) \subseteq \{+\}\,.$$

We use $\supseteq^-$ as the converse order to $\subseteq^-$. Define a new order, the *Scott* order, between configurations $x, y \in \mathcal{C}^\infty(A)$, by

$$x \sqsubseteq_A y \iff \exists z \in \mathcal{C}^\infty(A).\ x \supseteq^- z \subseteq^+ y\,.$$

It is an easy exercise to show that when such a z exists it is necessarily $x \cap y$.

Proposition 7.1. *Let A be an event structure with polarity.*
(i) If $x \subseteq^+ w \supseteq^- y$ in $\mathcal{C}^\infty(A)$, then $x \supseteq^- x \cap y \subseteq^+ y$ in $\mathcal{C}^\infty(A)$.
(ii) $(\mathcal{C}^\infty(A), \sqsubseteq_A)$ is a partial order.

Proof. (i) Assume $x \subseteq^+ w \supseteq^- y$ in $\mathcal{C}^\infty(A)$. Clearly $x \supseteq x \cap y$. Suppose $a \in x$ and $pol_A(a) = +$. Then $a \in w$, and because only $-$ve events are lost from w in $w \supseteq^- y$ we obtain $a \in y$, so $a \in x \cap y$. It follows that $x \supseteq^- x \cap y$, as required. Similarly, $x \cap y \subseteq^+ y$. Summed up diagrammatically:

$$\begin{tikzcd}
\cdot \arrow[r, dotted, dash, "\supseteq^-"] & \cdot \\
\cdot \arrow[u, dotted, dash, "\subseteq^+"] &
\end{tikzcd}
\implies
\begin{tikzcd}
\cdot \arrow[r, dash, "\supseteq^-"] & \cdot \\
\cdot \arrow[u, dash, "\subseteq^+"] \arrow[r, dotted, dash, "\supseteq^-"'] & \cdot \arrow[u, dotted, dash, "\subseteq^+"']
\end{tikzcd}$$

(ii) Clearly $\sqsubseteq$ is reflexive. Supposing $x \sqsubseteq y$, *i.e.* $x \supseteq^- z \subseteq^+ y$ in $\mathcal{C}^\infty(A)$ we see that the +ve events of x are included in y, and the −ve events of y are included in x. Hence if $x \sqsubseteq y$ and $y \sqsubseteq x$ in $\mathcal{C}^\infty(A)$ then x and y have the same +ve and −ve events and so are equal. Transitivity follows from (i):

$$\begin{tikzcd}
 & & z \\
 & y \arrow[r, dotted, dash, "\supseteq^-"] & \cdot \arrow[u, dotted, dash, "\subseteq^+"] \\
x \arrow[r, dotted, dash, "\supseteq^-"] & \cdot \arrow[u, dotted, dash, "\subseteq^+"] &
\end{tikzcd}
\quad \text{entails} \quad
\begin{tikzcd}
 & & & z \\
 & & y \arrow[r, phantom, "\supseteq^-"] & \cdot \arrow[u, dotted, dash, "\subseteq^+"] \\
 & x \arrow[r, phantom, "\supseteq^-"] & \cdot \arrow[u, phantom, "\subseteq^+"] \arrow[r, dotted, dash, "\supseteq^-"] & \cdot \arrow[u, dotted, dash, "\subseteq^+"]
\end{tikzcd}$$

□

An alternative proof of part (ii) of the proposition above, that $\sqsubseteq_A$ is a partial order, follows directly from the following proposition. (When x is a subset of events of an event structure with polarity, we use x^- and x^+ for its subset of events of the indicated polarity.)

Proposition 7.2. *Let A be an event structure with polarity. For $x, y \in \mathcal{C}^\infty(A)$,*

$$x \sqsubseteq_A y \iff y^- \subseteq x^- \ \&\ x^+ \subseteq y^+ .$$

Proof. We have

$$x \sqsubseteq_A y \iff x \supseteq^- x \cap y \subseteq^+ y .$$

But

$$x \supseteq^- x \cap y \iff x^+ \subseteq y^+$$

—argue contrapositively—and similarly

$$x \cap y \subseteq^+ y \iff y^- \subseteq x^- ,$$

whence the result. □

Proposition 7.3. $(\mathcal{C}^\infty(A), \sqsubseteq_A)$ *is a complete partial order: any ω-chain*

$$x_0 \sqsubseteq_A x_1 \sqsubseteq_A \cdots \sqsubseteq_A x_n \sqsubseteq_A \cdots$$

has a least upper bound

$$\bigsqcup_{n \in \omega} x_n = (\bigcap_{n \in \omega} x_n)^- \cup (\bigcup_{n \in \omega} x_n)^+ .$$

Proof. Consider an ω-chain

$$x_0 \sqsubseteq_A x_1 \sqsubseteq_A \cdots \sqsubseteq_A x_n \sqsubseteq_A \cdots .$$

From the definition of $\sqsubseteq_A$ we deduce

$$x_0^- \supseteq x_1^- \supseteq \cdots \supseteq x_n^- \supseteq \cdots \quad \text{and} \quad x_0^+ \subseteq x_1^+ \subseteq \cdots \subseteq x_n^+ \subseteq \cdots .$$

We first check that $\bigsqcup_{n\in\omega} x_n =_{\mathrm{def}} (\bigcap_{n\in\omega} x_n)^- \cup (\bigcup_{n\in\omega} x_n)^+$ is a configuration of A. Firstly, it is consistent: let $X \subseteq_{\mathrm{fin}} \bigsqcup_{n\in\omega} x_n$; then $X^- \subseteq \bigcap_{n\in\omega} x_n$ so $X^- \subseteq x_n$ for all $n \in \omega$, and $X^+ \subseteq \bigcup_{n\in\omega} x_n$ so, being finite, $X^+ \subseteq x_m$ for some $m \in \omega$; whence $X \subseteq x_m$ ensuring $X \in \mathrm{Con}_A$. Secondly, it is down-closed, so a configuration. Suppose $a' \leq_A a \in \bigsqcup_{n\in\omega} x_n$. If a is −ve, then $a \in \bigcap_{n\in\omega} x_n$ so $a \in x_n$ whence $a' \in x_n$, for all $n \in \omega$; it follows that whatever the polarity of a', we have $a' \in \bigsqcup_{n\in\omega} x_n$. If a is +ve, then $a \in \bigcup_{n\in\omega} x_n$ so $a \in x_n$ for all $n \geq m$, for some $m \in \omega$. As $a' \leq_A a$ we have $a' \in x_n$ for all $n \geq m$. If a' is +ve, clearly $a' \in (\bigcup_{n\in\omega} x_n)^+ \subseteq \bigsqcup_{n\in\omega} x_n$. If a' is −ve, we also have $a' \in a_n$ for all $n \leq m$, ensuring $a' \in (\bigcap_{n\in\omega} x_n)^- \subseteq \bigsqcup_{n\in\omega} x_n$.

Firstly, $\bigsqcup_{n\in\omega} x_n$ is an upper bound: $x_m \sqsubseteq_A \bigsqcup_{n\in\omega} x_n$, for any $m \in \omega$. Consider the configuration

$$x_m \cap \bigsqcup_{n\in\omega} x_n = (\bigcap_{n\in\omega} x_n)^- \cup x_m^+ ,$$

where the equality follows from the definition of $\bigsqcup_{n\in\omega} x_n$. Clearly

$$x_m \supseteq^- (\bigcap_{n\in\omega} x_n)^- \cup x_m^+ \ \text{ and } \ (\bigcap_{n\in\omega} x_n)^- \cup x_m^+ \subseteq^+ (\bigcap_{n\in\omega} x_n)^- \cup (\bigcup_{n\in\omega} x_n)^+ = \bigsqcup_{n\in\omega} x_n ,$$

from which $x_m \sqsubseteq_A \bigsqcup_{n\in\omega} x_n$.

To show $\bigsqcup_{n\in\omega} x_n$ is a least upper bound, suppose for $y \in \mathcal{C}^\infty(A)$ that $x_n \sqsubseteq_A y$ for all $n \in \omega$, *i.e.*,

$$x_n \supseteq^- x_n \cap y \subseteq^+ y ,$$

for all $n \in \omega$. Then,

$$\bigcup_{n\in\omega} x_n \supseteq^- \bigcup_{n\in\omega} x_n \cap y ,$$

so

$$(\bigcup_{n\in\omega} x_n)^+ = (\bigcup_{n\in\omega} x_n \cap y)^+ .$$

Hence

$$\bigsqcup_{n\in\omega} x_n = (\bigcup_{n\in\omega} x_n)^+ \cup (\bigcap_{n\in\omega} x_n)^- \ \supseteq^- \ (\bigcup_{n\in\omega} x_n \cap y)^+ \cup (\bigcap_{n\in\omega} x_n \cap y)^- = \bigsqcup_{n\in\omega} x_n \cap y .$$

Also,

$$\bigcap_{n\in\omega} x_n \cap y \subseteq^+ y ,$$

so

$$(\bigcap_{n\in\omega} x_n \cap y)^- = y^- ,$$

which yields

$$\bigsqcup_{n\in\omega} x_n \cap y = (\bigcup_{n\in\omega} x_n \cap y)^+ \cup (\bigcap_{n\in\omega} x_n \cap y)^- \ \subseteq^+ \ y .$$

We have obtained

$$\bigsqcup_{n\in\omega} x_n \supseteq^- \bigsqcup_{n\in\omega} x_n \cap y \ \subseteq^+ y ,$$

i.e., $\bigsqcup_{n\in\omega} x_n \sqsubseteq_A y$, as required. □

It is a tempting thought that the Scott order $(\mathcal{C}^\infty(A), \sqsubseteq_A)$ should be a Scott domain when A is countable (though a Scott domain without necessarily a bottom element). For this we would need $(\mathcal{C}^\infty(A), \sqsubseteq_A)$ to be ω-algebraic. This is not the case. Consider A comprising ω parallel copies $\oplus_n \rightarrowtriangle \ominus_n$. Let x be the configuration consisting of all its events. If $y \sqsubseteq_A x$ then $y = x$. To see this observe that $y \sqsubseteq_A x$ implies $y^- \supseteq x^-$ which by the downclosure of y implies $y = x$. If $(\mathcal{C}^\infty(A), \sqsubseteq_A)$ were to be algebraic x would be the directed union of isolated (finite) elements $\sqsubseteq_A$-below it; this could only be so were x isolated. Similarly any downclosed subset of x would be isolated, However there would then be uncountatbly many isolated elements of $(\mathcal{C}^\infty(A), \sqsubseteq_A)$, contradicting ω-algebraicity.

7.2 Strategies as presheaves

Let A be an event structure with polarity. We shall show how strategies in A correspond to certain fibrations, so presheaves, over the order $(\mathcal{C}(A), \sqsubseteq_A)$. We concentrate on discrete fibrations over partial orders.

Definition 7.4. A *discrete fibration* over a partial order $(Y, \sqsubseteq_Y)$ is a partial order $(X, \sqsubseteq_X)$ and an order-preserving function $f : X \to Y$ such that

$$\forall x \in X, y' \in Y.\ y' \sqsubseteq_Y f(x) \implies \exists! x' \sqsubseteq_X x.\ f(x') = y' .$$

Via the Scott order we can recast strategies $\sigma : S \to A$ as those discrete fibrations $F : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)$ which preserve $\varnothing$, $\supseteq^-$ and $\subseteq^+$ in the sense that $F(\varnothing) = \varnothing$ while $x \supseteq^- y$ implies $F(x) \supseteq^- F(y)$, and $x \subseteq^+ y$ implies $F(x) \subseteq^+ F(y)$, for $x, y \in \mathcal{C}(S)$:

Theorem 7.5. *(i) Let $\sigma : S \to A$ be a strategy in game A. The map $\sigma``$ taking a finite configuration $x \in \mathcal{C}(S)$ to $\sigma x \in \mathcal{C}(A)$ is a discrete fibration from $(\mathcal{C}(S), \sqsubseteq_S)$ to $(\mathcal{C}(A), \sqsubseteq_A)$ which preserves $\varnothing$, $\supseteq^-$ and $\subseteq^+$.*
(ii) Suppose $F : (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)$ is a discrete fibration which preserves $\varnothing$, $\supseteq^-$ and $\subseteq^+$. There is a unique strategy $\sigma : S \to A$ such that $F = \sigma``$.

Proof. (i) That $\sigma``$ forms a discrete fibration is a direct corollary of Lemma 4.21. As a map of event structures with polarity, $\sigma``$ automatically preserves $\varnothing$, $\supseteq^-$ and $\subseteq^+$. (ii) Assume F is a discrete fibration preserving $\varnothing$, $\supseteq^-$ and $\subseteq^+$. First observe a consequence, that if $x \subseteq^+ x'$ in $\mathcal{C}(S)$ and $F(x) \subseteq^+ y'' \subseteq F(x')$ in $\mathcal{C}(A)$, then there is a unique $x'' \in \mathcal{C}(S)$ such that $x \subseteq^+ x'' \subseteq x'$ and $F(x'') = y''$. (An analogous observation holds with $+$ replaced by $-$.) Suppose now $x \overset{+}{\longrightarrow\!\!\subset} x'$ in $\mathcal{C}(S)$—where we write $x \overset{+}{\longrightarrow\!\!\subset} x'$ to abbreviate $x \overset{s}{\longrightarrow\!\!\subset} x'$ for some +ve $s \in S$. As F preserves $\subseteq^+$, $F(x) \subseteq^+ F(x')$. The observation implies $F(x) \overset{+}{\longrightarrow\!\!\subset} F(x')$ in $\mathcal{C}(A)$. Similarly, $x \overset{-}{\longrightarrow\!\!\subset} x'$ implies $F(x) \overset{-}{\longrightarrow\!\!\subset} F(x')$.

Define the relation $\approx$ between prime intervals $[x, x']$, where $x \longrightarrow\!\!\subset x'$, as the least equivalence relation such that $[x, x'] \approx [y, y']$ if $x \longrightarrow\!\!\subset y$ and $x' \longrightarrow\!\!\subset y'$ with $y \neq x'$. For configurations of an event structure, $[x, x'] \approx [y, y']$ iff $x \overset{e}{\longrightarrow\!\!\subset} x'$ and $y \overset{e}{\longrightarrow\!\!\subset} y'$ for some common event e. As F preserves coverings it preserves $\approx$. Consequently we obtain a well-defined function $\sigma : S \to A$ by taking s to a if an instance $x \stackrel{s}{\longrightarrow\!\!\!\!\subset} x'$ is sent to $F(x) \stackrel{a}{\longrightarrow\!\!\!\!\subset} F(x')$. Clearly σ preserves polarities.

By induction on the length of covering chains $\varnothing \stackrel{s_1}{\longrightarrow\!\!\!\!\subset} x_1 \stackrel{s_2}{\longrightarrow\!\!\!\!\subset} \cdots \stackrel{s_n}{\longrightarrow\!\!\!\!\subset} x_n = x$ and the fact that F preserves $\varnothing$ and coverings, $\varnothing = F(\varnothing) \stackrel{\sigma(s_1)}{\longrightarrow\!\!\!\!\subset} F(x_1) \stackrel{\sigma(s_2)}{\longrightarrow\!\!\!\!\subset} \cdots \stackrel{\sigma(s_n)}{\longrightarrow\!\!\!\!\subset} F(x_n) = F(x)$ with $\sigma x = F(x) \in \mathcal{C}(A)$. Moreover we cannot have $\sigma(s_i) = \sigma(s_j)$ for distinct i, j without contradicting F preserving coverings. This establishes $\sigma : S \to A$ as a total map of event structures with polarity. The assumed properties of F directly ensure that σ satisfies the two conditions of Lemma 4.21 required of strategy. $\square$

As discrete fibrations correspond to presheaves, Theorem 7.5 entails that strategies $\sigma : S \to A$ correspond to (certain) presheaves over $(\mathcal{C}(A), \sqsubseteq_A)$—the presheaf for σ is a functor $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \to \mathbf{Set}$ sending y to the fibre $\{x \in \mathcal{C}(S) \mid \sigma x = y\}$.

7.3 Strategies as profunctors

A strategy

$$\sigma : A \mathrel{-\!\!\!+\!\!\!\twoheadrightarrow} B$$

determines a discrete fibration over

$$(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}) .$$

But

$$\begin{aligned} (\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}) &\cong (\mathcal{C}(A^\perp), \sqsubseteq_{A^\perp}) \times (\mathcal{C}(B), \sqsubseteq_B) && (1) \\ &\cong (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B) . && (2) \end{aligned}$$

The first step (1) relies on the correspondence

$$x \leftrightarrow (\{a \mid (1, a) \in x\},\ \{b \mid (2, b) \in x\})$$

between a configuration of $A^\perp \| B$ and a pair, with left component a configuration of $A^\perp$ and right component a configuration of B. In the last step (2) we are using the correspondence between configurations of $A^\perp$ and A induced by the correspondence $a \leftrightarrow \overline{a}$ between their events: a configuration x of $A^\perp$ corresponds to a configuration $\overline{x} =_{\mathrm{def}} \{\overline{a} \mid a \in x\}$ of A. Because $A^\perp$ reverses the roles of $+$ and $-$ in A, the order $x \sqsubseteq_{A^\perp} y$ in $\mathcal{C}(A^\perp)$,

$$\begin{tikzcd} & y \\ x \arrow[ur, dotted, no head, "\sqsubseteq"] \arrow[r, dotted, no head, "\supseteq^-"'] & x \cap y \arrow[u, dotted, no head, "\subseteq^+"'] \end{tikzcd} ,$$

corresponds to the order $\overline{y} \sqsubseteq_A \overline{x}$, *i.e.* $\overline{x} \sqsubseteq_A^{\mathrm{op}} \overline{y}$, in $\mathcal{C}(A)$,

$$\begin{tikzcd}
 & \overline{y} \arrow[d, dotted, no head, "\supseteq^-"] \\
\overline{x} \arrow[ur, dotted, no head, "\sqsubseteq"] \arrow[r, dotted, no head, "\supseteq^+"] & \overline{x} \cap \overline{y}\,.
\end{tikzcd}$$

It follows that a strategy

$$\sigma : S \to A^{\perp} \| B$$

determines a discrete fibration

$$\sigma``: (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$$

where

$$\sigma``(x) = (\overline{\sigma_1 x},\ \sigma_2 x)\,,$$

for $x \in \mathcal{C}(S)$. The fibration can be vewed as a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$—it assigns the set

$$\{x \in \mathcal{C}(S) \mid \overline{\sigma_1 x} = v \ \&\ \sigma_2 x = z\}$$

to the pair $(v, z) \in \mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(B)$. One way to define a *profunctor* from $(\mathcal{C}(A), \sqsubseteq_A)$ to $(\mathcal{C}(B), \sqsubseteq_B)$ is as a discrete fibration over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$. Hence the strategy σ determines a profunctor[1]

$$\sigma``: (\mathcal{C}(A), \sqsubseteq_A) \mathrel{\longrightarrow\mkern-14mu\shortmid\mkern8mu} (\mathcal{C}(B), \sqsubseteq_B)\,.$$

7.4 Composition of strategies and profunctors

The operation from strategies σ to profunctors $\sigma``$ preserves identities:

Lemma 7.6. *Let A be an event structure with polarity. For $x \in \mathcal{C}^{\infty}(A^{\perp} \| A)$,*

$$x \in \mathcal{C}^{\infty}(\mathbb{C}_A) \quad \textit{iff} \quad x_2 \sqsubseteq_A \overline{x}_1\,,$$

where $x_1 = \{a \in A^{\perp} \mid (1, a) \in x\}$ and $x_2 = \{a \in A \mid (2, a) \in x\}$.

Proof. Let $x \in \mathcal{C}^{\infty}(A^{\perp} \| A)$. From the dependency within copy-cat of the +ve events $a \in A$ on corresponding −ve events $\overline{a} \in A^{\perp}$, and *vice versa*, as expressed in Proposition 4.1, we deduce: $x \in \mathcal{C}^{\infty}(CC_A)$ iff

$$(i)\ \ \overline{x}_1^{+} \supseteq x_2^{+} \quad \text{and} \quad (ii)\ \ \overline{x}_1^{-} \subseteq x_2^{-}\,,$$

[1] Most often a profunctor from $(\mathcal{C}(A), \sqsubseteq_A)$ to $(\mathcal{C}(B), \sqsubseteq_B)$ is defined as a functor $(\mathcal{C}(A), \sqsubseteq_A) \times (\mathcal{C}(B), \sqsubseteq_B)^{\mathrm{op}} \to \mathbf{Set}$, *i.e.*, as a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$, and as such corresponds to a discrete fibration.

where $z^+ = \{a \in z \mid pol_A(a) = +\}$ and $z^- = \{a \in z \mid pol_A(a) = -\}$ for $z \in \mathcal{C}^\infty(A)$.

****THIS REPEATS PROP7.2**** It remains to argue that (i) and (ii) iff $x_2 \supseteq^- \overline{x}_1 \cap x_2 \subseteq^+ \overline{x}_1$. *"Only if":* Assume (i) and (ii). Clearly, $\overline{x}_1 \cap x_2 \subseteq \overline{x}_1$. Suppose $a \in \overline{x}_1$ with $pol_A(a) = -$. By (ii), $a \in x_2$. Consequently, $x_1 \cap x_2 \subseteq^+ \overline{x}_1$. Similarly, (i) entails $x_2 \supseteq^- \overline{x}_1 \cap x_2$. *"If":* To show (i), let $a \in x_2^+$. Then as $x_2 \supseteq^- \overline{x}_1 \cap x_2$ ensures only −ve events are lost in moving from x_2 to $\overline{x}_1 \cap x_2$, we see $a \in \overline{x}_1 \cap x_2$, so $a \in \overline{x}_1^+$. The proof of (ii) is similar. □

Corollary 7.7. *Let A be an event structure with polarity. The profunctor $\gamma_A``$ of the copy-cat strategy γ_A is an identity profunctor on $(\mathcal{C}(A), \sqsubseteq_A)$.*

Proof. The profunctor $\gamma_A`` : (\mathcal{C}(A), \sqsubseteq_A) \twoheadrightarrow (\mathcal{C}(A), \sqsubseteq_A)$ sends $x \in \mathcal{C}(\mathbb{C}_A)$ to $(\overline{x}_1, x_2) \in (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(A), \sqsubseteq_A)$ precisely when $x_2 \sqsubseteq_A \overline{x}_1$. It is thus an identity on $(\mathcal{C}(A), \sqsubseteq_A)$. □

We now relate the composition of strategies to the standard composition of profunctors. Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be strategies, so $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$. Abbreviating, for instance, $(\mathcal{C}(A), \sqsubseteq_A)$ to $\mathcal{C}(A)$, strategies σ and τ give rise to profunctors $\sigma`` : \mathcal{C}(A) \twoheadrightarrow \mathcal{C}(B)$ and $\tau`` : \mathcal{C}(B) \twoheadrightarrow \mathcal{C}(C)$. Their composition is the profunctor $\tau`` \circ \sigma`` : \mathcal{C}(A) \twoheadrightarrow \mathcal{C}(C)$ built as a discrete fibration from the discrete fibrations $\sigma`` : \mathcal{C}(S) \to \mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(B)$ and $\tau`` : \mathcal{C}(T) \to \mathcal{C}(B)^{\mathrm{op}} \times \mathcal{C}(C)$.

First, we define the set of *matching pairs*,

$$M =_{\mathrm{def}} \{(x,y) \in \mathcal{C}(S) \times \mathcal{C}(T) \mid \sigma_2 x = \overline{\tau_1 y}\},$$

on which we define ~ as the least equivalence relation for which

$$\begin{aligned}(x,y) \sim (x',y') \ \text{ if } \quad & x \sqsubseteq_S x' \ \& \ y' \sqsubseteq_T y \ \& \\ & \sigma_1 x = \sigma_1 x' \ \& \ \tau_2 y' = \tau_2 y .\end{aligned}$$

Define an order on equivalence classes $M/\sim$ by:

$$\begin{aligned} m \sqsubseteq m' \ \text{ iff } \ & m = \{(x,y)\}_\sim \ \& \ m' = \{(x',y')\}_\sim \ \& \\ & x \sqsubseteq_S x' \ \& \ y \sqsubseteq_T y' \ \& \\ & \sigma_2 x = \sigma_2 x' \ \& \ \tau_1 y = \tau_1 y' ,\end{aligned}$$

for some matching pairs $(x,y), (x',y')$—so then $\sigma_2 x = \sigma_2 x' = \overline{\tau_1 y} = \overline{\tau_1 y'}$.

Exercise 7.8. *Show that $\sqsubseteq$ above is transitive, so a partial order on $M/\sim$. Verify that $\tau`` \circ \sigma``$ is a discrete fibration.* □

Lemma 7.9. *On matching pairs, define*

$$(x,y) \sim_1 (x',y') \ \textit{ iff } \ \exists s \in S, t \in T.\ x \overset{s}{\longrightarrow\!\!\!\subset} x' \ \& \ y \overset{t}{\longrightarrow\!\!\!\subset} y' \ \& \ \sigma_2(s) = \overline{\tau_1(t)} .$$

The smallest equivalence relation including $\sim_1$ coincides with the relation $\sim$.

Proof. From their definitions, $\sim_1$ is included in $\sim$. To prove the converse, it suffices to show that matching pairs (x,y), (x',y') satisfying

$$x \sqsubseteq_S x' \ \&\ y' \sqsubseteq_T y\ \&$$
$$\sigma_1 x = \sigma_1 x' \ \&\ \tau_2 y' = \tau_2 y\,,$$

—the clause used in the definition $\sim$ —are in the equivalence relation generated by $\sim_1$. Take a covering chain

$$x {-\!\!\sqsubset}_S x_1 {-\!\!\sqsubset}_S \cdots x_m {-\!\!\sqsubset}_S x'$$

in $(\mathcal{C}(S), \sqsubseteq_S)$. Here ${-\!\!\sqsubset}_S$ is the covering relation w.r.t. the order $\sqsubseteq_s$, so $x {-\!\!\sqsubset}_S x_1$ means x, x_1 are distinct and $x \sqsubseteq_S x_1$ with nothing strictly in between. Via the map σ we obtain

$$\sigma_2 x {-\!\!\sqsubset}_B \sigma_2 x_1 {-\!\!\sqsubset}_B \cdots \sigma_2 x_m {-\!\!\sqsubset}_B \sigma_2 x'$$

in $\mathcal{C}(B)$ where $\sigma_2 x = \overline{\tau_1 y}$ and $\sigma_2 x' = \overline{\tau_1 y'}$. Via the discrete fibration $\tau``$ we obtain a covering chain in the reverse direction,

$$y {\sqsupset\!\!-}_T y_1 {\sqsupset\!\!-}_T \cdots y_m {\sqsupset\!\!-}_T y'$$

in $(\mathcal{C}(T), \sqsubseteq_T)$, where each each (x_i, y_i), for $1 \le i \le m$, is a matching pair. Moreover, $(x_i, y_i) \sim_1 (x_{i+1}, y_{i+1})$ at each i with $1 \le i \le m$. Hence (x,y) and (x',y') are in the equivalence relation generated by $\sim_1$. □

The profunctor composition $\tau`` \circ \sigma``$ is given as the discrete fibration

$$\tau`` \circ \sigma`` : M/\sim \ \to \ \mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(C)$$

acting so

$$\{(x,y)\}_\sim \mapsto (\overline{\sigma_1 x}, \tau_2 y)\,.$$

It is *not* the case that $(\tau \odot \sigma)``$ and $\tau`` \circ \sigma``$ coincide up to isomorphism. The profunctor composition $\tau`` \circ \sigma``$ will generally contain extra equivalence classes $\{(x,y)\}_\sim$ for matching pairs (x,y) which are "unreachable." Although $\sigma_2 x = z = \overline{\tau_1 y}$ automatically for a matching pair (x,y), the configurations x and y may impose incompatible causal dependencies on their interface z so never be realized as a configuration in the synchronized composition $\mathcal{C}(T) \circledast \mathcal{C}(S)$, used in building the composition of strategies $\tau \odot \sigma$.

Example 7.10. Let A and C both be the empty event structure $\varnothing$. Let B be the event structure consisting of the two concurrent events b_1, assumed $-$ve, and b_2, assumed $+$ve in B . Let the strategy $\sigma : \varnothing \longrightarrow\!\!\!\!\!\!\!\!\!\!\;+\;\,\, B$ comprise the event structure $s_1 \rightarrowtriangle s_2$ with s_1 $-$ve and s_2 $+$ve, $\sigma(s_1) = b_1$ and $\sigma(s_2) = b_2$. In $B^\perp$ the polarities are reversed so there is a strategy $\tau : B \longrightarrow\!\!\!\!\!\!\!\!\!\!\;+\;\,\, \varnothing$ comprising the event structure $t_2 \rightarrowtriangle t_1$ with t_2 $-$ve and t_1 $+$ve yet with $\tau(t_1) = \overline{b}_1$ and $\tau(t_2) = \overline{b}_2$. The equivalence class $\{(x,y)\}_\sim$, where $x = \{s_1, s_2\}$ and $y = \{t_1, t_2\}$, would be present in the profunctor composition $\tau`` \circ \sigma``$ whereas $\tau \odot \sigma$ would be the empty strategy and accordingly the profunctor $(\tau \odot \sigma)``$ only has a single element, $\varnothing$.

Definition 7.11. For (x, y) a matching pair, define

$$\begin{aligned} x \cdot y =_{\text{def}} & \{(s, *) \mid s \in x \ \& \ \sigma_1(s) \text{ is defined}\} \cup \\ & \{(*, t) \mid t \in y \ \& \ \tau_2(t) \text{ is defined}\} \cup \\ & \{(s, t) \mid s \in x \ \& \ t \in y \ \& \ \sigma_2(s) = \overline{\tau_1(t)}\} \end{aligned}$$

Say (x, y) is *reachable* if $x \cdot y \in \mathcal{C}(T) \circledast \mathcal{C}(S)$, and *unreachable* otherwise.

For $z \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ say a *visible prime* of z is a prime of the form $[(s, *)]_z$, for $(s, *) \in z$, or $[(*, t)]_z$, for $(*, t) \in z$.

Lemma 7.12. *(i) If (x, y) is a reachable matching pair and $(x, y) \sim (x', y')$, then (x', y') is a reachable matching pair;*
(ii) For reachable matching pairs (x, y), (x', y'), $(x, y) \sim (x', y')$ iff $x \cdot y$ and $x' \cdot y'$ have the same visible primes.

Proof. We use the characterization of $\sim$ in terms of the single-step relation $\sim_1$ given in Lemma 7.9.
(i) Suppose $(x, y) \sim_1 (x', y')$ or $(x', y') \sim_1 (x, y)$. By inspection of the construction of the product of stable families in Section 3.3.1, if $x \cdot y \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ then $x' \cdot y' \in \mathcal{C}(T) \circledast \mathcal{C}(S)$.

(ii) *"If":* Suppose $x \cdot y$ and $x' \cdot y'$ have the same visible primes, forming the set Q. Then $z =_{\text{def}} \bigcup Q \in \mathcal{C}(T) \circledast \mathcal{C}(S)$, being the union of a compatible set of configurations in $\mathcal{C}(T) \circledast \mathcal{C}(S)$. Moreover, $z \subseteq x \cdot y, x' \cdot y'$. Take a covering chain

$$z \xrightarrow{e_1}\!\!\subset \cdots z_i \xrightarrow{e_i}\!\!\subset z_{i+1} \xrightarrow{e_n}\!\!\subset x \cdot y$$

in $\mathcal{C}(T) \circledast \mathcal{C}(S)$. Each $(\pi_1 z_i, \pi_2 z_i)$ is a matching pair, from the definition of $\mathcal{C}(T) \circledast \mathcal{C}(S)$. Necessarily, $e_i = (s_i, t_i)$ for some $s_i \in S$, $t_i \in T$, with $\sigma_2(s_i) = \overline{\tau_1(t_i)}$, again by the definition of $\mathcal{C}(T) \circledast \mathcal{C}(S)$. Thus

$$(\pi_1 z_i, \pi_2 z_i) \sim_1 (\pi_1 z_{i+1}, \pi_2 z_{i+1}) .$$

Hence $(\pi_1 z, \pi_2 z) \sim (x, y)$, and similarly $(\pi_1 z, \pi_2 z) \sim (x', y')$, so $(x, y) \sim (x', y')$.

"Only if": It suffices to observe that if $(x, y) \sim_1 (x', y')$, then $x \cdot y$ and $x' \cdot y'$ have the same visible primes. But if $(x, y) \sim_1 (x', y')$ then $x \cdot y \xrightarrow{(s,t)}\!\!\subset x' \cdot y'$, for some $s \in S, t \in T$, and no visible prime in $x' \cdot y'$ contains (s, t). $\square$

Lemma 7.13. *Let $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$ be strategies. Defining*

$$\varphi_{\sigma,\tau} : \mathcal{C}(T \odot S) \to M/\sim \quad \text{by} \quad \varphi_{\sigma,\tau}(z) = \{(\Pi_1 z, \Pi_2 z)\}_\sim ,$$

where $\Pi_1 z = \pi_1 \bigcup z$ and $\Pi_2 z = \pi_2 \bigcup z$, yields an injective, order-preserving function from $(\mathcal{C}(T \odot S), \sqsubseteq_{T \odot S})$ to $(M/\sim, \sqsubseteq)$—its range is precisely the equivalence *classes* $\{(x,y)\}_\sim$ *for reachable matching pairs* (x,y). *The diagram*

$$
\begin{tikzcd}
(\mathcal{C}(T\odot S), \sqsubseteq_{T\odot S}) \arrow[r, "\varphi_{\sigma,\tau}"] \arrow[d, "(\tau\odot\sigma)``"'] & (M/\sim, \sqsubseteq) \arrow[dl, "\tau``\circ\sigma``"] \\
(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(C), \sqsubseteq_C) &
\end{tikzcd}
$$

commutes.

Proof. For $z \in \mathcal{C}(T\odot S)$, we obtain that $\varphi_{\sigma,\tau}(z) = (\Pi_1 z, \Pi_2 z) = (\pi_1 \bigcup z, \pi_2 \bigcup z)$ is a matching pair, from the definition of $\mathcal{C}(T) \circledast \mathcal{C}(S)$; it is clearly reachable as $\pi_1 \bigcup z \cdot \pi_2 \bigcup z = \bigcup z \in \mathcal{C}(T) \circledast \mathcal{C}(S)$. For any reachable matching pair (x,y) let z be the set of visible primes of $x \cdot y$. Then, $z \in \mathcal{C}(T\odot S)$ and, by Lemma 7.12(ii), $(\Pi_1 z, \Pi_2 z) \sim (x,y)$ so $\varphi_{\sigma,\tau}(z) = \{(x,y)\}_\sim$. Injectivity of $\varphi_{\sigma,\tau}$ follows directly from Lemma 7.12(ii).

To show that $\varphi_{\sigma,\tau}$ is order-preserving it suffices to show if $z \multimapdotboth z'$ in $(\mathcal{C}(T\odot S), \sqsubseteq)$ then $\varphi_{\sigma,\tau}(z) \sqsubseteq \varphi_{\sigma,\tau}(z')$ in $(M/\sim, \sqsubseteq)$. (The covering relation $\relbar\!\!\sqsubset$ is the same as that used in the proof of Lemma 7.9.) If $z \relbar\!\!\sqsubset z'$ then either $z \overset{p}{\relbar\!\!\sqsubset} z'$, with p +ve, or $z' \overset{p}{\relbar\!\!\sqsubset} z$, with p −ve, for p a visible prime of $\mathcal{C}(T) \circledast \mathcal{C}(S)$, *i.e.* with $top(p)$ of the form $(s,*)$ or $(*,t)$. We concentrate on the case where p is +ve (the proof when p is −ve is similar). In the case where p is +ve,

$$\Pi_1 z \cdot \Pi_2 z = \bigcup z \subseteq \bigcup z' = \Pi_1 z' \cdot \Pi_2 z'$$

in $\mathcal{C}(T) \circledast \mathcal{C}(S)$ and there is a covering chain

$$\bigcup z = w_0 \overset{(s_1,t_1)}{\relbar\!\!\sqsubset} w_1 \cdots \overset{(s_n,t_n)}{\relbar\!\!\sqsubset} w_n \overset{top(p)}{\relbar\!\!\sqsubset} \bigcup z'$$

in $\mathcal{C}(T) \circledast \mathcal{C}(S)$. Each w_i, for $0 \le i \le m$, is associated with a reachable matching pair $(\pi_1 w_i, \pi_2 w_i)$ where $\pi_1 w_i \cdot \pi_2 w_i = w_i$. Also $(\pi_1 w_i, \pi_2 w_i) \sim_1 (\pi_1 w_{i+1}, \pi_2 w_{i+1})$, for $0 \le i < m$. Hence $(\Pi_1 z, \Pi_2 z) \sim (\pi_1 w_n, \pi_2 w_n)$, by Lemma 7.9(ii). If $top(p) = (s,*)$ then $\pi_1 w_n \overset{s}{\relbar\!\!\sqsubset} \Pi_1 z'$, with s +ve, and $\pi_2 w_n = \Pi_2 z'$. If $top(p) = (*,t)$ then $\pi_1 w_n = \Pi_1 z'$ and $\pi_2 w_n \overset{t}{\relbar\!\!\sqsubset} \Pi_2 z'$, with t +ve. In either case $\pi_1 w_n \sqsubseteq_S \Pi_1 z'$ and $\pi_2 w_n \sqsubseteq_T \Pi_2 z'$ with $\sigma_2 \pi_1 w_n = \sigma_2 \Pi_1 z'$ and $\tau_1 \pi_2 w_n = \tau_1 \Pi_2 z'$. Hence, from the definition of $\sqsubseteq$ on $M/\sim$,

$$\varphi_{\sigma,\tau}(z) = \{(\Pi_1 z, \Pi_2 z)\}_\sim = \{(\pi_1 w_n, \pi_2 w_n)\}_\sim \sqsubseteq \{(\Pi_1 z', \Pi_2 z')\}_\sim = \varphi_{\sigma,\tau}(z').$$

It remains to show commutativity of the diagram. Let $z \in \mathcal{C}(T\odot S)$. Then,

$$(\tau`` \circ \sigma``)(\varphi_{\sigma,\tau}(z)) = (\tau`` \circ \sigma``)(\{(\Pi_1 z, \Pi_2 z)\}_\sim) = (\overline{\sigma_1 \Pi_1 z},\ \tau_2 \Pi_2 z) = (\tau\odot\sigma)``(z),$$

via the definition of $\tau\odot\sigma$—as required. ∎

Because $(-)``$ does not preserve composition up to isomorphism but only up to the transformation $\varphi$ of Lemma 7.13, $(-)``$ forms a *lax* functor from the bicategory of strategies to that of profunctors.

7.5 Games as factorization systems

The results of Section 7.1 show an event structure with polarity determines a factorization system; the 'left' maps are given by $\supseteq^-$ and the 'right' maps by $\subseteq^+$. More specifically they form an instance of a *rooted* factorization system $(\mathbb{X}, \to_L, \to_R, 0)$ where maps $f : x \to_L x'$ are the 'left' maps and $g : x \to_R x'$ the 'right' maps of a factorization system on a small category $\mathbb{X}$, with distinguished object 0, such that any object x of $\mathbb{X}$ is reachable by a chain of maps:

$$0 \leftarrow_L \cdot \to_R \cdots \leftarrow_L \cdot \to_R x\,;$$

and two 'confluence' conditions hold:

$$x_1 \to_R x \ \&\ x_2 \to_R x \implies \exists x_0.\ x_0 \to_R x_1 \ \&\ x_0 \to_R x_2\,, \quad \text{and its dual}$$
$$x \to_L x_1 \ \&\ x \to_L x_2 \implies \exists x_0.\ x_1 \to_L x_0 \ \&\ x_2 \to_R x_0\,.$$

Think of objects of $\mathbb{X}$ as configurations, the R-maps as standing for (compound) Player moves and L-maps for the reverse, or undoing, of (compound) Opponent moves in a game.

The characterization of strategy, Lemma 4.21, exhibits a strategy as a discrete fibration w.r.t. $\sqsubseteq$ whose functor preserves $\supseteq^-$ and $\subseteq^+$. This generalizes. Define a strategy in a rooted factorization system to be a functor from another rooted factorization system preserving L-maps, R-maps, 0 and forming a discrete fibration. To obtain strategies *between* rooted factorization systems we again follow the methodology of Joyal [6], and take a strategy from $\mathbb{X}$ to $\mathbb{Y}$ to be a strategy in the dual of $\mathbb{X}$ in parallel composition with $\mathbb{Y}$. Now the dual operation becomes the opposite construction on a factorization system, reversing the roles and directions of the 'left' and 'right' maps. The parallel composition of factorization systems is given by their product. Composition of strategies is given essentially as that of profunctors, but restricting to reachable elements. The confluence conditions are used here.

I thought at first that this work meant that bistructures, a way to present Berry's bidomains as factorization systems [24], inherited a reading as games. But unfortunately bistructures don't satisfy the confluence conditions above.

Chapter 8

A language for strategies

8.0.1 Affine maps

Notation 8.1. Let A be an event structure with polarity. Let $x \in \mathcal{C}^\infty(A)$. Write A/x for the event structure with polarity which remains after playing x. Precisely, ...

We extend the notation to configurations regarding them as elementary event structures. If $y \in \mathcal{C}^\infty(A)$ with $x \subseteq y$ then by y/x we mean the configuration $y \setminus x \in \mathcal{C}^\infty(A/x)$. In the case of a singleton configuration $\{a\}$ of A—when a is an *initial* event of A—we'll often write A/a and x/a instead of $A/\{a\}$ and $x/\{a\}$.

An *affine* map of event structures f from A to B comprises a pair (f_0, f_1) where $f_0 \in \mathcal{C}(B)$ and f_1 is a map of event structures $f_1 : A \to B/f_0$. It determines a function from $\mathcal{C}(A)$ to $\mathcal{C}(B)$ given by

$$f x = f_0 \cup f_1 x$$

for $x \in \mathcal{C}(A)$. The allied f_0 and f_1 can be recovered from the action of f on configurations: $f_0 = f\varnothing$ and f_1 is that unique map of event structures $f_1 : A \to B/f\varnothing$ which on configurations $x \in \mathcal{C}(A)$ returns $fx/f\varnothing$. It is simplest to describe the composition gf of affine maps $f = (f_0, f_1)$ from A to B and $g = (g_0, g_1)$ from B to C in terms of its action on configurations: the composition takes a configuration $x \in \mathcal{C}(A)$ to $g(f\,x)$. Alternatively, the composition gf can be described as comprising $(g_0 \cup g_1 f_0, h)$ where h is that unique map of event structures $h : A \to C/(g_0 \cup g_1 f_0)$ which sends $x \in \mathcal{C}(A)$ to $g_1(f_0 \cup f_1 x)/g_1 f_0$.

An *affine* map $f : A \to_a B$ of event structures with polarity is an affine map $f = (f_0, f_1)$ between the underlying event structures of which the allied map $f_1 : A \to B/f\varnothing$ of event structures preserves polarities.

8.1 A metalanguage for strategies

8.1.1 Types

Types are event structures with polarity $A, B, C, \cdots$ understood as games. We have type operations corresponding to the operations on games of forming the dual $A^\perp$, simple parallel composition $A \| B$, sum $\Sigma_{i \in I} A_i$ and, although largely ignored for the moment, recursively-defined types.

One way to relate types is through the affine maps between them. There will be operations for shifting between types related by affine maps (described by configuration expressions). These will enable us *e.g.* to pullback or 'relabel' a strategy across an affine map.

A *type environment* is a finite partial function from variables to types, for convenience written typically as $\Gamma \equiv x_1 : A_1, \cdots, x_m : A_m$, in which the (configuration) variables $x_1, \cdots, x_m$ are distinct. It denotes a (simple) parallel composition $\|_{x_i} A_i$ in which the set of events comprises the disjoint union $\bigcup_{1 \le i \le m} \{x_i\} \times A_i$. In describing the semantics we shall sometimes write Γ for the parallel composition it denotes.

8.1.2 Configuration expressions

Configuration expressions denote finite configurations of event structures. A typing judgement for a configuration expression p in a type environment Γ

$$\Gamma \vdash p : B$$

denotes an affine map of event structures with polarity from Γ to B.

In particular, the judgement

$$\Gamma, x : A \vdash x : A$$

denotes the partial map of event structures projecting to the single component A. The special case

$$x : A \vdash x : A$$

denotes the identity map.

We shall allow configuration expressions to be built from affine maps $f = (f_o, f_1) : A \to_a B$ in

$$\Gamma, x : A \vdash fx : B$$

and its equivalent

$$\Gamma, x : A \vdash f_0 \cup f_1 x : B \,.$$

In particular, f_1 may be completely undefined, allowing **configuration expressions** to be built from constant configurations, as *e.g.* in the judgement for the empty configuration

$$\Gamma \vdash \varnothing : A$$

or a singleton configuration

$$\Gamma \vdash \{a\} : A$$

when a is an initial event of A. In particular, the expression $\{a\} \cup x'$ associated with the judgement

$$\Gamma, x' : A/a \vdash \{a\} \cup x' : A\,,$$

where a is an initial event of A, is used later in the transition semantics.

For a sum $\Sigma_{i \in I} A_i$ there are configuration-expressions jp where $j \in J$ and p is a configuration-expression of type A_j:

$$\frac{\Gamma \vdash p : A_j}{\Gamma \vdash jp : \Sigma_{i \in I} A_i} \quad j \in I$$

In the rule for simple parallel composition we exploit the fact that configurations of simple parallel compositions are simple parallel compositions of configurations of the components:

$$\frac{\Gamma \vdash p : A \qquad \Delta \vdash q : B}{\Gamma, \Delta \vdash (p, q) : A \| B}$$

Configurations of $B^\perp$ can be taken to be the same as configurations of B, so another sound rule is

$$\frac{\Gamma \vdash p : B}{\Gamma^\perp \vdash p : B^\perp}$$

where $\Gamma^\perp$ is $x_1 : A_1^\perp, \cdots, x_m : A_m^\perp$.

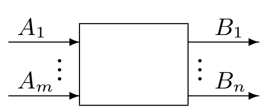
8.1.3 Terms for strategies

A language for both strategies is presented. Its terms denoting strategies are associated with typing judgements:

$$x_1 : A_1, \cdots, x_m : A_m \vdash \; t \; \dashv y_1 : B_1, \cdots, y_n : B_n \;,$$

where all the variables are distinct, interpreted as a strategy from the game $x_1 : A_1, \cdots, x_m : A_m$ denotes to the game $y_1 : B_1, \cdots, y_n : B_n$ denotes.

We can think of the term t as a box with input and output wires for the typed variables:



The duality of input and output is caught by the rules:

$$\frac{\Gamma, x : A \vdash t \dashv \Delta}{\Gamma \vdash t \dashv x : A^\perp, \Delta} \qquad \frac{\Gamma \vdash t \dashv x : A, \Delta}{\Gamma, x : A^\perp \vdash t \dashv \Delta}$$

Composition of strategies is described in the rule

$$\frac{\Gamma \vdash t \dashv \Delta \qquad \Delta \vdash u \dashv \mathrm{H}}{\Gamma \vdash \exists \Delta . [\, t \parallel u \,] \dashv \mathrm{H}}$$

which, in the picture of partial strategies as boxes, joins the input wires of one partial strategy to output wires of the other. The composition denotes the usual composition of strategies, in the case of strategies, and that described above, composition without hiding, in the case of partial strategies. Note that the simple parallel composition of strategies arises as a special case when Δ is empty. Via the alternative derivation

$$\dfrac{\dfrac{\mathrm{H}^{\perp} \vdash u \dashv \Delta^{\perp} \qquad \Delta^{\perp} \vdash t \dashv \Gamma^{\perp}}{\mathrm{H}^{\perp} \vdash \exists \Delta^{\perp}.[\, u \parallel t \,] \dashv \Gamma^{\perp}}}{\Gamma \vdash \exists \Delta^{\perp}.[\, u \parallel t \,] \dashv \mathrm{H}\,,}$$

we see an equivalent way to express the composition of strategies.

We can form the nondeterministic sum of strategies of the same type:

$$\dfrac{\Gamma \vdash t_i \dashv \Delta \quad i \in I}{\Gamma \vdash [\,]_{i \in I}\, t_i \dashv \Delta}$$

We shall use $\perp$ for the empty nondeterministic sum, when the rule above specialises to

$$\Gamma \vdash \perp \dashv \Delta\,.$$

The term $\perp$ denotes the minimum strategy in the game $\Gamma^{\perp} \| \Delta$—it essentially comprises the initial segment of the game $\Gamma^{\perp} \| \Delta$ consisting of all the initial $-$ve events of A.

We can also form the pullback of two strategies of the same type:

$$\dfrac{\Gamma \vdash t_1 \dashv \Delta \quad \Gamma \vdash t_2 \dashv \Delta}{\Gamma \vdash t_1 \wedge t_2 \dashv \Delta}$$

In the case where t_1 and t_2 denote the respective strategies $\sigma_1 : S_1 \to \Gamma^{\perp} \| \Delta$ and $\sigma_1 : S_1 \to \Gamma^{\perp} \| \Delta$ the strategy $t_1 \wedge t_2$ denotes the pullback

$$\begin{tikzcd}
 & S_1 \wedge S_2 \arrow[dl] \arrow[dr] \arrow[dd, dotted, "\sigma_1 \wedge \sigma_2"] & \\
S_1 \arrow[dr, "\sigma_1"'] & & S_2 \arrow[dl, "\sigma_2"] \\
 & \Gamma^{\perp} \| \Delta\,. &
\end{tikzcd}$$

Proposition 14.42 shows that pullbacks of strategies against maps of event structures are pullbacks.

Write $\varnothing_\Delta$ for the environment assigning the empty configuration $\varnothing$ to all configuration variables in a type environment Δ. If $\Delta \vdash p : C$, write $p[\varnothing_\Delta]$ for the configuration expression resulting from the substitution of $\varnothing$ for each variable in a configuration expression p. Later, we shall often write $p[\varnothing]$ for the substitution of the empty configuration $\varnothing$ for all configuration variables appearing in p. The hom-set rule

$$\frac{\Gamma \vdash p' : C \qquad \Delta \vdash p : C}{\Gamma \vdash p \sqsubseteq_C p' \dashv \Delta} \quad p[\varnothing_\Delta] \sqsubseteq_C p'[\varnothing_\Gamma]$$

introduces a term standing for the hom-set $(\mathcal{C}(C), \sqsubseteq_C)(p, p')$. It relies on configuration expressions p, p' and their typings. If $\Delta \vdash p : C$ denotes the affine map $g = (g_0, g_1)$ and $\Gamma \vdash p' : C$ the affine map $f = (f_0, f_1)$, the side condition of the rule ensures that $g_0 \sqsubseteq_C f_0$. Copy-cat is seen as a special case of the hom-set rule:

$$x : A \vdash y \sqsubseteq_A x \dashv y : A$$

W.r.t. affine maps $f = (f_0, f_1) : A \to_a C$ and $g = (g_0, g_1) : B \to_a C$, the judgement

$$x : A \vdash gy \sqsubseteq_C fx \dashv y : B$$

should be equivalent to the judgement

$$x : A \vdash \exists z : C/(f_0 \cap g_0).\,[\, gy \sqsubseteq_C (f_0 \cap g_0) \cup z \parallel (f_0 \cap g_0) \cup z \sqsubseteq_C fx \,] \dashv y : B$$

in the sense that the strategies they describe should be isomorphic (check!).

The Scott order $\sqsubseteq_{C^\perp}$ in $C^\perp$, the dual of a game C, is the opposite of the Scott order $\sqsubseteq_C$ of A. Correspondingly,

$$\frac{\Gamma \vdash p \sqsubseteq_C p' \dashv \Delta}{\Gamma \vdash p' \sqsubseteq_{C^\perp} p \dashv \Delta} \qquad \text{and} \qquad \frac{\Gamma \vdash p' \sqsubseteq_{C^\perp} p \dashv \Delta}{\Gamma \vdash p \sqsubseteq_C p' \dashv \Delta}\,.$$

In showing equivalences between strategies one needs basic facts about the Scott order. For example, assuming $z \subseteq x, y$ in $\mathcal{C}(A)$, we have

$$y \sqsubseteq_A x \quad \text{iff} \quad y/z \sqsubseteq_{A/z} x/z\,.$$

The precise definition of the strategy which the hom-set rule yields is given in the next section.

Example 8.2. The denotation of

$$x : A \vdash \varnothing \sqsubseteq_A \varnothing \dashv y : B$$

is the strategy in the game $A^\perp \| B$ given by the identity map $\mathrm{id}_{A^\perp \| B} : A^\perp \| B \to A^\perp \| B$. The denotation of

$$\vdash y \sqsubseteq_A \varnothing \dashv y : A$$

is $\bot_A$, the minimum strategy in the game A comprising just the initial −ve events of A.

The judgement

$$x : A_j \vdash y \sqsubseteq_{\Sigma_{i \in I} A_i} jx \dashv y : \Sigma_{i \in I} A_i$$

denotes the injection strategy—its application to a strategy in A_j fills out the strategy according to the demands of receptivity to a strategy in $\Sigma_{i\in I} A_i$. Its converse

$$x : \Sigma_{i\in I} A_i \vdash jy \sqsubseteq_{\Sigma_{i\in I} A_i} x \dashv y : A_j$$

applied to a strategy of $\Sigma_{i\in I} A_i$ projects, or restricts, the strategy to a strategy in A_j.

Assume $\vdash t \dashv y : B$. When $f : A \to B$ is a map of event structures with polarity, I believe that the composition

$$\vdash\ \exists y : B.\,[\, t \parallel fx \sqsubseteq_B y \,]\ \dashv x : A$$

denotes the pullback $f^* \sigma$ of the strategy σ denoted by t across the map $f : A \to B$.

In the case where a map of event structures with polarity $f : A \to B$ is innocent, I expect that the composition

$$\vdash\ \exists x : A.\,[\, y \sqsubseteq_B fx \parallel t \,]\ \dashv y : B$$

denotes the 'relabelling' $f_! \sigma$ of the strategy σ denoted by t. (Check!) $\square$

Via the hom-set rule we obtain

$$x : A, y : B \vdash\ z \sqsubseteq_{A\|B} (x, y)\ \dashv z : A\|B\ ,$$

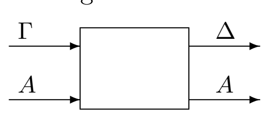
which joins two inputs to a common output. A great deal is achieved through basic manipulation of the input and output "wiring" afforded by the hom-set rules and input-output duality. For instance, the following achieves the effect of lambda abstraction:

$$\dfrac{\dfrac{\Gamma, x : A \vdash t \dashv y : B}{\Gamma \vdash t \dashv x : A^\perp, y : B} \qquad \dfrac{\dfrac{}{x : A^\perp, y : B \vdash (x, y) : A^\perp\|B} \qquad \dfrac{}{z : A^\perp\|B \vdash z : A^\perp\|B}}{x : A^\perp, y : B \vdash\ z \sqsubseteq_{A^\perp\|B} (x, y)\ \dashv z : A^\perp\|B}}{\Gamma \vdash\ \exists x : A^\perp, y : B.\,[\, t \parallel z \sqsubseteq_{A^\perp\|B} (x, y))\,]\ \dashv z : A^\perp\|B}$$

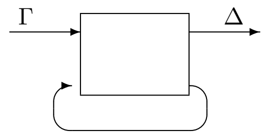
A trace, or feedback, operation is another effect of such "wiring'.' Given a strategy $\Gamma, x : A \vdash t \dashv y : A, \Delta$, we can derive

$$\dfrac{\dfrac{\dfrac{\dfrac{x : A^\perp \vdash y \sqsubseteq_{A^\perp} x \dashv y : A^\perp}{x : A^\perp \vdash x \sqsubseteq_A y \dashv y : A^\perp}}{\vdash x \sqsubseteq_A y \dashv x : A, y : A^\perp} \qquad \dfrac{\Gamma, x : A \vdash t \dashv y : A, \Delta}{x : A, y : A^\perp \vdash t \dashv \Gamma^\perp, \Delta}}{\vdash \exists x : A, y : A^\perp.\,[\, x \sqsubseteq_A y \parallel t \,] \dashv \Gamma^\perp, \Delta}}{\Gamma \vdash \exists x : A, y : A^\perp.\,[\, x \sqsubseteq_A y \parallel t \,] \dashv \Delta}$$

which denotes the *trace* of t. Its effect is to adjoin a feedback loop from $y : A$ to $x : A$. If t is represented by the diagram



then the diagram



represents its trace. The final judgement of the derivation may also be written

$$\Gamma \vdash \exists x : A^{\perp}, y : A.\, [\, t \parallel x \sqsubseteq_A y \,] \dashv \Delta$$

standing for the post-composition of

$$\Gamma, \Delta \vdash t \dashv x : A^{\perp}, y : A$$

with the term

$$x : A^{\perp}, y : A \vdash x \sqsubseteq_A y \dashv$$

denoting the copy-cat strategy $\gamma_{A^{\perp}}$. The composition introduces causal links from the +ve events of $y : A$ to the −ve events of $x : A$, and from the +ve events of $x : A$ to the −ve events of $y : A$—these are the usual links of copy-cat $\gamma_{A^{\perp}}$ as seen from the left of the turnstyle.

Duplication terms

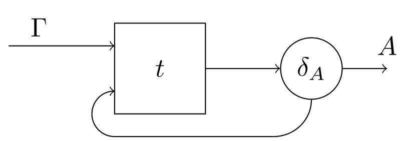
$$\frac{\Gamma \vdash p : C \quad \Delta_1 \vdash q_1 : C \quad \Delta_2 \vdash q_2 : C}{\Gamma \vdash \delta_C(p, q_1, q_2) \dashv \Delta_1, \Delta_2} \quad p[\varnothing_\Gamma], q_1[\varnothing_{\Delta_1}], q_2[\varnothing_{\Delta_2}] \text{ is balanced}\,,$$

where what it means for a triple of configurations $p[\varnothing_\Gamma], q_1[\varnothing_{\Delta_1}], q_2[\varnothing_{\Delta_2}]$ to be *balanced* is defined in Section 8.2.2. (The meaning of a triple of configurations x, y_1, y_2 of C being balanced is almost $y_1 \cup y_2 \sqsubseteq_C x$ but can't be this in general as $y_1 \cup y_2$ need not itself be a configuration of C.) The term for the duplication strategy is, in particular,

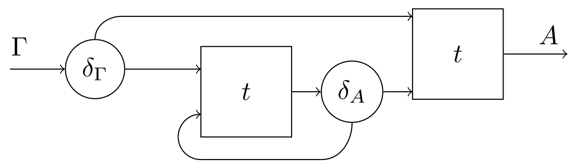
$$x : A \vdash \delta_A(x, y_1, y_2) \dashv y_1 : A, y_2 : A\,.$$

Their semantics rests on the strategy $\delta_A : A \rightarrow\!\!\!\!\!+\!\!\!> A \| A$ defined in Section 8.2.2. The operation δ_A forms a comonoid with counit $\perp : A \rightarrow\!\!\!\!\!+\!\!\!> \varnothing$.

Recursive definitions can be achieved from trace with the help of duplication terms, based on a strategy δ_A from a game A to $A \| A$, roughly, got by joining two copy-cat strategies together:



Provided the body t of the recursion respects δ_A the diagram above unfolds in the way expected of recursion, to:



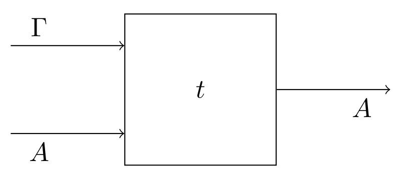
For those strategies which respect δ, *i.e.*

$$\delta_A \odot \sigma \;\cong\; (\sigma \| \sigma) \odot \delta_{\Gamma \| A}\,,$$

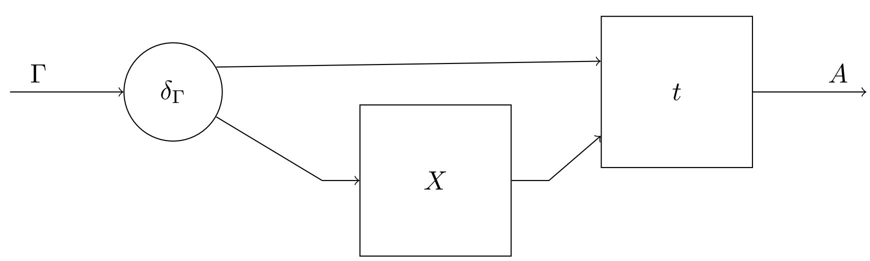
and in particular for strategies which are homomorphisms between δ-comonoids, the recursive definition does unfold in the way expected. This follows as a general fact from the properties of a trace monoidal category.

In fact, recursive definitions can made more generally, without the use of trace, by exploiting old techniques for defining event structures recursively. The substructure order $\trianglelefteq$ on event structures forms a "large complete partial order," continuous operations on which possess least fixed points — see [4, **?**]. Given $x : A, \Gamma \vdash t \dashv y : A$, the term $\Gamma \vdash \mu x : A.\, t \dashv y : A$ denotes the $\trianglelefteq$-least fixed point amongst strategies $X : \Gamma \longrightarrow\!\!\!\!\!\!+\,\, A$ of the $\trianglelefteq$-continuous operation $F(X) = t \odot (\mathrm{id}_\Gamma \| X) \odot \delta_\Gamma$; here $\sigma \trianglelefteq \sigma'$ between two strategies $\sigma : S \to \Gamma^\perp \| A$ and $\sigma' : S' \to \Gamma^\perp \| A$ signifies $S \trianglelefteq S'$ and that the associated inclusion map $i : S \to S'$ makes $\sigma = \sigma' i$. ****

Given $x : A, \Gamma \vdash\ t\ \dashv y : A$,



the term $\Gamma \vdash\ \mu x{:}A.\, t\ \dashv y : A$ denotes the $\trianglelefteq$-least fixed point amongst strategies $X : \Gamma \longrightarrow\!\!\!\!\!\!+\,\, A$ of $F(X) = t \odot (\mathrm{id}_\Gamma \| X) \odot \delta_\Gamma$:



8.2 Semantics

8.2.1 Hom-set terms

The definition of the strategy which

$$\Gamma \vdash p \sqsubseteq_C p' \dashv \Delta$$

denotes is quite involved. We first simplify notation. W.l.o.g. assume $\Delta \vdash p : C$ and $\Gamma \vdash p' : C$—using duality we can always rearrange the environment to achieve this. Write A for the denotation of the environment Γ and B for the denotation of Δ. Let $\Delta \vdash p : C$ and $\Gamma \vdash p' : C$ denote respectively the affine maps $g = (g_0, g_1) : B \to_a C$ and $f = (f_0, f_1) : A \to_a C$. Note, from the typing of $p \sqsubseteq_C p'$ we have that $g_0 \sqsubseteq_C f_0$. We build the strategy out of a rigid family $\mathcal{Q}$ with elements as follows. First, define a pre-element to be a finite preorder comprising a set

$$\{1\} \times \overline{x} \ \cup \ \{2\} \times y \, ,$$

for which

$$\overline{x} \in \mathcal{C}(A^\perp) \ \& \ y \in \mathcal{C}(B) \ \& \ gy \sqsubseteq_c fx \, ,$$

with order that induced by $\leq_{A^\perp}$ on $\overline{x}$, $\leq_B$ on y, with additional causal dependencies

$$(1, a) \leq (2, b) \text{ if } f_1(a) = g_1(b) \ \ \& \ b \text{ is +ve}$$

and

$$(2, b) \leq (1, a) \text{ if } f_1(a) = g_1(b) \ \ \& \ b \text{ is −ve} \, .$$

As elements of the rigid family $\mathcal{Q}$ we take those pre-elements for which the order $\leq$ is a partial order (*i.e.* is antisymmetric). The elements of $\mathcal{Q}$ are closed under rigid inclusions, so $\mathcal{Q}$ forms a rigid family—see Lemma 8.3 below. We now take $S =_{\text{def}} \Pr(\mathcal{Q})$; the events of S (those elements of $\mathcal{Q}$ with a top event) map to their top events in $A^\perp \| B$ from where they inherit polarities. This map can be checked to be a strategy: innocence follows directly from the construction, while receptivity follows from the constraint that $gy \sqsubseteq_c fx$.

It is quite easy to choose an example where antisymmetry fails in a pre-element, in other words, in which the preorder is not a partial order—see Example 8.4 below. However, when either p or p' is just a variable no nontrivial causal loops are introduced and all pre-elements are elements. More generally, if one of p or p' is associated with a partial rigid map (*i.e.* a map which preserves causal dependency when defined), then no nontrivial causal loops are introduced and all pre-elements are elements.

Lemma 8.3. *$\mathcal{Q}$ above is a rigid family.*

Proof. For $\mathcal{Q}$ to be a rigid family we require that its is closed under rigid inclusions, or equivalently, that any down-closed subset of any element q, with order the restriction of that of q, is itself an element of $\mathcal{Q}$.

Let $q =_{\text{def}} (\{1\} \times \overline{x} \ \cup \ \{2\} \times y, \leq)$ be an element of $\mathcal{Q}$, as constructed above. Suppose z is a $\leq$-down-closed subset of q. Let $z_1 =_{\text{def}} \{\overline{a} \mid (1, \overline{a}) \in z\} \subseteq \overline{x}$ and $z_2 =_{\text{def}} \{b \mid (2, b) \in z\} \subseteq y$. We first show

$$gz_2 \sqsubseteq_C f\overline{z}_1 \, ,$$

i.e. that $gz_2 \supseteq^- gz_2 \cap f\overline{z}_1 \subseteq^+ f\overline{z}_1$.

Suppose, to obtain a contradiction, that it is not the case that $gz_2 \cap f\overline{z}_1 \subseteq^+ f\overline{z}_1$. Then, there is some −ve event $c \in f\overline{z}_1$ with $c \notin gz_2$ (†). It immediately follows that $c \notin g_0$. As $c \in f\overline{z}_1$, there are now two cases to consider according as $c \in f_0$ or not. However, if $c \in f_0$ because c is −ve and $g_0 \sqsubseteq_C f_0$ we would obtain $c \in g_0$—a contradiction. Hence $c \notin f_0$, and there is $a \in \overline{z}_1$ with $c = f_1(a)$, and so +ve $\overline{a} \in z_1$. As we have $gy \sqsubseteq_C fx$,

$$gy \cap fx \subseteq^+ fx\,.$$

From this fact we see that because $c \in fx$ is −ve we must have $c \in gy$. So as $c \notin g_0$, we have $c = g_1(b)$ for some −ve $b \in y$. From the construction of q, we have $b \le \overline{a}$ in q. Hence $b \in z_2$, as z is down-closed. But now $c = g_1(b) \in gz_2$, contradicting (†) above.

Similarly, to obtain a contradiction, suppose that it is not the case that $gz_2 \supseteq^- gz_2 \cap f\overline{z}_1$. Then, there is some +ve $c \in gz_2$ with $c \notin f\overline{z}_1$ (‡). We immediately see $c \notin f_0$. As c is +ve and $g_0 \sqsubseteq_C f_0$, if $c \in g_0$ then $c \in f_0$—a contradiction. Therefore, as $c \in gz_2$, there is +ve $b \in z_2$ with $c = g_1(b)$. As we have $gy \sqsubseteq_C fx$,

$$gy \supseteq^- gy \cap fx\,.$$

Because $c \in gy$ is +ve we must have $c \in fx$. So $c = f_1(a)$ for some $a \in x$. From the construction of q, we have $\overline{a} \le b$. As z is down-closed, $\overline{a} \in z_1$. But now $c = f_1(a) \in f\overline{z}_1$, contradicting (‡) above.

To conclude, we now have $gz_2 \sqsubseteq_C f\overline{z}_1$, from which, according to the construction above, we obtain a pre-element $q_z = (z, \le_z)$. From the construction, the order $\le_z$ is included in $\le$, so in particular a partial order, ensuring q_z is an element of $\mathcal{Q}$. We require that q_z be rigidly included in q, for which we need that $\le_z$ is the restriction of $\le$ to z. Any ordering $e \le e'$ between events $e, e' \in z$ results from a chain of causal links in A or B or through the additional links of the construction above. Because z is a down-closed subset of q by the nature of the construction the same chain will be present in q_z. It follows that $\le_z$ is the restriction of $\le$ to z. Hence $\mathcal{Q}$ is closed under rigid inclusions. □

Example 8.4. Let A comprise $a_1 = \ominus \rightarrowtriangle \oplus = a_2$. Let B comprise $b_1 = \oplus \rightarrowtriangle \ominus = b_2$. Let C comprise the two concurrent events $c_1 = \ominus$ and $c_2 = \oplus$. Let $f : A \to C$ send a_1 to c_1 and a_2 to c_2. Let $g : B \to C$ send b_1 to c_2 and b_2 to c_1. Taking $x = \{a_1, a_2\}$ and $y = \{b_1, b_2\}$ we have $fx = gy$, so certainly $gy \sqsubseteq_C fx$. According to the construction of $\mathcal{Q}$ above, there is a pre-element comprising the set $\{1\} \times \overline{x} \cup \{2\} \times y$ with preorder in which $\overline{a}_1 \le \overline{a}_2$ (from $\le_{A^\perp}$), $\overline{a}_2 \le b_1$ (as $f(a_2) = g(b_1)$) , $b_1 \le b_2$ (from $\le_B$) and $b_2 \le \overline{a}_1$ (as $g(b_2) = f(a_1)$). The preorder clearly contains a loop so this pre-element is not an element of the constructed rigid family. □

8.2.2 Duplication

The definition of $\delta_A : A \rightarrow\!\!\!\!\!\rightarrow A \| A$ is via rigid families. For each triple

$$(x, y_1, y_2)$$

where $x \in \mathcal{C}(A^\perp)$, $y_1 \in \mathcal{C}(A)$ and $y_2 \in \mathcal{C}(A)$ which is *balanced*, *i.e.*

$$\begin{aligned}
&\forall a \in y_1.\ pol_A(a) = + \implies \overline{a} \in x\,,\\
&\forall a \in y_2.\ pol_A(a) = + \implies \overline{a} \in x \quad \text{and}\\
&\forall a \in x.\ pol_{A^\perp}(a) = + \implies \overline{a} \in y_1 \text{ or } \overline{a} \in y_2\,,
\end{aligned}$$

and *choice* function

$$\chi : x^+ \to \{1,2\}\,,$$

such that

$$\chi(a) = 1 \implies \overline{a} \in y_1 \quad \text{and} \quad \chi(a) = 2 \implies \overline{a} \in y_2\,,$$

the order $q(x, y_1, y_2; \chi)$ is defined to have underlying set

$$\{0\} \times x \cup \{1\} \times y_1 \cup \{2\} \times y_2$$

with order generated by that inherited from $A^\perp \| A \| A$ together with

$$\begin{aligned}
&\{((0,\overline{a}),\, (1,a)) \mid a \in y_1\} \cup \{((0,\overline{a}), (2,a)) \mid a \in y_2\} \cup\\
&\{((\chi(a),\overline{a}),\, (0,a)) \mid a \in x \ \&\ pol_{A^\perp}(a) = +\}\,.
\end{aligned}$$

The rigid family $\mathcal{Q}$ consists of all such $q(x, y_1, y_2; \chi)$ for balanced (x, y_1, y_2) and choice functions χ. From $\mathcal{Q}$ we obtain the event structure $\mathrm{Pr}(\mathcal{Q})$ in which events are prime orders, with a top element; events of $\mathrm{Pr}(\mathcal{Q})$ inherit the polarity of their top elements to obtain an event structure with polarity. We define the strategy $\delta_A : A \mathrel{-\!\!\!+\!\!\!\!\!\!\to} A \| A$ to be the map

$$\mathrm{Pr}(\mathcal{Q}) \to A^\perp \| A \| A$$

sending a prime to its top element. Of course, we had better check that $\mathcal{Q}$ is a rigid family, in particular that each $q(x, y_1, y_2; \chi)$ is a partial order, and that δ_A is indeed a strategy.

Lemma 8.5. *The family $\mathcal{Q}$ is rigid. The function δ_A taking an event of* $\mathrm{Pr}(\mathcal{Q})$ *to its top element is a strategy* $\mathrm{Pr}(\mathcal{Q}) \to A^\perp \| A \| A$.

Proof. That $\mathcal{Q}$ is closed under rigid inclusions follows straightforwardly; rigid inclusions ensure that choice functions restrict appropriately. □

Consider now the semantics of a term

$$\Gamma \vdash \delta_C(p, q_1, q_2) \dashv \Delta\,.$$

W.l.o.g. we may assume that the environment is arranged so $\Delta \equiv \Delta_1, \Delta_2$ with judgements $\Gamma \vdash p : C$, $\Delta_1 \vdash q_1 : C$ and $\Delta_2 \vdash q_2 : C$. To simplify notation assume the latter judgements for configuration expressions denote the respective affine maps $f = (f^0, f^1) : A \to_a C$, $g_1 = (g_1^0, {g_1}^1) : B_1 \to C$ and $g_2 = (g_2^0, {g_2}^1) : B_2 \to C$. From the typing of $\delta_C(p, q_1, q_2)$ we have that (f^0, g_1^0, g_2^0) forms a balanced triple in C. We build the strategy out of a rigid family $\mathcal{Q}$ with elements as follows.

We construct pre-elements from $x \in \mathcal{C}(A^\perp)$, $y_1 \in \mathcal{C}(B_1)$ and $y_2 \in \mathcal{C}(B_2)$ where (fx, g_1y_1, g_2y_2) is a balanced triple in C with a choice function χ. There are three kinds of elements of x:

$$\begin{aligned}
x^- &= \{a \in x \mid pol_{A^\perp}(a) = -\}\,,\\
x_0^+ &= \{a \in x \mid pol_{A^\perp}(a) = + \ \&\ f^1(a) \in g^0_{\chi(f^1(a))}\} \text{ and}\\
x_1^+ &= \{a \in x \mid pol_{A^\perp}(a) = + \ \&\ f^1(a) \in g^1_{\chi(f^1(a))} y_{\chi(f^1(a))}\}
\end{aligned}$$

We define a typical pre-element to be a finite preorder on the set

$$\{0\} \times (x^- \cup x_1^+ \cup \{(\chi(f^1(a)), a) \mid a \in x_0^+\}) \ \cup\ \{1\} \times y_1 \ \cup\ \{2\} \times y_2\,,$$

with order that induced by that of the game $A^\perp \| B_1 \| B_2$—each event of the set is clearly associated with a unique event of the game—with additional causal dependencies

$$\begin{aligned}
(0, a) &\le (1, b) \text{ if } f^1(a) = g_1^1(b) \ \&\ \text{ b is +ve in } B_1,\\
(0, a) &\le (2, b) \text{ if } f^1(a) = g_2^1(b) \ \&\ \text{ b is +ve in } B_2,
\end{aligned}$$

and

$$(\chi(f^1(a)), b) \le (0, a) \text{ if } a \in x_1^+ \ \&\ f^1(a) = g^1_{\chi(f^1(a))}(b)\,, \text{ for } b \text{ a } -\text{ve in } B_{\chi(f^1(a))}\,.$$

As elements of the rigid family $\mathcal{Q}$ we take those pre-elements for which the order $\le$ is a partial order (*i.e.* is antisymmetric). Once $\mathcal{Q}$ is checked to be a rigid family—see Lemma 8.6 below—we can take $S =_{\mathrm{def}} \Pr(\mathcal{Q})$; the events of S map to the events in the game $A^\perp \| B_1 \| B_2$ associated with their top events, from where they inherit polarities. This map defines the strategy denoting the original duplication term.

Lemma 8.6. *The family $\mathcal{Q}$ is rigid. The function taking events of* $\Pr(\mathcal{Q})$ *to their top elements defines a strategy from A to $B_1 \| B_2$.*

Proof. For $\mathcal{Q}$ to be a rigid family we require that any down-closed subset of any element q, with order the restriction of that of q, is itself an element of $\mathcal{Q}$.

Let $q =_{\mathrm{def}} (\{0\} \times x \cup \{1\} \times y_1 \cup \{2\} \times y_2, \le)$ be an element of $\mathcal{Q}$, as constructed above. Suppose z is a $\le$-down-closed subset of q. Let $z_0 =_{\mathrm{def}} \{a \mid (0, a) \in z\} \subseteq x$, $z_1 =_{\mathrm{def}} \{b \mid (1, b) \in z\} \subseteq y_1$ and $z_2 =_{\mathrm{def}} \{b \mid (2, b) \in z\} \subseteq y_2$. We first show

$$(fz_0, g_1z_1, g_2z_2)$$

is balanced. ****** □

Chapter 9

Winning ways

What does it mean to win a nondeterministic concurrent game and what is a winning strategy? This chapter extends the work on games and strategies to games with winning conditions and winning strategies. Without winning conditions Player and Opponent can elect to not make any moves. For example, there is always a minimum strategy in a game in which Player makes no moves whatsoever. Winning conditions in a game provide an incentive with respect to which Player or Opponent can be encouraged to make moves in order to avoid losing and win.

9.1 Winning strategies

A *game with winning conditions* comprises $G = (A, W)$ where A is an event structure with polarity and $W \subseteq \mathcal{C}^\infty(A)$ consists of the *winning configurations* for Player. We define the *losing conditions* to be $L =_{\mathrm{def}} \mathcal{C}^\infty(A) \setminus W$. Clearly a game with winning conditions is determined once we specify either its winning or losing conditions, and we can define such a game by specifying its losing conditions.

A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes Player moves to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all $+$-maximal configurations $x \in \mathcal{C}^\infty(S)$—a configuration x is $+$-maximal if whenever $x \xrightarrow{s}\subset$ then the event s has $-$ve polarity. Any achievable position $z \in \mathcal{C}^\infty(S)$ of the game can be extended to a $+$-maximal, so winning, configuration (via Zorn's Lemma). So a strategy prescribes Player moves to reach a winning configuration whatever state of play is achieved following the strategy. Note that for a game A, if winning conditions $W = \mathcal{C}^\infty(A)$, *i.e.* every configuration is winning, then any strategy in A is a winning strategy.

In the special case of a deterministic strategy $\sigma : S \to A$ in G it is winning iff $\sigma\varphi(x) \in W$ for all $x \in \mathcal{C}^\infty(S)$, where φ is the closure operator $\varphi : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S)$

determined by σ or, equivalently, the images under σ of fixed points of φ lie outside L. Recall from Section 6.2.3 that a deterministic strategy $\sigma : S \to A$ determines a closure operator φ on $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \&\ Neg[\{s\}] \subseteq x\}\,.$$

Clearly, we can equivalently say a strategy $\sigma : S \to A$ in G is winning if it always prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent; a strategy $\sigma : S \to A$ in G is winning if $\sigma x \notin L$ for all $+$-maximal configurations $x \in \mathcal{C}^\infty(S)$

Informally, we can also understand a strategy as winning for Player if when played against any counter-strategy of Opponent, the final result is a win for Player. Suppose $\sigma : S \to A$ is a strategy in a game (A, W). A counter-strategy is strategy of Opponent, so a strategy $\tau : T \to A^\perp$ in the dual game. We can view σ as a strategy $\sigma : \varnothing \mathrel{-\!\!\!+\!\!\!\rightarrow} A$ and τ as a strategy $\tau : A \mathrel{-\!\!\!+\!\!\!\rightarrow} \varnothing$. Their composition $\tau \odot \sigma : \varnothing \mathrel{-\!\!\!+\!\!\!\rightarrow} \varnothing$ is not in itself so informative. Rather it is the status of the configurations in $\mathcal{C}^\infty(A)$ their full interaction induces which decides which of Player or Opponent wins. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,

$$\begin{tikzcd} & P \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, "\vee" near start] & \\ S \arrow[dr, "\sigma"'] & & T \arrow[dl, "\tau"] \\ & A\,, & \end{tikzcd}$$

to obtain the event structure P resulting from the interaction of σ and τ. (Note $P \cong \Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, in the terms of Chapter 4, by the remarks of Section 4.3.3.) Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}^\infty(P)$. A maximal configuration z in $\mathcal{C}^\infty(P)$ images to a configuration $\sigma\Pi_1 z = \tau\Pi_2 z$ in $\mathcal{C}^\infty(A)$. Define the set of *results* of the interaction of σ and τ to be

$$\langle \sigma, \tau \rangle =_{\mathrm{def}} \{\sigma\Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^\infty(P)\}\,.$$

We shall show the strategy σ is a winning for Player iff all the results of the interaction $\langle \sigma, \tau \rangle$ lie within the winning configurations W, for any counter-strategy $\tau : T \to A^\perp$ of Opponent.

It will be convenient later to have proved facts about $+$-maximality in the broader context of the composition of receptive pre-strategies.

Convention 9.1. Refer to the construction of the composition of pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : B^\perp \| C$ in Chapter 4 We shall say a configuration x of either $\mathcal{C}^\infty(S)$, $\mathcal{C}^\infty(T)$ or $(\mathcal{C}(T) \circledast \mathcal{C}(S))^\infty$ is $+$-maximal if whenever $x \stackrel{e}{\longrightarrow\!\!\!\subset}$ then the event e has $-$ve polarity. In the case of $(\mathcal{C}(T) \circledast \mathcal{C}(S))^\infty$ an event of $-$ve polarity is deemed to be one of the form $(s,*)$, with s −ve in S, or $(*,t)$, with t −ve in T. We shall say a configuration z of $\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T)\circledast\mathcal{C}(S)))$ is +-maximal if whenever $z \xrightarrow{p}\!\!\subset$ then $top(p)$ has −ve polarity.

Lemma 9.2. *Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be receptive pre-strategies. Then,*

$$
\begin{aligned}
& z \in (\mathcal{C}(T)\circledast\mathcal{C}(S))^\infty \textit{ is +-maximal iff}\\
& \pi_1 z \in \mathcal{C}^\infty(S) \textit{ is +-maximal } \& \ \pi_2 z \in \mathcal{C}^\infty(T) \textit{ is +-maximal.}
\end{aligned}
$$

Proof. Let $z \in (\mathcal{C}(T)\circledast\mathcal{C}(S))^\infty$. *"Only if":* Assume z is +-maximal. Suppose, for instance, $\pi_1 z$ is not +-maximal. Then, $\pi_1 z \xrightarrow{s}\!\!\subset$ for some +ve event $s \in S$. Consider the two cases. *Case $\sigma_1(s)$ is defined:* Form the configuration $z \cup \{(s,*)\} \in (\mathcal{C}(T)\circledast\mathcal{C}(S))^\infty$, to contradict the +-maximality of z. *Case $\sigma_2(s)$ is defined:* As s is +-ve by the receptivity of τ there is $t \in T$ such that $\pi_2 z \xrightarrow{t}\!\!\subset$ and $\tau_1(t) = \overline{\sigma_2(s)}$. Form the configuration $z \cup \{(s,t)\} \in (\mathcal{C}(T)\circledast\mathcal{C}(S))^\infty$, to contradict the +-maximality of z. The argument showing $\pi_2 z$ is +-maximal is similar.

"If": Assume both $\pi_1 z$ and $\pi_2 z$ are +-maximal. Suppose z were not +-maximal. Then, either

- $z \xrightarrow{(s,*)}\!\!\subset$ or $z \xrightarrow{(s,t)}\!\!\subset$ with s a +ve event of S, or
- $z \xrightarrow{(*,t)}\!\!\subset$ or $z \xrightarrow{(s,t)}\!\!\subset$ with t a +ve event of T.

But then either $\pi_1 z \xrightarrow{s}\!\!\subset$, contradicting the +-maximality of $\pi_1 z$, or $\pi_2 z \xrightarrow{t}\!\!\subset$, contradicting the +-maximality of $\pi_2 z$. □

Corollary 9.3. *Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be receptive pre-strategies. Then,*

$$
\begin{aligned}
& x \in \mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T)\circledast\mathcal{C}(S))) \textit{ is +-maximal iff}\\
& \Pi_1 x \in \mathcal{C}^\infty(S) \textit{ is +-maximal } \& \ \Pi_2 x \in \mathcal{C}^\infty(T) \textit{ is +-maximal.}
\end{aligned}
$$

Proof. From Lemma 9.2, noting the order isomorphism $\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T)\circledast\mathcal{C}(S))) \cong (\mathcal{C}(T)\circledast\mathcal{C}(S))^\infty$ given by $x \mapsto \bigcup x$ and that $\Pi_1 x = \pi_1 \bigcup x$, $\Pi_2 x = \pi_2 \bigcup x$. □

Remark. In fact the proof of Lemma 9.2 above only relies on the existence part of receptivity.

Lemma 9.4. *Let $\sigma : S \to A$ be a strategy in a game (A,W). The strategy σ is winning for Player iff $\langle\sigma,\tau\rangle \subseteq W$ for all (deterministic) strategies $\tau : T \to A^\perp$.*

Proof. "Only if": Suppose σ is winning, *i.e.* $\sigma x \in W$ for all +-maximal $x \in \mathcal{C}^\infty(S)$. Let $\tau : T \to A^\perp$ be a strategy. By Corollary 9.3,

$$
\begin{aligned}
& x \in \mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T)\circledast\mathcal{C}(S))) \text{ is +-maximal}\\
\text{iff}\quad & \\
& \Pi_1 x \in \mathcal{C}^\infty(S) \text{ is +-maximal } \& \ \Pi_2 x \in \mathcal{C}^\infty(T) \text{ is +-maximal.}
\end{aligned}
$$

Letting x be maximal in $\mathcal{C}^\infty(\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S)))$ it is certainly +-maximal, whence $\Pi_1 x$ is +-maximal in $\mathcal{C}^\infty(S)$. It follows that $\sigma \Pi_1 x \in W$ as σ is winning. Hence $\langle \sigma, \tau \rangle \subseteq W$.

"If": Assume $\langle \sigma, \tau \rangle \subseteq W$ for all strategies $\tau : T \to A^\perp$. Suppose x is +-maximal in $\mathcal{C}^\infty(S)$. Define T to be the event structure given as the restriction

$$T =_{\text{def}} A^\perp \upharpoonright \sigma x \cup \{a \in A^\perp \mid pol_{A^\perp} = -\}\,.$$

Let $\tau : T \to A^\perp$ be the inclusion map $T \hookrightarrow A^\perp$. The pre-strategy τ can be checked to be receptive and innocent, so a strategy. (In fact, τ is a *deterministic* strategy as all its +ve events lie within the configuration σx.) One way to describe a pullback of τ along σ is as the "inverse image" $P =_{\text{def}} S \upharpoonright \{s \in S \mid \sigma(s) \in T\}$:

$$\begin{tikzcd}
 & P \arrow[dl, hook'] \arrow[dr, "\sigma\upharpoonright P"] \arrow[dd, phantom, "\lrcorner" very near start, rotate=-45] & \\
S \arrow[dr, "\sigma"'] & & T \arrow[dl, hook', "\tau"] \\
 & A &
\end{tikzcd}$$

From the definition of T and P we see $x \in \mathcal{C}^\infty(P)$; and moreover that x is maximal in $\mathcal{C}^\infty(P)$ as x is +-maximal in $\mathcal{C}^\infty(S)$. Hence $\sigma x \in \langle \sigma, \tau \rangle$ ensuring $\sigma x \in W$, as required.

The proof is unaffected if we restrict to *deterministic* counter-strategies $\tau : T \to A^\perp$. □

Corollary 9.5. *There are the following four equivalent ways to say that a strategy $\sigma : S \to A$ is winning in (A, W)—we write L for the losing configurations $\mathcal{C}^\infty(A) \setminus W$:*

1. *$\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$, i.e. the strategy prescribes Player moves to reach a winning configuration, no matter what the activity or inactivity of Opponent;*

2. *$\sigma x \notin L$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$, i.e. the strategy prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent;*

3. *$\langle \sigma, \tau \rangle \subseteq W$ for all strategies $\tau : T \to A^\perp$, i.e. all plays against counter-strategies of the Opponent result in a win for Player;*

4. *$\langle \sigma, \tau \rangle \subseteq W$ for all deterministic strategies $\tau : T \to A^\perp$, i.e. all plays against deterministic counter-strategies of the Opponent result in a win for Player.*

****ADD σ WINNING IFF ... FOR ALL RECEPTIVE PRE-STRATEGIES τ - A COROLL OF PF ABOVE****

Not all games with winning conditions have winning strategies. Consider the game A consisting of one player move $\oplus$ and one opponent move $\ominus$ inconsistent with each other, with $\{\{\oplus\}\}$ as its winning conditions. This game has no winning strategy; any strategy $\sigma : S \to A$, being receptive, will have an event $s \in S$ with $\sigma(s) = \ominus$, and so the losing $\{s\}$ as a $+$-maximal configuration.

9.2 Operations

9.2.1 Dual

There is an obvious dual of a game with winning conditions $G = (A, W_G)$:

$$G^\perp = (A^\perp, W_{G^\perp})$$

where, for $x \in \mathcal{C}^\infty(A)$,

$$x \in W_{G^\perp} \quad \text{iff} \quad \overline{x} \notin W_G \,.$$

We are using the notation $a \leftrightarrow \overline{a}$, giving the correspondence between events of A and $A^\perp$, extended to their configurations: $\overline{x} =_{\mathrm{def}} \{\overline{a} \mid a \in x\}$, for $x \in \mathcal{C}^\infty(A)$. As usual the dual reverses the roles of Player and Opponent and correspondingly the roles of winning and losing conditions.

9.2.2 Parallel composition

The parallel composition of two games with winning conditions $G = (A, W_G)$, $H = (B, W_H)$ is

$$G \| H =_{\mathrm{def}} (A \| B,\ W_G \| \mathcal{C}^\infty(B) \cup \mathcal{C}^\infty(A) \| W_H)$$

where $X \| Y = \{\{1\} \times x \cup \{2\} \times y \mid x \in X \ \&\ y \in Y\}$ when X and Y are subsets of configurations. In other words, for $x \in \mathcal{C}^\infty(A \| B)$,

$$x \in W_{G\|H} \quad \text{iff} \quad x_1 \in W_G \text{ or } x_2 \in W_H \,,$$

where $x_1 = \{a \mid (1, a) \in x\}$ and $x_2 = \{b \mid (2, b) \in x\}$. To win in $G \| H$ is to win in either game. Its losing conditions are $L_A \| L_B$—to lose is to lose in both games G and H.[1] The unit of $\|$ is $(\varnothing, \varnothing)$. In order to disambiguate the various forms of parallel composition, we shall sometimes use the linear-logic notation $G \invamp H$ for the parallel composition $G \| H$ of games with winning strategies.

9.2.3 Tensor

Defining $G \otimes H =_{\mathrm{def}} (G^\perp \| H^\perp)^\perp$ we obtain a game where to win is to win in both games G and H—so to lose is to lose in either game. More explicitly,

$$(A, W_A) \otimes (B, W_B) =_{\mathrm{def}} (A \| B,\ W_A \| W_B) \,.$$

The unit of $\otimes$ is $(\varnothing, \{\varnothing\})$.

[1] I'm grateful to Nathan Bowler, Pierre Clairambault and Julian Gutierrez for guidance in the definition of parallel composition of games with winning conditions.

9.2.4 Function space

With $G \multimap H =_{\text{def}} G^{\perp} \| H$ a win in $G \multimap H$ is a win in H conditional on a win in G.

Proposition 9.6. *Let $G = (A, W_G)$ and $H = (B, W_H)$ be games with winning conditions. Write $W_{G \multimap H}$ for the winning conditions of $G \multimap H$, so $G \multimap H = (A^{\perp} \| B, W_{G \multimap H})$. For $x \in \mathcal{C}^{\infty}(A^{\perp} \| B)$,*

$$x \in W_{G \multimap H} \quad \textit{iff} \quad \overline{x_1} \in W_G \implies x_2 \in W_H \,.$$

Proof. Letting $x \in \mathcal{C}^{\infty}(A^{\perp} \| B)$,

$$\begin{aligned} x \in W_{G \multimap H} \ &\text{iff} \ \ x \in W_{G^{\perp} \| H} \\ &\text{iff} \ \ x_1 \in W_{G^{\perp}} \text{ or } x_2 \in W_H \\ &\text{iff} \ \ \overline{x_1} \notin W_G \text{ or } x_2 \in W_H \\ &\text{iff} \ \ \overline{x_1} \in W_G \implies x_2 \in W_H \,. \end{aligned}$$

□

9.3 The bicategory of winning strategies

We can again follow Joyal and define strategies between games now with winning conditions: a (winning) strategy from G, a game with winning conditions, to another H is a (winning) strategy in $G \multimap H = G^{\perp} \| H$. We compose strategies as before. We first show that the composition of winning strategies is winning.

Lemma 9.7. *Let σ be a winning strategy in $G^{\perp} \| H$ and τ be a winning strategy in $H^{\perp} \| K$. Their composition $\tau \odot \sigma$ is a winning strategy in $G^{\perp} \| K$.*

Proof. Let $G = (A, W_G)$, $H = (B, W_H)$ and $K = (C, W_K)$.

Suppose $x \in \mathcal{C}^{\infty}(T \odot S)$ is $+$-maximal. Then $\bigcup x \in (\mathcal{C}(T) \circledast \mathcal{C}(S))^{\infty}$. By Zorn's Lemma we can extend $\bigcup x$ to a maximal configuration $z \supseteq \bigcup x$ in $(\mathcal{C}(T) \circledast \mathcal{C}(S))^{\infty}$ with the property that all events of $z \setminus \bigcup x$ are synchronizations of the form (s, t) for $s \in S$ and $t \in T$. Then, z will be $+$-maximal in $(\mathcal{C}(T) \circledast \mathcal{C}(S))^{\infty}$ with

$$\sigma_1 \pi_1 z = \sigma_1 \pi_1 \bigcup x \quad \& \quad \tau_2 \pi_2 z = \tau_2 \pi_2 \bigcup x \,. \tag{1}$$

By Lemma 9.2,

$$\pi_1 z \text{ is } +\text{-maximal in } S \quad \& \quad \pi_2 z \text{ is } +\text{-maximal in } T \,.$$

As σ and τ are winning,

$$\sigma \pi_1 z \in W_{G^{\perp} \| H} \quad \& \quad \tau \pi_2 z \in W_{H^{\perp} \| K} \,.$$

Now $\sigma \pi_1 z \in W_{G^{\perp} \| H}$ expreses that

$$\overline{\sigma_1 \pi_1 z} \in W_G \implies \sigma_2 \pi_1 z \in W_H \tag{2}$$

and $\tau\pi_2 z \in W_{H^\perp \| K}$ that

$$\overline{\tau_1\pi_2 z} \in W_H \implies \tau_2\pi_2 z \in W_K\,, \tag{3}$$

by Proposition 9.6. But $\sigma_2\pi_1 z = \overline{\tau_1\pi_2 z}$, so (2) and (3) yield

$$\overline{\sigma_1\pi_1 z} \in W_G \implies \tau_2\pi_2 z \in W_K\,.$$

By (1)

$$\overline{\sigma_1\pi_1 \bigcup x} \in W_G \implies \tau_2\pi_2 \bigcup x \in W_K\,,$$

*i.e.*by Proposition 4.2,

$$\overline{\upsilon_1 x} \in W_G \implies \upsilon_2 x \in W_K$$

in the span of the composition $\tau\odot\sigma$. Hence $x \in W_{G^\perp \| K}$, as required. □

For a general game with winning conditions (A, W) the copy-cat strategy need not be winning, as shown in the following example.

Example 9.8. Let A consist of two events, one +ve event $\oplus$ and one −ve event $\ominus$, inconsistent with each other. Take as winning conditions the set $W = \{\{\oplus\}\}$. The event structure $\mathbb{C}\mathbb{C}_A$:

$$\begin{array}{cccc} A^\perp & \ominus \rightarrowtriangle \oplus & A \\ & \oplus \leftarrowtriangle \ominus & \end{array}$$

To see $\mathbb{C}\mathbb{C}_A$ is not winning consider the configuration x consisting of the two −ve events in $\mathbb{C}\mathbb{C}_A$. Then x is +-maximal as any +ve event is inconsistent with x. However, $\overline{x}_1 \in W$ while $x_2 \notin W$, failing the winning condition of $(A, W) \multimap (A, W)$.

Recall from Chapter 7, that each event structure with polarity$A$ possesses a Scott order on its configurations $\mathcal{C}^\infty(A)$:

$$x' \sqsubseteq x \;\text{ iff }\; x' \supseteq^- x \cap x' \subseteq^+ x\,.$$

Hence a necessary and sufficient for copy-cat to be winning w.r.t. a game (A, W):

$$\begin{array}{ll} \forall x, x' \in \mathcal{C}^\infty(A). & \text{if } x' \sqsubseteq x \;\&\; \overline{x}\|x' \text{ is +-maximal in } \mathcal{C}^\infty(\mathbb{C}\mathbb{C}_A) \\ & \text{then } \; x \in W \implies x' \in W\,. \end{array} \qquad \textbf{(Cwins)}$$

Proposition 9.9. *Let (A, W) be a game with winning conditions. The copy-cat strategy $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ is winning iff (A, W) satisfies* **(Cwins)***.*

Proof. **(Cwins)** expresses precisely that copy-cat is winning. □

A robust sufficient condition on an event structure with polarity A which ensures that copy-cat is a winning strategy for all choices of winning conditions is the property

$$\forall x \in \mathcal{C}(A).\; x \overset{a}{\longrightarrow\!\!\subset} \;\&\; x \overset{a'}{\longrightarrow\!\!\subset} \;\&\; pol(a) = + \;\&\; pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,.$$

(race-free)

This property, which says immediate conflict respects polarity, is seen earlier in Lemma 5.3 (characteriziing those A for which copy-cat is deterministic).

Lemma 9.10. *Assume A is race-free. If $x' \sqsubseteq x$ in $\mathcal{C}^\infty(A)$ and $\overline{x} \| x'$ is $+$-maximal in $\mathcal{C}^\infty(\mathbb{C}_A)$, then $x = x'$.*

Proof. Assume A is race-free and $x' \sqsubseteq x$ and $\overline{x} \| x'$ is $+$-maximal in $\mathcal{C}^\infty(\mathbb{C}_A)$. Then $x \supseteq^+ x \cap x' \subseteq^- x'$. There are covering chains associated with purely $+$ve and $-$ve events from $x \cap x'$ to x and x', respectively:

$$x \cap x' \overset{+}{-\!\!\!\subset} x_1 \cdots \overset{+}{-\!\!\!\subset}\ x\,,$$
$$x \cap x' \overset{-}{-\!\!\!\subset} x_1' \cdots \overset{-}{-\!\!\!\subset}\ x'\,.$$

If one of the covering chains is of zero length, *i.e.* $x \supseteq^+ x'$ or $x \subseteq^- x'$, then so must the other be—otherwise we contradict the maximality assumption. On the other hand, if both are nonempty, by repeated use of (**race-free**) we again contradict the maximality assumption, *e.g.*

$$\begin{array}{ccccccc}
x_1' & \overset{+}{-\!\!\!\subset} & x_1 \cup x_1' & \overset{+}{-\!\!\!\subset} & \cdots & \overset{+}{-\!\!\!\subset} & x \cup x_1' \\
{\scriptstyle -}\,\Big\uparrow & & {\scriptstyle -}\,\Big\uparrow & & & & {\scriptstyle -}\,\Big\uparrow \\
x \cap x' & \overset{+}{-\!\!\!\subset} & x_1 & \overset{+}{-\!\!\!\subset} & \cdots & \overset{+}{-\!\!\!\subset} & x
\end{array}$$

so $x' \sqsubseteq x \cup x_1'$ and $(\overline{x} \| x') \overset{+}{-\!\!\!\subset} (\overline{x \cup x_1'} \| x')$, showing how a repeated use of (**race-free**) contradicts the $+$-maximality of $\overline{x} \| x'$. We conclude $x = x \cap x' = x'$ $\square$

Proposition 9.11. *Let A be an event structure with polarity. Copy-cat is a winning strategy for all games (A, W) with winning conditions W iff A satisfies* (**race-free**)*.*

Proof. "If": Assume (**race-free**). Suppose $\overline{x} \| x'$ is a $+$-maximal configuration in $\mathcal{C}^\infty(\mathbb{C}_A)$. Then, by Lemma 9.10, $x = x'$. Let $W \subseteq \mathcal{C}^\infty(A)$. Certainly $x \in W \implies x' \in W$, as required to fulfil (**Cwins**).

"Only if": Suppose A failed (**race-free**), *i.e.* $x \overset{a}{-\!\!\!\subset} x_1$ & $x \overset{a'}{-\!\!\!\subset} x_2$ with $x_1 \not\uparrow x_2$ and $pol_A(a) = +$ and $pol(a') = -$ within the finite configurations of A. The set $\overline{x}_1 \| x_2 =_{\rm def} \{1\} \times \overline{x}_1 \cup \{2\} \times x_2$ is certainly a finite configuration of $A^\perp \| A$ and is easily checked to also be a configuration of $\mathbb{C}_A$. Define winning conditions by

$$W = \{x \in \mathcal{C}^\infty(A) \mid a \in x\}\,.$$

Let $z \in \mathcal{C}^\infty(\mathbb{C}_A)$ be a $+$-maximal extension of $\overline{x}_1 \| x_2$ (the maximal extension exists by Zorn's Lemma). Take $z_1 = \{a \mid (1, a) \in z\}$ and $z_2 = \{a \mid (2, a) \in z\}$. Then $\overline{z}_1 \supseteq x_1$ and $z_2 \supseteq x_2$. As $a \in \overline{z}_1$ we obtain $\overline{z}_1 \in W$, whereas $z_2 \notin W$ because z_2 extends x_2 which is inconsistent with a. Hence copy-cat is not winning in $(A, W)^\perp \| (A, W)$. $\square$

We can now refine the bicategory of strategies **Strat** to the bicategory **WGames** with objects games with winning conditions $G, H, \cdots$ satisfying (**Cwins**) and arrows winning strategies $G \longrightarrow\!\!\!\!\!\!+\ H$; 2-cells, their vertical and horizontal composition is as before. Its restriction to deterministic strategies yields a bicategory **WDGames** equivalent to a simpler order-enriched category.

9.4 Total strategies

As an application of winning conditions we apply them to pick out a subcategory of "total strategies," informally strategies in which Player can always answer a move of Opponent.[2]

We restrict attention to 'simple games' (games and strategies are alternating and begin with opponent moves—see Section 6.2.4). Here a strategy is *total* if all its finite maximal sequences are even, so ending in a +ve move, *i.e.* a move of Player. In general, the composition of total strategies need not be total—see the Exercise below. However, as we will see, we can pick out a subcategory of 'simple games' with suitable winning conditions. Within this full subcategory of games with winning conditions winning strategies will be total and moreover compose.

Exercise 9.12. *Exhibit two total strategies whose composition is not total.* □

As objects of the subcategory we choose simple games with winning strategies,

$$(A, W_A)$$

where A is a simple game and W_A is a subset of possibly infinite sequences $s_1 s_2 \cdots$ satisfying

$$W_A \cap \mathrm{Finite}(A) = \mathrm{Even}(A) \qquad \textbf{(Tot)}$$

i.e. the finite sequences in W_A are precisely those of even length. Note that winning strategies in such a game will be total. (Below we use 'sequence' to mean allowable finite or infinite sequences of the appropriate simple game.)

The function space $(A, W_A) \multimap (B, W_B)$, given as $(A, W_A)^\perp \| (B, W_B)$, has winning conditions W such that

$$s \in W \text{ iff } s \upharpoonright A \in W_A \implies s \upharpoonright B \in W_B\,.$$

Lemma 9.13. *For s a sequence of $A^\perp \| B$, s is even iff $s \upharpoonright A$ is odd or $s \upharpoonright B$ is even.*

Proof. By parity, considering the final move of the sequence.
"Only if": Assume s is even, *i.e.* its final event is +ve. If s ends in B, $s \upharpoonright B$ ends in + so is even. If s ends in A, $s \upharpoonright A$ ends in − so is odd.
"If": Assume $s \upharpoonright A$ is odd or $s \upharpoonright B$ is even. Suppose, to obtain a contradiction, that s is not even, *i.e.* s is odd so ends in −. If s ends in B, $s \upharpoonright B$ ends in − so is odd and consequently $s \upharpoonright A$ even (as the length of s is the sum of the lengths of $s \upharpoonright A$ and $s \upharpoonright B$). Similarly, if s ends in A, $s \upharpoonright A$ ends in + so $s \upharpoonright A$ is even and $s \upharpoonright B$ is odd. Either case contradicts the initial assumption. Hence s is even. □

[2] This section is inspired by [25], though differs in several respects.

It follows that W, the winning conditions of the function space, satisfies **(Tot)**: Let s be a finite sequence of a strategy in $A^\perp \| B$. Then,

$$\begin{aligned} s \in W &\text{ iff } s \upharpoonright A \in W_A \implies s \upharpoonright B \in W_B \\ &\text{ iff } s \upharpoonright A \notin W_A \text{ or } s \upharpoonright B \in W_B \\ &\text{ iff } s \upharpoonright A \text{ is odd or } s \upharpoonright B \text{ is even} \\ &\text{ iff } s \text{ is even.} \end{aligned}$$

All maps in the subcategory (which are winning strategies in its function spaces $(A, W_A) \multimap (B, W_B)$) compose (because winning strategies do) and are total (because winning conditions of its function spaces satisfy **(Tot)**).

9.5 On determined games

A game with winning conditions G is said to be *determined* when either Player or Opponent has a winning strategy, *i.e.* either there is a winning strategy in G or in $G^\perp$.[3] Not all games are determined. Neither the game G consisting of one player move $\oplus$ and one opponent move $\ominus$ inconsistent with each other, with $\{\{\oplus\}\}$ as winning conditions, nor the game $G^\perp$ have a winning strategy.

Notation 9.14. *Let $\sigma : S \to A$ be a strategy. We say $y \in \mathcal{C}^\infty(A)$ is σ-*reachable *iff $y = \sigma x$ for some $x \in \mathcal{C}^\infty(S)$. Let $y' \subseteq y$ in $\mathcal{C}^\infty(A)$. Say y' is* $-$-maximal in y *iff $y \xrightarrow{-}\!\!\subset y''$ implies $y'' \notin y$. Similarly, say y' is* $+$-maximal in y *iff $y \xrightarrow{+}\!\!\subset y''$ implies $y'' \notin y$.*

Lemma 9.15. *Let (A, W) be a game with winning conditions. Let $y \in \mathcal{C}^\infty(A)$. Suppose*

$$\begin{aligned} &\forall y' \in \mathcal{C}^\infty(A). \\ &y' \subseteq y \;\&\; y' \textit{ is } -\textit{-maximal in } y \;\&\; \textit{not } +\textit{-maximal in } y \\ &\implies \\ &\{y'' \in \mathcal{C}(A) \mid y' \subseteq^+ y'' \;\&\; (y'' \setminus y') \cap y = \varnothing\} \cap W = \varnothing\,. \end{aligned}$$

Then y is σ-reachable in all winning strategies σ.

Proof. Assume the property above of $y \in \mathcal{C}^\infty(A)$. Suppose, to obtain a contradiction, that y is not σ-reachable in a winning strategy $\sigma : S \to A$.

Let $x' \in \mathcal{C}^\infty(A)$ be $\subseteq$-maximal such that $\sigma x' \subseteq y$ (this uses Zorn's lemma).

By the receptivity of σ, the configuration $\sigma x'$ is $-$-maximal in y. By supposition, $\sigma x' \subsetneq y$, so we must therefore have $\sigma x' \xrightarrow{+}\!\!\subset y_0 \subseteq y$ in $\mathcal{C}^\infty(A)$, *i.e.* $\sigma x'$ is not $+$-maximal in y. From the property assumed of y we deduce both

$$\sigma x' \notin W \quad \& \quad (\forall y'' \in W.\ \sigma x' \subseteq^+ y'' \implies (y'' \setminus \sigma x') \cap y \neq \varnothing)\,.$$

As σ is winning, there is $+$-maximal extension $x' \subseteq^+ x''$ in $\mathcal{C}^\infty(S)$ such that $\sigma x'' \in W$. Hence

$$(\sigma x'' \setminus \sigma x') \cap y \neq \varnothing\,.$$

[3]This section is based on work with Julian Gutierrez.

Taking a $\leq_A$-minimal event a_1, necessarily +ve, in the above set we obtain

$$\sigma x' \xrightarrow{a_1}\!\!\!\subset y_1 \subseteq^+ \sigma x''\,.$$

By Corollary 4.23, $y_1 = \sigma x_1$ for some $x_1 \in \mathcal{C}^\infty(S)$ with $x' \stackrel{+}{\longrightarrow}\!\!\!\subset x_1 \subseteq x''$. But this contradicts the choice of x' as $\subseteq$-maximal such that $\sigma x' \subseteq y$. Hence the original assumption that y is not σ-reachable must be false. □

Recall the property (**race-free**) of an event structure with polarity A, first seen in Lemma 5.3, though here rephrased a little:

$$\forall y, y_1, y_2 \in \mathcal{C}(A).\ y \stackrel{-}{\longrightarrow}\!\!\!\subset y_1 \ \&\ y \stackrel{+}{\longrightarrow}\!\!\!\subset y_2 \implies y_1 \uparrow y_2\,. \qquad \textbf{(race-free)}$$

Corollary 9.16. *If A, an event structure with polarity, fails to satisfy* (**race-free**), *then there are winning conditions W, for which the game (A, W) is not determined.*

Proof. Suppose (**race-free**) failed, that $y \stackrel{-}{\longrightarrow}\!\!\!\subset y_1$ and $y \stackrel{+}{\longrightarrow}\!\!\!\subset y_2$ and $y_1 \not\uparrow y_2$ in $\mathcal{C}(A)$. Assign configurations $\mathcal{C}^\infty(A)$ to winning conditions W or its complement as follows:

(i) for y'' with $y_1 \subseteq^+ y''$, assign $y'' \notin W$;

(ii) for y'' with $y_2 \subseteq^- y''$, assign $y'' \in W$;

(iii) for y'' with $y' \subseteq^+ y''$ and $(y'' \setminus y') \cap y = \varnothing$, for some sub-configuration y' of y with y' $-$-maximal and not +-maximal in y, assign $y'' \notin W$;

(iv) for y'' with $y' \subseteq^- y''$ and $(y'' \setminus y') \cap y = \varnothing$, for some sub-configuration y' of y with y' +-maximal and not $-$-maximal in y, assign $y'' \in W$;

(v) assign arbitrarily in all other cases.

We should check the assignment is well-defined, that we do not assign a configuration both to W and its complement.

Clearly the first two cases (i) and (ii) are disjoint as $y_1 \not\uparrow y_2$.

The two cases (iii) and (iv) are also disjoint. Suppose otherwise, that both (iii) and (iv) hold for y'', *viz.*

$$\begin{aligned} &y_1' \subseteq^+ y'' \ \&\ (y'' \setminus y_1') \cap y = \varnothing \ \&\\ &\qquad\qquad y_1' \text{ is } -\text{-maximal} \ \&\ \text{not } +\text{-maximal in } y\,, \text{ and}\\ &y_2' \subseteq^- y'' \ \&\ (y'' \setminus y_2') \cap y = \varnothing \ \&\\ &\qquad\qquad y_2' \text{ is } +\text{-maximal} \ \&\ \text{not } -\text{-maximal in } y\,. \end{aligned}$$

As

$$y_1' \subseteq^+ y'' \supseteq^- y_2'$$

we deduce ${y_2'}^- \subseteq y_1'$, *i.e.* all the $-$ve events of y_2' are in y_1'. Now let $a \in {y_2'}^+$. Then $a \in y$ as $y_2' \subseteq y$. Therefore $a \notin y'' \setminus y_1'$, by assumption. But $a \in y''$ as $y_2' \subseteq^- y''$, so $a \in y'_1$. We conclude $y'_2 \subseteq y'_1$. A similar dual argument shows $y'_1 \subseteq y'_2$. Thus $y'_1 = y'_2$. But this implies that y'_1 is both −-maximal and not −maximal in y —a contradiction.

Suppose both the conditions (i) and (iv) are met by y''. From (vi), as y' is +-maximal & not −-maximal in y,

$$y' \overset{a}{\longrightarrow\!\!\!\!\subset} y_0 \subseteq y\,,$$

for some event a with $pol_A(a) = -$ and $y_0 \in \mathcal{C}^\infty(A)$. From (i), $y \subseteq y''$, so

$$y' \overset{a}{\longrightarrow\!\!\!\!\subset} y_0 \subseteq y''\,.$$

Therefore

$$a \in y'' \setminus y' \;\&\; a \in y\,,$$

which contradicts (iv). Similarly the cases (ii) and (iii) are disjoint.

We conclude that the assignment of winning conditions is well-defined.

Then y is reachable for both winning strategies in (A, W) and winning strategies in $(A, W)^\perp$. Suppose σ is a winning strategy σ in (A, W). By (iii) and Lemma 9.15, y is σ-reachable. From receptivity y_1 is σ-reachable, say $y_1 = \sigma x_1$ for some $x_1 \in \mathcal{C}(S)$. There is a +-maximal extension x'_1 of x_1 in $\mathcal{C}^\infty(S)$. By (i), $\sigma x'_1$ cannot be a winning configuration. Hence there can be no winning strategy in (A, W). In a dual fashion, there can be no winning strategy in $(A, W)^\perp$. □

It is tempting to believe that a nondeterministic winning strategy always has a winning (weakly-)deterministic sub-strategy. However, this is not so, as the following examples show.

Example 9.17. A winning strategy need not have a winning deterministic sub-strategy. Consider the game (A, W) where A consists of two inconsistent events $\ominus$ and $\oplus$, of the indicated polarity, and $W = \{\{\ominus\}, \{\oplus\}\}$. Consider the strategy σ in A given by the identity map $\mathrm{id}_A : a \to A$. Then σ is a nondeterministic winning strategy—all +-maximal configurations in A are winning. However any sub-strategy must include $\ominus$ by receptivity and cannot include $\oplus$ if it is to be deterministic, wherepon it has $\varnothing$ as a +-maximal configuration which is not winning.

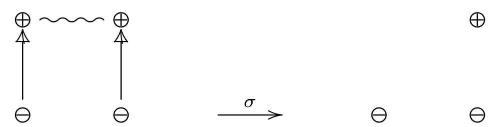
Example 9.18. Observe that the strategy σ of Example 9.17 is already weakly-deterministic—*cf.* Corollary 5.7. A winning strategy need not have a winning *weakly*-deterministic sub-strategy. Consider the game (A, W) where A consists of two −ve events $1, 2$ and one +ve event 3 all consistent with each other and

$$W = \{\varnothing, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Let S be the event structure

$$\begin{tikzcd} \oplus \arrow[r, no head, squiggly] & \oplus \\ \ominus \arrow[u] & \ominus \arrow[u] \end{tikzcd}$$

and $\sigma : S \to A$ the only possible total map of event structures with polarity:



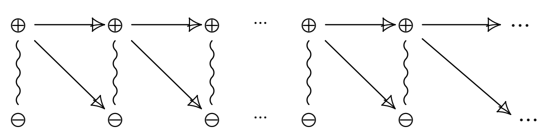
Then σ is a winning strategy for which there is no weakly-deterministic sub-strategy.

The following example shows that for games where configurations can have infinitely many events, race-freeness is not sufficient to ensure determinacy. It also shows that the existence of a winning receptive pre-strategy does not imply that there is a winning strategy.

Example 9.19. Consider the infinite game A comprising the event structure with polarity

$$\ominus \qquad \oplus \rightarrowtriangle \oplus \rightarrowtriangle \oplus \rightarrowtriangle \cdots \rightarrowtriangle \oplus \rightarrowtriangle \cdots$$

where Player wins iff
(i) Player plays all $\oplus$ moves and Opponent does nothing, or
(ii) Player plays finitely many $\oplus$ moves and Opponent plays $\ominus$.
In this case there is a winning *pre-strategy* for Player. Informally, this is to continue playing moves until Opponent moves, then stop. Formally, it is described by the event structure with polarity S



with pre-strategy the unique total map to A. The pre-strategy is receptive and winning in the sense that its $+$-maximal configurations image to winning configurations in A. It follows that there is no winning strategy for Opponent: if σ is a winning receptive pre-strategy then $\langle \sigma, \tau \rangle$ will be a subset of winning configurations, exactly as in the proof of Lemma 9.4, so must result in a loss for τ, which cannot be winning. Nor is there a winning strategy for Player. Suppose $\sigma : S \to A$ was a winning strategy for Player; for σ to win against the empty strategy there must be $x \in S$ such that σx comprises all +ve events of A. But now, using receptivity and $-$-innocence, there must be $s \in S$ such that $\sigma(s) = \ominus$ with $x \cup \{s\} \in \mathcal{C}^\infty(S)$ losing and $+$-maximal—a contradiction. □

9.6 Determinacy for well-founded games

Definition 9.20. *A game A is* well-founded *if every configuration in $\mathcal{C}^\infty(A)$ is finite.*

It is shown that any well-founded concurrent game satisfying (**race-free**) is determined.

9.6.1 Preliminaries

Proposition 9.21. *Let $\mathcal{Q}$ be a non-empty family of finite partial orders closed under rigid inclusions, i.e. if $q \in \mathcal{Q}$ and $q' \hookrightarrow q$ is a rigid inclusion (regarded as a map of event structures) then $q' \in \mathcal{Q}$. The family $\mathcal{Q}$ determines an event structure $(P, \leq, \mathrm{Con})$ as follows:*

- *the events P are the prime partial orders in $\mathcal{Q}$, i.e. those finite partial orders in $\mathcal{Q}$ with a top element;*
- *the causal dependency relation $p' \leq p$ holds precisely when there is a rigid inclusion from $p' \hookrightarrow p$;*
- *a finite subset $X \subseteq P$ is consistent, $X \in \mathrm{Con}$, iff there is $q \in \mathcal{Q}$ and rigid inclusions $p \hookrightarrow q$ for all $p \in X$.*

If $x \in \mathcal{C}(P)$ then $\bigcup x$, the union of the partial orders in x, is in $\mathcal{Q}$. The function $x \mapsto \bigcup x$ is an order-isomorphism from $\mathcal{C}(P)$, ordered by inclusion, to $\mathcal{Q}$, ordered by rigid inclusions.

Call a non-empty family of finite partial orders closed under rigid inclusions a *rigid family.* Observe:

Proposition 9.22. *Any stable family $\mathcal{F}$ determines a rigid family: its configurations x possess a partial order $\leq_x$ such that whenever $x \subseteq y$ in $\mathcal{F}$ there is a rigid inclusion $(x, \leq_x) \hookrightarrow (y, \leq_y)$ between the corresponding partial orders.*

Notation 9.23. *We shall use $\mathrm{Pr}(\mathcal{Q})$ for the construction described in Proposition 9.21. The construction extends that on stable families with the same name.*

Lemma 9.24. *Let $\sigma : S \to A$ be a strategy. Letting $x, y \in \mathcal{C}(S)$,*

$$x^+ \subseteq y^+ \ \&\ \sigma x \subseteq \sigma y \implies x \subseteq y\,.$$

Proof. The proof relies on Lemma 4.21, characterising strategies. We first prove two special cases of the lemma.
Special case $\sigma x \subseteq^- \sigma y$. By assumption $x^+ \subseteq y^+$. Supposing $s \in y^+ \setminus x^+$, via the injectivity of σ on y, we obtain $\sigma y \setminus \sigma x$ contains $\sigma(s)$ a +ve event—a contradiction. Hence $x^+ = y^+$.

From Lemma 4.21(ii), as $\sigma x \subseteq^- \sigma y$, we obtain (a unique) $x' \in \mathcal{C}(S)$ such that $x \subseteq x'$ and $\sigma x' = \sigma y$:

$$\begin{tikzcd} x \arrow[r, phantom, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, mapsto, dotted, "\sigma"] \\ \sigma x \arrow[r, phantom, "\subseteq^-"] & \sigma y\,. \end{tikzcd}$$

Now $[x^+] \subseteq^- x$, from which

$$\begin{tikzcd} {[x^+]} \arrow[r, phantom, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x \arrow[d, mapsto, "\sigma"] \\ \sigma[x^+] \arrow[r, phantom, "\subseteq^-"] & \sigma x\,. \end{tikzcd}$$

Combining the two diagrams:

$$\begin{tikzcd}
{[x^+]} \arrow[r, phantom, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, mapsto, "\sigma"] \\
\sigma[x^+] \arrow[r, phantom, "\subseteq^-"] & \sigma y\,.
\end{tikzcd}$$

As $[y^+] \subseteq^- y$,

$$\begin{tikzcd}
{[y^+]} \arrow[r, phantom, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & y \arrow[d, mapsto, "\sigma"] \\
\sigma[y^+] \arrow[r, phantom, "\subseteq^-"] & \sigma y\,.
\end{tikzcd}$$

where, by Lemma 4.21(ii), y is the unique such configuration of S. But $y^+ = x^+$ so this same property is shared by x'. Hence $x' = y$ and $x \subseteq y$.

Thus

$$x^+ \subseteq y^+ \;\&\; \sigma x \subseteq^- \sigma y \implies x \subseteq y\,. \tag{1}$$

Note that, in particular,

$$x^+ = y^+ \;\&\; \sigma x = \sigma y \implies x = y\,. \tag{2}$$

Special case $\sigma x \subseteq^+ \sigma y$. By Lemma 4.21(i), there is (a unique) $y_1 \in \mathcal{C}(S)$ with $y_1 \subseteq y$ such that $\sigma y_1 = \sigma x$:

$$\begin{tikzcd}
y_1 \arrow[r, dotted, no head, "\subseteq"] \arrow[d, mapsto, dotted, "\sigma"'] & y \arrow[d, mapsto, "\sigma"] \\
\sigma x \arrow[r, phantom, "\subseteq^+"] & \sigma y\,,
\end{tikzcd}$$

Now $x^+, y_1^+ \subseteq y$ and $\sigma\, x^+ = (\sigma x)^+ = \sigma\, y_1^+$. So by the local injectivity of σ we obtain $x^+ = y_1^+$. By (2) above, $x = y_1$, whence $x \subseteq y$. Thus

$$x^+ \subseteq y^+ \;\&\; \sigma x \subseteq^+ \sigma y \implies x \subseteq y\,. \tag{3}$$

Any inclusion $\sigma x \subseteq \sigma y$ can be built as a composition of inclusions $\subseteq^-$ and $\subseteq^+$, so the lemma follows from the special cases (1) and (3). □

Lemma 9.25. *Let $\sigma : S \to A$ be a strategy for which no +ve event of S appears as a −ve event in A. Defining*

$$\mathcal{F}_\sigma \;=_{\text{def}}\; \{x^+ \cup (\sigma x)^- \mid x \in \mathcal{C}(S)\}$$

yields a stable family for which

$$\alpha_\sigma(s) = \begin{cases} s & \text{if } s \text{ is +ve,} \\ \sigma(s) & \text{if } s \text{ is −ve.} \end{cases}$$

is a map of stable families $\alpha_\sigma : \mathcal{C}(S) \to \mathcal{F}_\sigma$ *which induces an order-isomorphism*

$$(\mathcal{C}(S), \subseteq) \cong (\mathcal{F}_\sigma, \subseteq)$$

taking $x \in \mathcal{C}(S)$ *to* $\alpha_\sigma\, x = x^+ \cup (\sigma x)^-$. *Defining*

$$f_\sigma(e) = \begin{cases} \sigma(e) & \text{if } e \text{ is +ve,} \\ e & \text{if } e \text{ is −ve} \end{cases}$$

on events e *of* $\mathcal{F}_\sigma$ *yields a map of stable families* $f_\sigma : \mathcal{F}_\sigma \to \mathcal{C}(A)$ *such that*

$$\begin{tikzcd} \mathcal{C}(S) \arrow[r, "\alpha_\sigma"] \arrow[dr, "\sigma"'] & \mathcal{F}_\sigma \arrow[d, "f_\sigma"] \\ & \mathcal{C}(A) \end{tikzcd}$$

commutes.

Proof. A configuration $x \in \mathcal{C}(S)$ has direct image

$$\alpha_\sigma x = x^+ \cup (\sigma x)^-$$

under the function α_σ. Direct image under α_σ is clearly surjective and preserves inclusions, and by Lemma 9.24 yields an order-isomorphism $(\mathcal{C}(S), \subseteq) \cong (\mathcal{F}_\sigma, \subseteq)$: if $\alpha_\sigma x \subseteq \alpha_\sigma y$, for $x, y \in \mathcal{C}(S)$, then $x^+ \subseteq y^+$ and $(\sigma x)^- \subseteq (\sigma y)^-$ by the disjointness of S^+ and A, whence $\sigma x \subseteq \sigma y$ so $x \subseteq y$.

It is now routine to check that $\mathcal{F}_\sigma$ is a stable family and α_σ is a map of stable families. For instance to show the stability property required of $\mathcal{F}_\sigma$, assume $\alpha_\sigma x, \alpha_\sigma y \subseteq \alpha_\sigma z$. Then $x, y \subseteq z$ so $\sigma\, x \cap y = (\sigma x) \cap (\sigma y)$ as σ is a map of event structures, and consequently $(\sigma\, x \cap y)^- = (\sigma x)^- \cap (\sigma y)^-$. Now reason

$$\begin{aligned} (\alpha_\sigma x) \cap (\alpha_\sigma y) =& (x^+ \cup (\sigma x)^-) \cap (y^+ \cup (\sigma y)^-) \\ =& (x^+ \cap y^+) \cup ((\sigma x)^- \cap (\sigma y)^-) \\ & \text{—by distributivity with the disjointness of } S^+ \text{ and } A\,, \\ =& (x \cap y)^+ \cup (\sigma\, x \cap y)^- \\ =& (\alpha_\sigma\, x \cap y) \in \mathcal{F}_\sigma\,. \end{aligned}$$

From the definitions of α_σ and f_σ it is clear that $f_\sigma \alpha_\sigma(s) = \sigma(s)$ for all events of S. Any configuration of $\mathcal{F}_\sigma$ is sent under f_σ to a configuration in $\mathcal{C}(A)$ in a locally injective fashion, making f_σ a map of stable families; this follows from the matching properties of σ. □

When we "glue" strategies together it can be helpful to assume that all the initial −ve moves of the strategies are exactly the same:

Lemma 9.26. *Let* $\sigma : S \to A$ *be a strategy. Then* $\sigma \cong \sigma'$, *a strategy* $\sigma' : S' \to A$ *for which*

$$\forall s' \in S'.\ pol_{S'}[s']_{S'} = \{-\} \implies s' = [\sigma(s')]_A\,.$$

Proof. Without loss of generality we may assume no +ve event of S appears as a −ve event in A. Take $f_\sigma : \mathcal{F}_\sigma \to \mathcal{C}(A)$ given by Lemma 9.26 and construct σ' as the composite map

$$\mathrm{Pr}(\mathcal{F}_\sigma) \xrightarrow{\mathrm{Pr}(\sigma)} \mathrm{Pr}(\mathcal{C}(A)) \overset{top}{\cong} A$$

—recall *top* takes a prime $[a]_A$ to a, where $a \in A$. □

9.7 Determinacy proof

Definition 9.27. *Let A be an event structure with polarity. Let $W \subseteq \mathcal{C}^\infty(A)$. Let $y \in \mathcal{C}^\infty(A)$. Define A/y to be the event structure with polarity comprising events*

$$\{a \in A \setminus y \mid y \cup [a]_A \in \mathcal{C}^\infty(A)\}\,,$$

also called A/y, with consistency relation

$$X \in \mathrm{Con}_{A/y} \text{ iff } X \subseteq_{\text{fin}} A/y \ \&\ y \cup [X]_A \in \mathcal{C}^\infty(A)\,,$$

and causal dependency the restriction of that on A. Define $W/y \subseteq \mathcal{C}^\infty(A/y)$ by

$$z \in W/y \text{ iff } z \in \mathcal{C}^\infty(A/y) \ \&\ y \cup z \in W\,.$$

Finally, define $(A, W)/y =_{\text{def}} (A/y, W/y)$.

Proposition 9.28. *Let A be an event structure with polarity and $y \in \mathcal{C}^\infty(A)$. Then,*

$$z \in \mathcal{C}^\infty(A/y) \text{ iff } z \subseteq A/y \ \&\ y \cup z \in \mathcal{C}^\infty(A)\,.$$

Assume A is a *well-founded* event structure with polarity with winning conditions $W \subseteq \mathcal{C}(A)$. Assume the property (**race-free**) of A:

$$\forall y, y_1, y_2 \in \mathcal{C}(A).\ y \overset{-}{\longrightarrow\!\!\subset} y_1 \ \&\ y \overset{+}{\longrightarrow\!\!\subset} y_2 \implies y_1 \uparrow y_2\,. \qquad \textbf{(race-free)}$$

Observe that by repeated use of (**race-free**), if $x, y \in \mathcal{C}(A)$ with $x \cap y \subseteq^+ x$ and $x \cap y \subseteq^- y$, then $x \cup y \in \mathcal{C}(A)$.

We show that the game (A, W) is determined. Assuming Player has no winning strategy we build a winning (counter) strategy for Opponent based on the following lemma.

Lemma 9.29. *Assume game A is well-founded and satisfies* (**race-free**). *Let $W \subseteq \mathcal{C}(A)$. Assume (A, W) has no winning strategy (for Player). Then,*

$$\begin{array}{l} \forall x \in \mathcal{C}(A).\ \varnothing \subseteq^+ x \ \&\ x \in W \\ \Longrightarrow \\ \exists y \in \mathcal{C}(A).\ x \subseteq^- y \ \&\ y \notin W \ \&\ (A, W)/y \textit{ has no winning strategy.} \end{array}$$

Proof. Suppose otherwise, that under the assumption that (A, W) has no winning strategy, there is some $x \in \mathcal{C}(A)$ such that

$$\begin{aligned}&\varnothing \subseteq^+ x \ \&\ x \in W\\ &\&\\ &\forall y \in \mathcal{C}(A).\ x \subseteq^- y \ \&\ y \notin W \implies (A, W)/y \text{ has a winning strategy.}\end{aligned}$$

We shall establish a contradiction by constructing a winning strategy for Player.

For each $y \in \mathcal{C}(A)$ with $x \subseteq^- y$ and $y \notin W$, choose a winning strategy

$$\sigma_y : S_y \to A/y\,.$$

By Lemma 9.26, we can replace σ_y by a stable family $\mathcal{F}_y$ with all −ve events in A and a map of stable families $f_y : \mathcal{F}_y \to \mathcal{C}(A)$. It is easy to arrange that, within the collection of all such stable families, $\mathcal{F}_{y_1}$ and $\mathcal{F}_{y_2}$ are disjoint on +ve events whenever y_1 and y_2 are distinct. We build a putative stable family as

$$\begin{aligned}\mathcal{F} \ =_{\text{def}}\ &\{y \in \mathcal{C}(A) \mid pol_A(y \setminus x) \subseteq \{-\}\} \ \cup\\ &\{y \cup v \mid y \in \mathcal{C}(A) \ \&\ pol_A(y \setminus x) \subseteq \{-\} \ \&\ x \cup y \notin W \ \&\\ &\qquad v \in \mathcal{F}_{x\cup y} \ \&\ + \in pol\, v \ \&\ y \cup f_{x\cup y} v \in \mathcal{C}(A)\}\,.\end{aligned}$$

[Note, in the second set-component, that $x \cup y$ is a configuration by (**race-free**).] We assign events of $\mathcal{F}$ the same polarities they have in A and the families $\mathcal{F}_y$.

We check that $\mathcal{F}$ is indeed a stable family.

Clearly $\varnothing \in \mathcal{F}$. Assuming $z_1, z_2 \subseteq z$ in $\mathcal{F}$, we require $z_1 \cup z_2$, $z_1 \cap z_2 \in \mathcal{F}$.

It is easily seen that if both z_1 and z_2 belong to the first set-component, so do their union and intersection. Suppose otherwise, without loss of generality, that z_2 belongs to the second set-component. Then, necessarily, z is in the second set-component of $\mathcal{F}$ and has the form $z = y \cup v$ described there.

Consider the case where $z_1 = y_1 \cup v_1$ and $z_2 = y_2 \cup v_2$, both belonging to the second set-component of $\mathcal{F}$. Then

$$x \cup y_1 = x \cup y_2 = x \cup y\,,$$

from the assumption that families $\mathcal{F}_y$ are disjoint on +ve events for distinct y, and

$$v_1, v_2 \subseteq v \text{ in } \mathcal{F}_{x\cup y}\,.$$

It follows that $x \cup (y_1 \cup y_2) = x \cup y \notin W$ and $v_1 \cup v_2 \in \mathcal{F}_{x\cup y} = \mathcal{F}_{x\cup(y_1\cup y_2)}$. As $z_1, z_z \subseteq z$,

$$(y_1 \cup f_{x\cup y} v_1),\ (y_2 \cup f_{x\cup y} v_2) \subseteq (y \cup f_{x\cup y} v)$$

so

$$(y_1 \cup y_2) \cup f_{x\cup y}(v_1 \cup v_2) = (y_1 \cup f_{x\cup y} v_1) \cup (y_2 \cup f_{x\cup y} v_2) \in \mathcal{C}(A)\,.$$

This ensures $z_1 \cup z_2 = (y_1 \cup y_2) \cup (v_1 \cup v_2) \in \mathcal{F}$. Similarly, $x \cup (y_1 \cap y_2) = (x \cup y_1) \cap (x \cup y_2) = x \cup y \notin W$ and $v_1 \cap v_2 \in \mathcal{F}_{x\cup y} = \mathcal{F}_{x\cup(y_1\cap y_2)}$. Checking

$$(y_1 \cap y_2) \cup f_{x\cup y}(v_1 \cap v_2) = (y_1 \cup f_{x\cup y} v_1) \cap (y_2 \cup f_{x\cup y} v_2) \in \mathcal{C}(A)$$

ensures $z_1 \cap z_2 = (y_1 \cap y_2) \cup (v_1 \cap v_2) \in \mathcal{F}$.

Consider the case where $z_1 \in \mathcal{C}(A)$ belongs to the first and $z_2 = y_2 \cup v_2$ to the second set-component of $\mathcal{F}$. As $z_1 \subseteq y \cup v$ it has the form $z_1 = y_1 \cup v_1$ where $y_1 \in \mathcal{C}(A)$ with $y_1 \subseteq y$ and $v_1 \in \mathcal{F}_{x\cup y}$ with $v_1 \subseteq v$; all the events of $v_1 = z_1 \setminus (x \cup y)$ have $-$ve polarity which ensures $v_1 \in \mathcal{F}_{x\cup y}$ by the receptivity of σ_y. Because v_2 and v have +ve events in common,

$$x \cup y_2 = x \cup y\,,$$

while clearly

$$v_1, v_2 \subseteq v \text{ in } \mathcal{F}_{x\cup y}\,.$$

We deduce $x \cup (y_1 \cup y_2) = x \cup y \notin W$ and $v_1 \cup v_2 \in \mathcal{F}_{x\cup y} = \mathcal{F}_{x\cup(y_1\cup y_2)}$ whence $z_1 \cup z_2 = (y_1 \cup y_2) \cup (v_1 \cup v_2) \in \mathcal{F}$ after an easy check that $(y_1 \cup y_2) \cup f_{x\cup y}(v_1 \cup v_2) \in \mathcal{C}(A)$. We have $y_2 \cup f_{x\cup y} v_2 \in \mathcal{C}(A)$. But $f_{x\cup y}$ is constant on $-$ve events so

$$z_1 \cap z_2 = z_1 \cap (y_2 \cup v_2) = z_1 \cap (y_2 \cup f_{x\cup y} v_2) \in \mathcal{C}(A)\,,$$

and $z_1 \cap z_2$ belongs to the first set-component of $\mathcal{F}$.

A routine check establishes that $\mathcal{F}$ is coincidence-free, and uses that each family $\mathcal{F}_y$ is coincidence-free when considering configurations of the second set-component.

Having established that $\mathcal{F}$ is a stable family, we define a total map of stable families

$$f : \mathcal{F} \to \mathcal{C}(A)$$

by taking

$$f(e) = \begin{cases} e & \text{if } e \in x \text{ or } e \text{ is } -\text{ve}, \\ f_y(e) & \text{if } e \text{ is a +ve event of } \mathcal{F}_y. \end{cases}$$

Defining σ to be the composite map of stable families

$$\mathcal{C}(\Pr(\mathcal{F})) \xrightarrow{top} \mathcal{F} \xrightarrow{f} \mathcal{C}(A)$$

we also obtain a map of event structures

$$\sigma : \Pr(\mathcal{F}) \to A$$

as the embedding of event structures in stable families is full and faithful. Ascribe to events p of $\Pr(\mathcal{F})$ the same polarities as events $top(p)$ of $\mathcal{F}$. Clearly σ preserves polarities as f does, so σ is a total map of event structures with polarity. In fact, σ is a winning strategy for (A, W).

To show receptivity of σ it suffices to show for all $z \in \mathcal{F}$ that $fz \overset{-}{\longrightarrow\!\!\!\!\subset} y'$ in $\mathcal{C}(A)$ implies $z \overset{z}{\longrightarrow\!\!\!\!\subset}{}'$ with $\sigma z' = z$ for some unique $z' \in \mathcal{F}$. If z belongs to the first set-component of $\mathcal{F}$ this is obvious—take $z' = y'$. Otherwise z belongs to the second set-component, and takes the form $y \cup v$, when receptivity follows from the receptivity of $\sigma_{x\cup y}$. No extra causal dependencies, over those of A, are introduced into y in the first set-component of $\mathcal{F}$. Considering $y \cup v$ in the second set-component of $\mathcal{F}$, the only extra causal dependencies introduced in $y \cup v$, above those inherited from its image $y \cup f_{x\cup y}v$ in A, are from v in $\mathcal{F}_{x\cup y}$ and those making a +ve event of v in $y \cup v$ depend on −ve events $y \setminus x$. For these reasons σ is also innocent, and a strategy in A.

To show σ is a winning strategy for (A, W) it suffices to show that $fz \in W$ for every $+$-maximal configuration $z \in \mathcal{F}$. Let z be a $+$-maximal configuration of $\mathcal{F}$.

Suppose that z belongs to the first set-component of $\mathcal{F}$ and, to obtain a contradiction, that $fz \notin W$. Then $z = fz \in \mathcal{C}(A)$ and $pol\, z \setminus x \subseteq \{-\}$. By axiom (**race-free**), $x \uparrow y$, so $x \subseteq z$ from the $+$-maximality of z. As $x \subseteq^- z$ and $z \notin W$ the strategy σ_z is winning in $(A, W)/z$. Because z is $+$-maximal in $\mathcal{F}$ we must have $\varnothing$ is $+$-maximal in $\mathcal{F}_z$. It follows that $\varnothing \in W/z$, *i.e.* $z \in W$—a contradiction.

Suppose that z belongs to the second set-component of $\mathcal{F}$, so that z has the form $y \cup v$ with $y \in \mathcal{C}(A)$ and $v \in \mathcal{F}_{x\cup y}$. By (**race-free**), $x \subseteq y$, as z is $+$-maximal in $\mathcal{F}$. Hence $v \in \mathcal{F}_y$ and is necessarily $+$-maximal in $\mathcal{F}_y$, again from the $+$-maximality of z. As σ_y is winning, $f_y v \in W/y$. Therefore $fz = y \cup f_y v \in W$.

Finally, we have constructed a winning strategy σ in (A, W)—the contradiction required to establish the lemma. □

Remark. In the proof above we could instead build the strategy for Player, on which the proof by contradiction depends, out of a rigid family of finite partial orders. Recall that stable families, including configurations of event structures, are rigid families w.r.t. the order induced on configurations; finite configurations x determine finite partial orders $(x, \leq_x)$, which we call $q(x)$ in the construction below. Define

$$\begin{aligned}\mathcal{Q} \;=_{\mathrm{def}}\; & \{q(y) \mid y \in \mathcal{C}(A) \;\&\; pol_A(y \setminus x) \subseteq \{-\}\} \;\cup \\ & \{q(y); q(v) \mid y \in \mathcal{C}(A) \;\&\; pol_A(y \setminus x) \subseteq \{-\} \;\&\; x \cup y \notin W \;\& \\ & \qquad v \in \mathcal{F}_{x\cup y} \;\&\; + \in pol\, v \;\&\; y \cup f_{x\cup y}v \in \mathcal{C}(A)\}\end{aligned}$$

where above $q(y); q(v)$ is the least partial order on $y \cup v$ in which events inherit causal dependencies from $q(v)$, from their images in $q(y \cup f_{x\cup y}v)$ and in addition have the causal dependencies $y^- \times v^+$. The family $\mathcal{Q}$ can be shown to be closed under rigid inclusions, and so a rigid family. □

Theorem 9.30. *Assume game A is well-founded, satisfies* (**race-free**) *and has winning conditions $W \subseteq \mathcal{C}(A)$. If (A, W) has no winning strategy for Player, then there is a winning (counter) strategy for Opponent.*

Proof. Assume (A, W) has no winning strategy for Player.

We build a winning counter-strategy for Opponent out of a rigid family of partial orders, themselves constructed from 'alternating sequences' of configurations of A.

Define an *alternating sequence* to be a sequence

$$x_1, y_1, x_2, y_2, \cdots, x_i, y_i, \cdots, x_k, y_k, x_{k+1}$$

of length $k+1 \geq 1$ of configurations of A such that

$$\varnothing \subseteq^+ x_1 \subseteq^- y_1 \subseteq^+ x_2 \subseteq^- y_2 \subseteq^- \cdots \subseteq^+ x_i \subseteq^- y_i \subseteq^+ \cdots \subseteq^+ x_k \subseteq^- y_k \subseteq^+ x_{k+1}$$

with

$$x_i \in W \ \& \ y_i \notin W \ \& \ (A,W)/y_i \text{ has no winning strategy,}$$

when $1 \leq i \leq k$. It is important that x_{k+1}, which may be $\varnothing$, need not be in W. In particular, we allow the alternating singleton sequence x_1 comprising a single configuration of A with $\varnothing \subseteq^+ x_1$ without necessarily having $x_1 \in W$.

For each alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$ define the partial order $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ to comprise the partial order on x_{k+1} inherited from A together with additional causal dependencies given by the pairs in

$$x_i^+ \times (y_i \smallsetminus x_i)\,, \text{ where } 1 \leq i \leq k.$$

We define $\mathcal{Q}$ to be the rigid family comprising the set of all partial orders got from alternating sequences, closed under rigid inclusions.

Form the event structure $\mathrm{Pr}(\mathcal{Q})$ as described in Proposition 9.21. Assign the same polarity to an event in $\mathrm{Pr}(\mathcal{Q})$ as its top event in A. Recall from Proposition 9.21 the order-isomorphism $\mathcal{C}(\mathrm{Pr}(\mathcal{Q})) \cong \mathcal{Q}$ given by $x \mapsto \bigcup x$ for $x \in \mathcal{C}(\mathrm{Pr}(\mathcal{Q}))$. The map

$$\tau : \mathrm{Pr}(\mathcal{Q}) \to A$$

taking $p \in \mathrm{Pr}(\mathcal{Q})$ to its top event is a total map of event structures with polarity. Writing $T : \mathcal{Q} \to \mathcal{C}(A)$ for the function taking $q \in \mathcal{Q}$ to its set of underlying events, $\tau x = T(\bigcup x)$ for all $x \in \mathcal{C}(\mathrm{Pr}(\mathcal{Q}))$, *i.e.* the diagram

$$\begin{tikzcd}
\mathcal{C}(\mathrm{Pr}(\mathcal{Q})) \arrow[r, phantom, "\cong"] \arrow[dr, "\tau"'] & \mathcal{Q} \arrow[d, "T"] \\
 & \mathcal{C}(A)
\end{tikzcd}$$

commutes. We shall reason about order-properties of τ via the function T.

We claim that τ is a winning counter-strategy, in other words a winning strategy for Opponent, in which the roles of $+$ and $-$ are reversed.

Because the construction of the partial orders in $\mathcal{Q}$ only introduces extra causal dependencies of $-$ve events on $+$ve events, τ is innocent (remember the reversal of polarities). To check receptivity of τ it suffices to show that for $q \in \mathcal{Q}$ assuming $T(q) \overset{a}{-\!\!\!\subset} z'$ in $\mathcal{C}(A)$, where $pol_A(a) = +$, there is a unique $q' \in \mathcal{Q}$ such that $q \,-\!\!\!\subset\, q'$ and $T(q') = z'$. Any such extension q' must comprise the partial order q extended by the event a. As a is $+$ve the events on which it immediately depends in q' will coincide with those on which a immediately depends in z', guaranteeing the uniqueness of q'. It remains to show the existence of q'.

By assumption, q rigidly embeds in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ for some alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$. In the case where q consists of purely

+ve events, take $q' =_{\mathrm{def}} Q(z')$. Otherwise, consider the largest i for which $T(q) \cap (y_i \smallsetminus x_i) \neq \varnothing$. Then,

$$pol_A\, T(q) \smallsetminus y_i \subseteq \{+\}\,. \tag{1}$$

From the construction of $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ and the rigidity of the inclusion of q in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ we obtain

$$x_i^+ \subseteq T(q)\,. \tag{2}$$

From (2), $T(q) \subseteq^- T(q) \cup y_i$ and, by assumption, $T(q) \xrightarrow{a}\!\!\subset z'$ with $pol_A(a) = +$. Using (**race-free**), their union remains in $\mathcal{C}(A)$, and we can define

$$x' =_{\mathrm{def}} T(q) \cup y_i \cup \{a\} \in \mathcal{C}(A)\,.$$

Note that

$$x_1, y_1, \cdots, x_i, y_i, x'$$

is an alternating sequence because $y_i \subseteq^+ x'$ by (1) and it is built from an alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$. Restricting $Q(x_1, y_1, \cdots, x_i, y_i, x')$ to events z we obtain a partial order q' for which $q \longrightarrow\!\!\subset q'$ in $\mathcal{Q}$ and $T(q') = z$.

We now show that τ is winning for Opponent. For this it suffices to show that if $q \in \mathcal{Q}$ is $-$-maximal then $T(q) \notin W$. Assume $q \in \mathcal{Q}$ is $-$-maximal in $\mathcal{Q}$. Necessarily q embeds rigidly in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ for some alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$.

In the case where q consists of purely +ve events

$$\varnothing \subseteq^+ T(q) \text{ in } \mathcal{C}(A)\,.$$

Suppose $T(q) \in W$. By Lemma 9.29, for some $y \in \mathcal{C}(A)$,

$$T(q) \subseteq^- y \ \&\ y \notin W\,.$$

But then there is a strict extension $q \hookrightarrow Q(T(q), y, \varnothing)$ of q by $-$ve events in $\mathcal{Q}$, and q is not $-$-maximal—a contradiction.

In the case where q has $-$ve events, we may take the largest i for which $T(q) \cap (y_i \smallsetminus x_i) \neq \varnothing$. As earlier,

$$(1)\ pol_A\, T(q) \smallsetminus y_i \subseteq \{+\} \quad \& \quad (2)\ x_i^+ \subseteq T(q)\,.$$

As q is $-$-maximal, $y_i \subseteq T(q)$, whence by (1),

$$y_i \subseteq^+ T(q)\,.$$

Suppose, to obtain a contradiction, that $T(q) \in W$. The game $(A, W)/y_i$ has no winning strategy. By Lemma 9.29, given

$$\varnothing \subseteq^+ x =_{\mathrm{def}} T(q) \smallsetminus y_i$$

in $\mathcal{C}((A,W)/y_i)$ there is $y \in \mathcal{C}((A,W)/y_i)$ with

$$x \subseteq^- y \quad \& \quad y \notin W/y_i\,.$$

Let $x'_{i+1} =_{\mathrm{def}} T(q)$ and $y'_{i+1} =_{\mathrm{def}} y_i \cup y \notin W$. Then,

$$x_1, y_1, \cdots, x_i, y_i, x'_{i+1}, y'_{i+1}, \varnothing$$

is an alternating sequence which strictly extends q by −ve events, contradicting its −-maximality.

We conclude that τ is a winning strategy for Opponent. □

Corollary 9.31. *If a well-founded game A satisfies* (**race-free**) *then* (A,W) *is determined for any winning conditions W.*

9.8 Satisfaction in the predicate calculus

The syntax for predicate calculus: formulae are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists x.\ \phi \mid \forall x.\ \phi$$

where R ranges over basic relation symbols of a fixed arity and $x, x_1, x_2, \cdots, x_k$ over variables.

A model M for the predicate calculus comprises a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M. Following Tarski we can then define by structural induction the truth of a formula of predicate logic w.r.t. an assignment of values in V_M to the variables of the formula. We write

$$\rho \vDash_M \phi$$

iff formula ϕ is true in M w.r.t. environment ρ; we take an environment to be a function from variables to values.

W.r.t. a model M and an environment ρ, we can denote a formula ϕ by $[\![\phi]\!]_M\rho$, a concurrent game with winning conditions, so that $\rho \vDash_M \phi$ iff the game $[\![\phi]\!]_M\rho$ has a winning strategy.

The denotation as a game is defined by structural induction:

$$
\begin{aligned}
[\![R(x_1, \cdots, x_k)]\!]_M\rho &= \begin{cases} (\varnothing, \{\varnothing\}) & \text{if } \rho \vDash_M R(x_1, \cdots, x_k)\,, \\ (\varnothing, \varnothing) & \text{otherwise.} \end{cases} \\
[\![\phi \wedge \psi]\!]_M\rho &= [\![\phi]\!]_M\rho \otimes [\![\psi]\!]_M\rho \\
[\![\phi \vee \psi]\!]_M\rho &= [\![\phi]\!]_M\rho \parr [\![\psi]\!]_M\rho \\
[\![\neg\phi]\!]_M\rho &= ([\![\phi]\!]_M\rho)^\perp \\
[\![\exists x.\ \phi]\!]_M\rho &= \bigoplus_{v \in V_M} [\![\phi]\!]_M\rho[v/x] \\
[\![\forall x.\ \phi]\!]_M\rho &= \bigominus_{v \in V_M} [\![\phi]\!]_M\rho[v/x]\,.
\end{aligned}
$$

We use $\rho[v/x]$ to mean the environment ρ updated to assign value v to variable x. The game $(\varnothing, \{\varnothing\})$ the unit w.r.t. $\otimes$ is the game used to denote true and the game $(\varnothing, \{\varnothing\})$ the unit w.r.t. $\parr$ to denote false. Denotations of conjunctions and disjunctions are denoted by the operations of $\otimes$ and $\parr$ on games, while negations denote dual games. Universal and existential quantifiers denote *prefixed sums* of games, operations which we now describe.

The prefixed game $\oplus.(A, W)$ comprises the event structure with polarity $\oplus.A$ in which all the events of A are made to causally depend on a fresh +ve event $\oplus$. Its winning conditions are those configurations $x \in \mathcal{C}^\infty(\oplus.A)$ of the form $\{\oplus\} \cup y$ for some $y \in W$. The game $\bigoplus_{v\in V}(A_v, W_v)$ has underlying event structure with polarity the sum (=coproduct) $\sum_{v\in V} \oplus.A_v$ with a configuration winning iff it is the image of a winning configuration in a component under the injection to the sum. Note in particular that the empty configuration of $\bigoplus_{v\in V} G_v$ is not winning—Player must make a move in order to win. The game $\bigominus_{v\in V} G_v$ is defined dually, as $(\bigoplus_{v\in V} G_v^\perp)^\perp$. In this game the empty configuration is winning but Opponent gets to make the first move. More explicitly, the prefixed game $\ominus.(A, W)$ comprises the event structure with polarity $\ominus.A$ in which all the events of A are made to causally depend on the previous occurrence of an opponent event $\ominus$, with winning configurations either the empty configuration or of the form $\{\ominus\} \cup y$ where $y \in W$. Writing $G_v = (A_v, W_v)$, the underlying event structure of $\bigominus_{v\in V} G_v$ is the sum $\sum_{v\in V} \ominus.A_v$ with a configuration winning iff it is empty or the image under injection of a winning configuration in a prefixed component.

It is easy to check by structural induction that:

Proposition 9.32. *For any formula ϕ the game $[\![\phi]\!]_M \rho$ is well-founded and race-free (i.e. satisfies Axiom* **(race-free)***), so a determined game by the result of the last section.*

The following facts are useful for building strategies.

Proposition 9.33.

(i) If $\sigma : S \to A$ is a strategy in A and $\tau : T \to B$ is a strategy in B, then $\sigma \| \tau : S \| T \to A \| B$ is a strategy in $A \| B$.

(ii) If $\sigma : S \to T$ is a strategy in T and $\tau : T \to B$ is a strategy in B, then their composition as maps of event structures with polarity $\tau\sigma : S \to B$ is a strategy in B.

Proof. It is easy to check that the properties of receptivity and innocence are preserved by parallel composition and composition of maps. $\square$

There are 'projection' strategies from a tensor product of games to its components:

Proposition 9.34. *Let $G = (A, W_G)$ and $H = (B, W_H)$ be race-free games with winning conditions. The map of event structures with polarity*

$$\mathrm{id}_{A^\perp} \| \gamma_B : A^\perp \| \mathbb{C}_B \to A^\perp \| B^\perp \| B$$

is a winning strategy $p_H : G \otimes H \twoheadrightarrow\!\!\!\!\!+ H$. *The map of event structures with polarity*

$$\mathrm{id}_{B^\perp} \| \gamma_A : B^\perp \| \mathbb{C}_A \to B^\perp \| A^\perp \| A \cong A^\perp \| B^\perp \| A$$

is a winning strategy $p_G : G \otimes H \twoheadrightarrow\!\!\!\!\!+ G$.

Proof. By Proposition 9.33, as $\mathrm{id}_{A^\perp}$ is a strategy in $A^\perp$ and γ_B is a strategy in $B^\perp \| B$ the map $p_H = \mathrm{id}_{A^\perp} \| \gamma_B$ is certainly a strategy in $A^\perp \| B^\perp \| B$.

We need to check that p_H is a winning strategy in $G \otimes H \multimap H$. Consider x, a $+$-maximal configuration of $A^\perp \| \mathbb{C}_B$. As B is race-free, the copy-cat strategy γ_B is winning in $H \multimap H$. Consequently if x images to a winning configuration in $G \otimes H$ on the left of $G \otimes H \multimap H$ it will image to a winning configuration in H on the right of $G \otimes H \multimap H$. (Recall a winning configuration of $G \otimes H$ is essentially the union of a winning configuration in G together with a winning configuration in H.) Consequently, x images to a winning configuration in $G \otimes H \multimap H$, as is required for p_H to be a winning strategy.

The strategy p_G is defined analogously but for the isomorphism $B^\perp \| A^\perp \| A \cong A^\perp \| B^\perp \| A$ which does not disturb its winning nature. □

The following lemma is used to build and deconstruct strategies in prefixed sums of games. The lemma concerns the more basic prefixed sums of event structures. These are built as coproducts $\sum_{i \in I} \bullet.B_i$ of event structures $\bullet.B_i$ in which an event $\bullet$ is prefixed to B_i, making all the events in B_i causally depend on $\bullet$.

Lemma 9.35. *Suppose* $f : A \to \sum_{i \in I} \bullet.B_i$ *is a total map of event structures, with codomain a prefixed sum. Then,* A *is isomorphic to an prefixed sum,* $A \cong \sum_{j \in J} \bullet.A_j$, *and there is a function* $r : J \to I$ *and total maps of event structures* $f_j : A_j \to B_{r(j)}$ *for which*

$$\begin{tikzcd} \sum_{j \in J} \bullet.A_j \arrow[d, "{[\bullet.f_j]_{j \in J}}"'] \arrow[r, phantom, "\cong"] & A \arrow[dl, "f"] \\ \sum_{i \in I} \bullet.B_i & \end{tikzcd}$$

commutes.

Proof. Let J be the subset of events of A whose images are prefix events $\bullet$ in $\sum_{i \in I} \bullet.B_i$. As f is a map of event structures any distinct pairs of events in J are inconsistent. Moreover, every event of A is $\le_A$-above a necessarily unique event in J. It follows that the events of J are $\le_A$-minimal with $A \cong \sum_{j \in J} \bullet.A_j$; the event structure A_j is $A/\{j\}$, that part of the event structure strictly above the event j. Each event $j \in J$ is sent to a unique prefix event $f(j)$ in $\sum_{i \in I} \bullet.B_i$. Thus f determines a function $r : J \to I$ and maps $f_j : A_j \to B_{r(i)}$ for all $j \in J$. By construction the map f is reassembled, up to isomorphism, as the unique mediating map $[\bullet.f_j]_{j\in J}$ for which

$$
\begin{tikzcd}
\bullet.A_j \arrow[r, "in_j^A"] \arrow[d, "\bullet.f_j"'] & \sum_{j\in J}\bullet.A_j \arrow[d, "{[\bullet.f_j]_{j\in J}}"'] \arrow[r, phantom, "\cong"] & A \arrow[dl, "f"] \\
\bullet.B_{r(j)} \arrow[r, "in^B_{r(j)}"'] & \sum_{i\in I}\bullet.B_i &
\end{tikzcd}
$$

commutes for all $j \in J$. $\square$

Lemma 9.36. *Let G, H, G_v, where $v \in V$, be race-free games with winning conditions. Then,*

(i) $G \otimes H$ has a winning strategy iff G has a winning strategy and H has a winning strategy.

(ii) $\bigoplus_{v\in V} G_v$ has a winning strategy iff G_v has a winning strategy for some $v \in V$.

(iii) $\ominus_{v\in V} G_v$ has a winning strategy iff G_v has a winning strategy for all $v \in V$.

If in addition G and H are determined,

(iv) $G \parr H$ has a winning strategy iff G has a winning strategy or H has a winning strategy.

Proof. Throughout write $G_v = (A_v, W_v)$, where $v \in V$.

(i) *'Only if':* If $G \otimes H$ has a winning strategy $\sigma : (\varnothing, \{\varnothing\}) \mathrel{-\!\!\!+\!\!\!\!\rightarrow} G \otimes H$, then the compositions $p_G \odot \sigma$ and $p_H \odot \sigma$ provide winning strategies in G and H, respectively. *'If':* If $G = (A, W_G)$ and $H = (B, W_H)$ have winning strategies given as maps of event structures with polarity $\sigma : S \to A$ and $\tau : T \to B$ then the map $\sigma \| \tau : S \| T \to A \| B$ is a winning strategy in $G \otimes H$.

(ii) *'Only if':* Suppose $\sigma : S \to \sum_{v\in V} \oplus.A_v$ is a winning strategy in $\bigoplus_{v\in V} G_v$. As $\varnothing$ is not winning in the game, S must be nonempty. By Lemma 9.35, S decomposes into a prefixed sum necessarily nonempty and of the form $\sum_{j\in J} \oplus.S_j$ with maps, now necessarily total maps of event structures with polarity, $\sigma_j : S_j \to A_{v(j)}$. Because σ is winning any such map will be a winning strategy in $G_{v(j)}$. *'If':* Suppose $\sigma_v : S_v \to A_v$ is a winning strategy in G_v. Prefixing we obtain $\oplus.\sigma_v : \oplus.S_v \to \oplus.A_v$, a winning strategy in $\oplus.G_v$. Composing with the winning 'injection' strategy $In_v : \oplus.G_v \mathrel{-\!\!\!+\!\!\!\!\rightarrow} \sum_{v\in V} \oplus.G_v$ defined below we obtain a winning strategy in $\bigoplus_{v\in V} G_v$. The injection strategy is built from the injection map of event structures with polarity

$$in_v : \oplus.A_v \to \sum_{v\in V} \oplus.A_v \,.$$

as the composite map

$$In_v : \mathbb{C}_{\oplus.A_v} \xrightarrow{\gamma_{\oplus.A_v}} (\oplus.A_v)^\perp \| \oplus.A_v \xrightarrow{\mathrm{id}_{(\oplus.A_v)^\perp} \| in_v} (\oplus.A_v)^\perp \| \textstyle\sum_{v\in V} \oplus.A_v \, .$$

Proposition 9.33 is used to show In_v is a strategy. It can be seen that in_v is both receptive and innocent so a strategy in $\sum_{v\in V} \oplus.A_v$. The map $\mathrm{id}_{(\oplus.A_v)^\perp}$ is a strategy. Hence $\mathrm{id}_{(\oplus.A_v)^\perp} \| in_v$ is a strategy. As the composition of two strategy maps, In_v is a strategy in $(\oplus.A_v)^\perp \| \sum_{v\in V} \oplus.A_v$. It is a winning strategy because, as is easily seen from the explicit composite form of In_v, the image under In_v of a $+$-maximal configuration in $\mathbb{C}_{\oplus.A_v}$ is winning.

(iii) *'Only if':* Defining $P_v =_{\mathrm{def}} In_v^\perp$, where $In_v : \oplus.G_v^\perp \twoheadrightarrow \bigoplus_{v\in V} G_v^\perp$ is an instance of an injection strategy defined above, we obtain by duality a winning strategy

$$P_v : \bigominus_{v\in V} G_v \twoheadrightarrow \ominus.G_v \, ,$$

for any $v \in V$. Let $v \in V$. By composition with P_v a winning strategy in $\bigominus_{v\in V} G_v$ yields a winning strategy in the component $\ominus.G_v$. By Lemma 9.35 in a strategy $\sigma : S \to \ominus.A_v$ the event structure S decomposes into a prefixed sum, where the prefixing events are necessarily all $-$ve. As σ is receptive the sum must be a unary prefixed sum of the form $\ominus.S'$. Lemma 9.35 provides a map $\sigma' : S' \to A_v$. From σ being winning the map σ' will be a winning strategy in G_v. *'If':* Suppose $\sigma_v : S_v \to A_v$ is a winning strategy in G_v, for all $v \in V$. Prefixing we obtain winning strategies $\ominus.\sigma_v : \ominus.S_v \to \ominus.A_v$ in $\ominus.G_v$. Forming the sum $\sum_{v\in V} \ominus.\sigma_v : \sum_{v\in V} \ominus.S_v \to \ominus.\sigma_v : \sum_{v\in V} \ominus.A_v$ we obtain a strategy winning in $\bigominus_{v\in V} G_v$.

(iv) Now suppose G and H are determined. *'If':* The dual winning strategies $p_{G^\perp}^\perp : G \twoheadrightarrow G \parr H$ and $p_{H^\perp}^\perp : H \twoheadrightarrow G \parr H$ compose with a winning strategy $(\varnothing, \{\varnothing\}) \twoheadrightarrow G$, or respectively a winning strategy $(\varnothing, \{\varnothing\}) \twoheadrightarrow H$, to yield a winning strategy $(\varnothing, \{\varnothing\}) \twoheadrightarrow G \parr H$. *'Only if':* Suppose $G \parr H$ has a winning strategy. Then $G^\perp \otimes H^\perp = (G \parr H)^\perp$ has no winning strategy. Hence by (i), $G^\perp$ has no winning strategy or $H^\perp$ has no winning strategy. From determinacy, G has a winning strategy or H has a winning strategy. □

Theorem 9.37. *For all predicate-calculus formulae ϕ and environments ρ, $\rho \vDash_M \phi$ iff the game $[\![\phi]\!]_M \rho$ has a winning strategy.*

Proof. By Proposition 9.32 the games $[\![\phi]\!]_M \rho$ obtained from formulae ϕ are race-free and determined. The proof is by structural induction on ϕ.

The base case where ϕ is $R(x_1, \cdots, x_k)$ is obvious; the game $(\varnothing, \{\varnothing\})$ has as (unique) winning strategy the map $\varnothing \to \varnothing$, while $(\varnothing, \varnothing)$ has no winning strategy.

For the case $\phi \wedge \psi$, reason

$$\begin{aligned}
\rho \vDash_M \phi \wedge \psi &\iff \rho \vDash_M \phi \ \&\ \rho \vDash_M \psi \\
&\iff [\![\phi]\!]_M \rho \text{ has a winning strategy } \& \ [\![\psi]\!]_M \rho \text{ has a winning strategy, by induction,} \\
&\iff [\![\phi]\!]_M \rho \otimes [\![\psi]\!]_M \rho \text{ has a winning strategy, by Lemma 9.36(i),} \\
&\iff [\![\phi \wedge \psi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\phi \vee \psi$,

$$\begin{aligned}
\rho \vDash_M \phi \vee \psi &\iff \rho \vDash_M \phi \text{ or } \rho \vDash_M \psi \\
&\iff [\![\phi]\!]_M \rho \text{ has a winning strategy or } [\![\psi]\!]_M \rho \text{ has a winning strategy, by induction,} \\
&\iff [\![\phi]\!]_M \rho \parr [\![\psi]\!]_M \rho \text{ has a winning strategy, by Lemma 9.36(iv),} \\
&\iff [\![\phi \wedge \psi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\neg\phi$,

$$\begin{aligned}
\rho \vDash_M \neg\phi &\iff \rho \nvDash_M \phi \\
&\iff [\![\phi]\!]_M \rho \text{ has no winning strategy, by induction,} \\
&\iff ([\![\phi]\!]_M \rho)^\perp \text{ has a winning strategy, by determinacy.}
\end{aligned}$$

In the case $\exists x.\ \phi$,

$$\begin{aligned}
\rho \vDash_M \exists x.\phi &\iff \rho[v/x] \vDash_M \phi \text{ for some } v \in V \\
&\iff [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, for some } v \in V\text{, by induction,} \\
&\iff \bigoplus_{v \in V} [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, by Lemma 9.36(ii),} \\
&\iff [\![\exists x.\phi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\forall x.\ \phi$,

$$\begin{aligned}
\rho \vDash_M \forall x.\phi &\iff \rho[v/x] \vDash_M \phi \text{ for all } v \in V \\
&\iff [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, for all } v \in V\text{, by induction,} \\
&\iff \bigominus_{v \in V} [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, by Lemma 9.36(iii),} \\
&\iff [\![\forall x.\phi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

$\square$

Chapter 10

Borel determinacy

10.1 Introduction

We show the determinacy of concurrent games with Borel sets as winning conditions, provided they are race-free and bounded-concurrent. Both restrictions are necessary. The proof of determinacy of concurrent games proceeds via a reduction to the determinacy of tree games, and the determinacy of these in turn reduces to the determinacy of traditional Gale-Stewart games.

10.2 Tree games and Gale-Stewart games

We introduce tree games as a special case of concurrent games, traditional Gale-Stewart games as a variant, and show how to reduce the determinacy of tree games to that of Gale-Stewart games. Via Martin's theorem for the determinacy of Gale-Stewart games with Borel winning conditions we show that tree games with Borel winning conditions are determined.

10.2.1 Tree games

Definition 10.1. Say E, an event structure with polarity, is *tree-like* iff it is race-free, has empty concurrency relation (so $\leq_E$ forms a forest) and is such that polarities alternate along branches, *i.e.* if $e \rightarrowtriangle e'$ then $pol_E(e) \neq pol_E(e')$.

A *tree game* is (E, W), a concurrent game with winning conditions, in which E is tree-like.

Proposition 10.2. *Let E be a tree-like event structure with polarity. Then, its configurations $\mathcal{C}(E)$ form a tree w.r.t. $\subseteq$. Its root is the empty configuration $\varnothing$. Its (maximal) branches may be finite or infinite; finite sub-branches correspond to finite configurations of E; infinite branches correspond to infinite configurations of E. Its arcs, associated with $x \overset{e}{\longrightarrow\!\!\subset} x'$, are in 1-1 correspondence with events $e \in E$. The events e associated with initial arcs $\varnothing \overset{e}{\longrightarrow\!\!\subset} x$ all share the same* *polarity. Along a branch*

$$\varnothing \overset{e_1}{\longrightarrow\!\!\subset} x_1 \overset{e_2}{\longrightarrow\!\!\subset} x_2 \overset{e_3}{\longrightarrow\!\!\subset} \cdots \overset{e_i}{\longrightarrow\!\!\subset} x_i \overset{e_{i+1}}{\longrightarrow\!\!\subset} \cdots$$

the polarities of the events $e_1, e_2, \ldots, e_i, \ldots$ *alternate.*

Proposition 10.2 gives the precise sense in which 'arc,' 'sub-branch' and 'branch' are synonyms for 'events,' 'configurations' and 'maximal configurations' when an event structure is tree-like. Notice that for a non-empty tree-like event structure with polarity, all the events that can occur initially share the same polarity.

Definition 10.3. We say a a non-empty tree game (E, W) has polarity $+$ or $-$ according as its initial events are $+$ve or $-$ve. It is convenient to adopt the convention that the empty game $(\varnothing, \varnothing)$ has polarity $+$, and the empty game $(\varnothing, \{\varnothing\})$ has polarity $-$.

Observe that:

Proposition 10.4. *Let* $f : S \to A$ *be a total map of event structures with polarity, where* A *is tree-like. Then,* S *is also tree-like and the map* f *is innocent. The map* f *is a strategy iff it is receptive.*

Proof. As f preserves the concurrency relation, being a map of event structures, S must be tree-like. Innocence of f now follows so that only its receptivity is required for it to be a strategy. □

10.2.2 Gale-Stewart games

For the sake of uniformity we shall present Gale-Stewart games as a slight variant of tree games, a variant in which all maximal configurations of the tree game are infinite, and where Player and Opponent must play to a maximal, infinite configuration.

Definition 10.5. A *Gale-Stewart* game (G, V) comprises

- a tree-like event structure G for which all maximal configurations are infinite, and
- a subset V of infinite configurations—the *winning* configurations.

A *winning strategy* in a Gale-Stewart game (G, V) is a deterministic strategy $\sigma : S \to G$ such that $\sigma x \in V$ for all maximal configurations x of S.

This is not how a Gale-Stewart game and, particularly, a winning strategy in a Gale-Stewart game are traditionally defined. However, because the strategy σ is deterministic it is injective as a map on configurations, so corresponds to the subfamily of configurations $T = \{\sigma x \mid x \in \mathcal{C}^\infty(S)\}$ of $\mathcal{C}^\infty(G)$. The family T forms a subtree of the tree of configurations of G. Its properties, detailed below, reconcile our definition with the traditional one.

Proposition 10.6. *A winning strategy in a Gale-Stewart game* (G, V) *corresponds to a non-empty subset* $T \subseteq \mathcal{C}^\infty(G)$ *such that*

(i) $\forall x, y \in \mathcal{C}^\infty(G).\ \ y \subseteq x \in T \implies y \in T\,,$

(ii) $\forall x, y \in \mathcal{C}(G).\ \ x \in T\ \&\ x \overset{-}{\longrightarrow\!\!\!\subset} y \implies y \in T\,,$

(iii) $\forall x, y_1, y_2 \in T.\ x \overset{+}{\longrightarrow\!\!\!\subset} y_1\ \&\ x \overset{+}{\longrightarrow\!\!\!\subset} y_2 \implies y_1 = y_2\,,$ *and*

(iv) all $\subseteq$*-maximal members of* T *are infinite and in* V.

Proof. Given σ, a winning strategy in the Gale-Stewart game we define T as above. Then, (i) follows because σ is a map of event structures and G is tree-like; (ii) and (iii) follow from σ being receptive and deterministic; (iv) is a consequence of all winning configurations being infinite. Conversely, given T a subfamily of $\mathcal{C}^\infty(G)$ satisfying (i)-(iv) it is a relatively routine matter to construct a tree-like event structure S and map $\sigma : S \to G$ which is a winning strategy in (G, V). $\square$

A Gale-Stewart game (G, V) has a *dual* game $(G, V)^* =_{\mathrm{def}} (G^\perp, V^*)$, where V^* is the set of all maximal configurations in $\mathcal{C}^\infty(G)$ not in V. A winning strategy for Opponent in (G, V) is a winning strategy (for Player) in the dual game $(G, V)^*$.

For any event structure A there is a topology on $\mathcal{C}^\infty(A)$ given by the Scott open subsets. The $\subseteq$-maximal configurations in $\mathcal{C}^\infty(A)$ inherit a sub-topology from that on $\mathcal{C}^\infty(A)$. The Borel subsets of a topological space are those subsets of configurations in the sigma-algebra generated by the Scott open subsets. Donald Martin proved in his celebrated theorem [26] that Gale-Stewart games (G, V) are determined, *i.e.* there is a either a winning strategy for Player or a winning strategy for Opponent, when V is a Borel subset of the maximal configurations of $\mathcal{C}^\infty(A)$.

10.2.3 Determinacy of tree games

We show the determinacy of tree games with Borel winning conditions through a reduction of the determinacy of tree games to the determinacy of Gale-Stewart games.

Let (E, W) be a tree game. We construct a Gale-Stewart game $\mathrm{GS}(E, W) = (G, V)$ and a partial map $\mathit{proj} : G \to E$. The events of G are built as sequences of events in E together with two new symbols δ^- and δ^+ decreed to have polarity $-$ and $+$, respectively; the symbols δ^- and δ^+ represent delay moves by Opponent and Player, respectively.

Precisely, an event of G is a non-empty finite sequence

$$[e_1, \cdots, e_k]$$

of symbols from $E \cup \{\delta^-, \delta^+\}$ where: e_1 has the same polarity as (E, W); polarities alternate along the sequence; and for all subsequences $[e_1, \cdots, e_i]$, with

$i \le k$,

$$\{e_1, \cdots, e_i\} \cap E \in \mathcal{C}(E)\,.$$

The immediate causal dependency relation of G is given by

$$[e_1, \cdots, e_k] \lessdot_G [e_1, \cdots, e_k, e_{k+1}]$$

and consistency by compatibility w.r.t. $\le_G$. Events $[e_1, \cdots, e_k]$ of G have the same polarity as their last entry e_k. It is easy to see that G is tree-like, and that the only maximal configurations are infinite (because of the possibility of delay moves).

The map $\mathit{proj} : G \to E$ takes an event $[e_1, \cdots, e_k]$ of G to e_k if $e_k \in E$, and is undefined otherwise. The winning set V consists of all those infinite configurations x of G for which $\mathit{proj}\, x \in W$.

We have constructed a Gale-Stewart game $\mathrm{GS}(E, W) = (G, V)$. The construction respects the duality on games.

Lemma 10.7. *Letting (E, W) be a tree game,*

$$\mathrm{GS}((E, W)^\perp) = (\mathrm{GS}(E, W))^*\,.$$

Proof. Directly from the definition of the operation GS. □

Suppose $\sigma : S \to G$ is a winning strategy for (G, V). The composite

$$S \xrightarrow{\sigma} G \xrightarrow{\mathit{proj}} E \qquad (F1)$$

is a partial map of event structures with polarity. Letting $D \subseteq S$ be the subset of events on which $\mathit{proj} \circ \sigma$ is defined, the map $\mathit{proj} \circ \sigma$ factors as

$$S \longrightarrow S{\downarrow}D \xrightarrow{\sigma_0} E \qquad (F2)$$

where: the first partial map acts like the identity on events in D and is undefined otherwise—it sends a configuration $x \in \mathcal{C}^\infty(S)$ to $x \cap D \in \mathcal{C}^\infty(S{\downarrow}D)$; and σ_0 is the total map that acts like σ on D. We shall show that σ_0 is a (possibly nondeterministic) winning strategy for (E, W).

Lemma 10.8. *The map σ_0 is a winning strategy for (E, W).*

Proof. Write $S_0 =_{\mathrm{def}} S{\downarrow}D$. By Proposition 10.4, for $\sigma_0 : S_0 \to E$ to be a strategy we only require its receptivity. From the construction of G and proj,

$$\mathit{proj}\, x \,{-\!\!\subset}\, y \text{ in } \mathcal{C}(E) \implies \exists! x' \in \mathcal{C}(G).\ x \,{-\!\!\subset}\, x' \ \&\ \mathit{proj}\, x' = y\,.$$

This together with the receptivity of σ entails the receptivity of σ_0.

To show σ_0 is winning, suppose z is a $+$-maximal configuration of S_0; we require $\sigma_0 z \in W$. We will show this by exhibiting an infinite configuration $x \in \mathcal{C}^\infty(S)$ such that $x \cap D = z$. Then, according to the factorisation $(F2)$, $x \mapsto z \mapsto \sigma_0 z$, so we will have $\sigma_0 z = \mathit{proj}\, \sigma x$. The configuration x being infinite will ensure $\sigma x \in V$ because σ is winning in the Gale-Stewart game (G, V). By definition, $\sigma x \in V$ implies $proj\, \sigma x \in W$, so $\sigma_0 z \in W$.

It remains to exhibit an infinite configuration $x \in \mathcal{C}^\infty(S)$ such that $x \cap D = z$. When z is infinite this is readily achieved by defining $x =_{\text{def}} [z]_S \in \mathcal{C}^\infty(S)$. Suppose z is finite. Define $x_0 =_{\text{def}} [z]_S \in \mathcal{C}(S)$, ensuring $x_0 \cap D = z$. We inductively build an infinite chain

$$x_0 \overset{s_1}{\longrightarrow\!\!\!\subset} x_1 \overset{s_2}{\longrightarrow\!\!\!\subset} \cdots \overset{s_n}{\longrightarrow\!\!\!\subset} x_n \overset{s_{n+1}}{\longrightarrow\!\!\!\subset} \cdots$$

in $\mathcal{C}(S)$ where all the events s_n are 'delay' moves not in D. Then $x_n \cap D = z$ for all $n \in \omega$. By the definition of a winning strategies in Gale-Stewart games, no x_n can be $\subseteq$-maximal in $\mathcal{C}(S)$. For each Opponent move s_n choose to delay—as we may do by the receptivity of σ. For each Player move s_n we have no choice as only a delay move is possible—otherwise we would contradict the $+$-maximality assumed of z. Taking $x =_{\text{def}} \bigcup_n x_n$ produces an infinite configuration $x \in \mathcal{C}^\infty(S)$ such that $x \cap D = z$, as required. □

Corollary 10.9. *Let H be a tree game. If the Gale-Stewart game* GS(H) *has a winning strategy, then H has a winning strategy.*

Theorem 10.10. *Tree games with Borel winning conditions are determined.*

Proof. Assume (E, W) is a tree game where W is a Borel set. Construct $\mathrm{GS}(E, W) = (G, V)$ as above. The function *proj*, acting as $x \mapsto proj\, x$ on configurations, is easily seen to be a Scott-continuous function from $\mathcal{C}^\infty(G) \to \mathcal{C}^\infty(E)$. It restricts to a continuous function from the subspace of maximal configurations in $\mathcal{C}^\infty(G)$. Hence V, as the inverse image of W under this restricted function, is a Borel subset. By Martin's Borel-determinacy theorem [26], the game (G, V) is determined, so has either a winning strategy for Player or a winning strategy for Opponent.

Suppose first that $\mathrm{GS}(E, W)$ has a winning strategy (for Player). By Corollary 10.9 we obtain a winning strategy for (E, W). Suppose, on the other hand, that $\mathrm{GS}(E, W)$ has a winning strategy for Opponent, *i.e.* there is a winning strategy in the dual game $\mathrm{GS}(E, W)^*$. By Lemma 10.7, $\mathrm{GS}((E, W)^\perp) = \mathrm{GS}(E, W)^*$ has a winning strategy. By Corollary 10.9, $(E, W)^\perp$ has a winning strategy, *i.e.* there is a winning strategy for Opponent in (E, W). □

10.3 Race-freeness and bounded-concurrency

Not all games are determined; We have seen the necessity of race-freeness for the determinacy of well-founded games. However, a determinacy theorem holds for well-founded games (games where all configurations are finite) which are **(race – free)**

$$x \overset{a}{\longrightarrow\!\!\!\subset}\ \&\ x \overset{a'}{\longrightarrow\!\!\!\subset}\ \&\ pol(a) \neq pol(a') \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(Race – free)}$$

However race-freeness is not sufficient to ensure determinacy when the game is not well-founded, as is illustrated in the following example.

Example 10.11. Let A be the event structure with polarity consisting of one positive event $\oplus$ which is concurrent with an infinite chain of alternating negative and positive events, *i.e.* for each i we have both $\oplus\ co \oplus_i$ and $\oplus\ co \ominus_i$, $i \in \mathbb{N}$,

$$A = \quad \oplus \qquad \ominus_1 \rightarrowtriangle \oplus_1 \rightarrowtriangle \ominus_2 \rightarrowtriangle \oplus_2 \rightarrowtriangle \cdots$$

and Borel winning conditions (for Player) given by

$$W = \{\varnothing, \{\ominus_1, \oplus_1\}, ..., \{\ominus_1, \oplus_1, ..., \ominus_i, \oplus_i\}, ..., A\}.$$

So, Player wins if (i) no event is played, or (ii) the event $\oplus$ is not played and the play is finite and finishes in some $\oplus_i$, or (iii) all of the events in A are played. Otherwise, Opponent wins.

Player does not have a winning strategy because Opponent has an infinite family of *spoiler* strategies, not all be dominated by a single strategy of Player. The inclusion maps $\tau_\infty : T_\infty \to A^\perp$ and $\tau_i : T_i \to A^\perp$, $i \in \mathbb{N}$, are strategies for Opponent where $T_\infty^\perp =_{\text{def}} A$ and $T_i^\perp =_{\text{def}} A \smallsetminus \{e' \in A \mid \ominus_i \le e'\}$, for $i \in \mathbb{N}$.

Any strategy for Player that plays $\oplus$ is dominated by some strategy τ_i for Opponent; likewise, any strategy for Player that does not play $\oplus$ and plays only finitely many positive events $\oplus_i$ is also dominated by some strategy τ_i for Opponent. Moreover, a strategy for Player that does not play $\oplus$ and plays all of the events $\oplus_i$ in A is dominated by τ_∞. So, Player does not have a winning strategy in this game. Similarly, Opponent does not have a winning strategy in A because Player has two strategies that cannot be both dominated by any strategy for Opponent. Let $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ and $\sigma_\oplus : S_\oplus \to A$ be strategies for Player such that $S_{\overline{\oplus}} =_{\text{def}} A \smallsetminus \{\oplus\}$ and $S_\oplus =_{\text{def}} A$.

On the one hand, any strategy for Opponent that plays only finitely many (possibly zero) negative events $\ominus_i$ is dominated by $\sigma_{\overline{\oplus}}$; on the other, any strategy for Opponent that plays all of the negative events $\ominus_i$ in A is dominated by $\sigma_\oplus$. Thus neither player has a winning strategy in this game! $\square$

In the above example, to win Player should only make the move $\oplus$ when Opponent has played an infinite number of moves. We can banish such difficulties by insisting that in a game no event is concurrent with infinitely many events of the opposite polarity. This property is called *bounded-concurrency*:

$$\forall y \in \mathcal{C}^\infty(A).\ \forall e \in y.\ \{e' \in y \mid e\ co\ e'\ \&\ pol(e) \ne pol(e')\} \text{ is finite.} \qquad \textbf{(Bounded – concurrent)}$$

Bounded concurrency is in fact a *necessary* structural condition for determinacy with respect to Borel winning conditions.

Notation 10.12. *For a concurrent game A with configurations y, y', write $max_+(y', y)$ iff y' is $\oplus$-maximal in y, i.e. $y' \overset{e}{\longrightarrow\!\!\subset}\ \&\ pol(e) = + \implies e \notin y$; in a dual way, we write $\overline{max}_+(y', y)$ iff y' is not $\oplus$-maximal in y. We use max_- analogously when $pol(e) = -$.*

We show that if a countable, race-free A is not bounded-concurrent, then there is Borel W so that the game (A, W) is not determined. Since A is not bounded-concurrent, there is $y \in \mathcal{C}^\infty(A)$ and $e \in y$ such that e is concurrent with infinitely many events of opposite polarity in y. W.l.o.g. assume that $pol(e) = +$, that $y \setminus \{e\}$ is a configuration and that $y = [e] \cup [\{a \in y \mid pol_A(a) = -\}]$. The following rules determine whether $y' \in \mathcal{C}^\infty(A)$ is in W or L:

1. $y' \supseteq y \Longrightarrow y' \in W$;
2. $y' \subset y \;\&\; e \in y' \Longrightarrow y' \in L$;
3. $y' \subset y \;\&\; e \notin y' \;\&\; max_+(y', y \setminus \{e\}) \;\&\; \overline{max}_-(y', y \setminus \{e\}) \Longrightarrow y' \in W$;
4. $y' \subset y \;\&\; e \notin y' \;\&\; \overline{max}_+(y', y \setminus \{e\})$ or $max_-(y', y \setminus \{e\}) \Longrightarrow y' \in L$;
5. $y' \not\supseteq y \;\&\; (y' \cap y) \subset^- y' \Longrightarrow y' \in W$;
6. $y' \not\supseteq y \;\&\; (y' \cap y) \subset^+ y' \Longrightarrow y' \in L$;
7. otherwise assign y' (arbitrarily) to W.

No y' is assigned as winning for both Player and Opponent: the implications' antecedents are all pair-wise mutually exclusive.[1] The countability of A is important in showing that W is Borel.

Lemma 10.13. *Let A be a countable race-free game. If A is not bounded-concurrent, then there is Borel $W \subseteq \mathcal{C}^\infty(A)$ such that the game (A, W) is not determined.*

Proof. The set W is Borel because it is defined by clauses such as $y' \subset y$ which have extensions, in this case $\{y' \in \mathcal{C}^\infty(A) \mid y' \subset y\}$, which are Borel sets by virtue of the countability of A. For instance, a clause such as $e \in y'$ has extension

$$\{y' \in \mathcal{C}^\infty(A) \mid e \in y'\} = \widehat{[e]}\,,$$

a basic open set. In general, for $x \in \mathcal{C}(A)$, we use $\widehat{x}$ to denote the basic open set $\{x' \in \mathcal{C}^\infty(A) \mid x \subseteq x'\}$. The clause $y' \supseteq y$, equivalent to $\forall a \in y.\, a \in y'$, has extension

$$\{y' \in \mathcal{C}^\infty(A) \mid y' \supseteq y\} = \bigcap_{a \in y} \widehat{[a]}\,;$$

because A is assumed countable so is y and the intersection is an intersection of countably many open sets. To see that $\{y' \in \mathcal{C}^\infty(A) \mid y' \subset y\}$ is Borel is a bit more complicated. Observe that

$$\{y' \in \mathcal{C}^\infty(A) \mid y' \subset y\} = \bigcap_{a \notin y} (\mathcal{C}^\infty(A) \setminus \widehat{[a]}) \cap \bigcup_{a \in y} (\mathcal{C}^\infty(A) \setminus \widehat{[a]})\,;$$

the big intersection is the extension of $y' \subseteq y$ and the big union that of $\exists a \in y.\, a \notin y'$—because A is assumed countable the intersection and union are countable.

We first show:

[1] The winning conditions W in Example 10.11 are instance of this scheme.

(i) If σ is a winning strategy for Player then y is σ-reachable, *i.e.* $\sigma : S \to A$, there is $x \in \mathcal{C}^\infty(S)$ s.t. $\sigma x = y$.
(ii) If τ is a winning strategy for Opponent then y is τ-reachable.
Write $y_e =_{\mathrm{def}} y \setminus \{e\}$.

(i) This part uses rules (2), (4) and (6). Suppose $\sigma : S \to A$ is a winning strategy for Player. There is a $\subseteq$-maximal configuration of S s.t. $\sigma x_0 \subseteq y$ (via Zorn's lemma). By receptivity, σx_0 is $-$-maximal in y. As σ is winning, there is a $+$-maximal $x \in \mathcal{C}^\infty(S)$ with $x_0 \subseteq^+ x$ and $\sigma x \in W$ (Zorn).

If $\sigma x \supseteq y$ then necessarily $\sigma x \supseteq^+ y$ and by a general property of strategies we obtain y is σ-reachable. For completeness we include the argument. Take $x' =_{\mathrm{def}} \{s \in x \mid \sigma(s) \notin (\sigma x) \setminus y\}$. Suppose $s' \twoheadrightarrow s$ in x. Then

$$\sigma(s') \in (\sigma x) \setminus y \implies \sigma(s) \in (\sigma x) \setminus y$$

by $+$-innocence. Hence its contrapositive, *viz.*

$$\sigma(s) \notin (\sigma x) \setminus y \implies \sigma(s') \notin (\sigma x) \setminus y\,,$$

so that $s \in x'$ implies $s' \in x'$. Thus, being down-closed and consistent, $x' \in \mathcal{C}^\infty(S)$, with $\sigma x' = y$ from the definition of x'.

The remaining case $\sigma x \not\supseteq y$ is impossible. Suppose $x_0 \neq x$, so $x_0 \subset x$. Then we also have $(\sigma x) \cap y \subset^+ \sigma x$, using the $\subseteq$-maximality of x_0. By (6), $\sigma x \in L$—a contradiction. Suppose, on the other hand, that $x_0 = x$. If $e \in \sigma x$, by (2) we obtain the contradiction $\sigma x \in L$. If $e \notin \sigma x$, by (4) we obtain the contradiction $\sigma x \in L$; recall $\sigma x = \sigma x_0$ is $-$-maximal in y so in y_e when $e \notin \sigma x$.

(ii) This part uses rules (1), (3) and (5). Suppose $\tau : T \to A^\perp$ is a winning strategy for Opponent. It is sufficient to show y_e is τ-reachable as then y will also be τ-reachable by receptivity. Assume to obtain a contradiction that y_e is not τ-reachable. Then there is a $\subseteq$-maximal $x_0 \in \mathcal{C}^\infty(T)$ s.t. $\tau x_0 \subseteq y$ (via Zorn's lemma). By assumption, $\tau x_0 \subset y$. By receptivity, τx_0 is $+$-maximal in y_e and necessarily τx_0 is not $-$-maximal in y_e. By (3), $\tau x_0 \in W$. As τ is winning, there is a $-$-maximal $x \in \mathcal{C}^\infty(T)$ with $x_0 \subseteq^- x$ and $\tau x \in L$ (Zorn); from the latter $x_0 \subset x$. We claim that by (1)&(5), $\tau x \subseteq y_e$, contradicting the $\subseteq$-maximality of x_0. To show the claim, suppose to obtain a contradiction that $\tau x \nsubseteq y_e$. Then $\tau x \nsubseteq y$, as e is $+$ve , so $(\tau x) \cap y \subset^- \tau x$. By (1), $\tau x \not\supseteq y$. Now by (5), $\tau x \in W$, the required contradiction.

To conclude the proof we show there is no winning strategy for either player.

If σ is a winning strategy for Player then by (i) there is $x \in \mathcal{C}^\infty(S)$ s.t. $\sigma x = y$; in particular there is $s_e \in x$ s.t. $\sigma(s_e) = e$. Define the inclusion map $\tau_0 : A^\perp \upharpoonright (\sigma[s_e]_S \cup \{a \in A^\perp \mid pol_A(a) = +\} \hookrightarrow A^\perp$. Then τ_0 s a strategy for Opponent for which there is $y' \in \langle \sigma, \tau_0 \rangle$ with $e \in y'$ and where y' only contains finitely many $-$-events. Either $y' \subset y$ whence $y' \in L$ by (2), or $y' \nsubseteq y$ whereupon $(y' \cap y) \subset^+ y'$ so $y' \in L$ by (6). Hence as τ_0 is a strategy for Opponent not dominated by σ the latter cannot be a winning strategy for Player.

If τ is a winning strategy for Opponent then y is τ-reachable. Define the inclusion map $\sigma_0 : A \upharpoonright (y \cup \{a \in A \mid pol_A(a) = -\} \hookrightarrow A$. Then σ_0 is a strategy for Player for which there is $y' \in \langle \sigma_0, \tau \rangle$ with $y' \supseteq y$. By (1) $y' \in W$, so σ_0 is not dominated by τ, which cannot be a winning strategy for Opponent. $\square$

10.4 Determinacy of concurrent games

We now construct a tree game $\mathrm{TG}(A, W)$ from a concurrent game (A, W). We can think of the events of $\mathrm{TG}(A, W)$ as corresponding to (non-empty) *rounds* of −ve or +ve events in the original concurrent game (A, W). When (A, W) is race-free and bounded-concurrent, a winning strategy for $\mathrm{TG}(A, W)$ will induce a winning strategy for (A, W). In this way we reduce determinacy of concurrent games to determinacy of tree games.

10.4.1 The tree game of a concurrent game

From a concurrent game (A, W) we construct a tree game

$$\mathrm{TG}(A, W) = (TA, TW)\,.$$

The construction of TA depends on whether $\varnothing \in W$.

In the case where $\varnothing \in W$, define an alternating sequence of (A, W) to be a sequence

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^+ x_{2i} \subset^- x_{2i+1} \subset^+ x_{2i+2} \subset^- \cdots$$

of configurations in $\mathcal{C}^\infty(A)$—the sequence need not be maximal. Define the −ve events of $\mathrm{TG}(W, A)$ to be

$$[\varnothing, x_1, x_2, \ldots, x_{2k-2}, x_{2k-1}]\,,$$

finite alternating sequences of the form

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^+ x_{2k-2} \subset^- x_{2k-1}\,,$$

and the +ve events to be

$$[\varnothing, x_1, x_2, \ldots, x_{2k-1}, x_{2k}]\,,$$

finite alternating sequences

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^- x_{2k-1} \subset^+ x_{2k}\,,$$

where $k \geq 1$. The causal dependency relation on TA is given by the relation of initial sub-sequence, with a finite subset of events being consistent iff the events are all initial sub-sequences of a common alternating sequence.

It is easy to see that a configuration of TA corresponds to an alternating sequence, the −ve events of TA matching arcs $x_{2k-2} \subset^- x_{2k-1}$ and the +ve events arcs $x_{2k-1} \subset^+ x_{2k}$. As such, we say a configuration $y \in \mathcal{C}^\infty(TA)$ is winning, and in TW, iff y corresponds to an alternating sequence

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots$$

for which $\bigcup_i x_i \in W$.

In the case where $\varnothing \notin W$, we define an alternating sequence of (A, W) as a sequence

$$\varnothing \subset^+ x_1 \subset^- x_2 \subset^+ \cdots \subset^- x_{2i} \subset^+ x_{2i+1} \subset^- x_{2i+2} \subset^+ \cdots$$

of configurations in $\mathcal{C}^\infty(A)$. In this case, the −ve events of $\mathrm{TG}(W, A)$ are finite alternating sequences ending in x_{2k}, while the +ve events end in x_{2k-1}, for $k \geq 1$. The remaining parts of the definition proceed analogously.

We have constructed a tree game $\mathrm{TG}(A, W)$ from a concurrent game (A, W). The construction respects the duality on games.

Lemma 10.14. *Let (A, W) be a concurrent game.*

$$\mathrm{TG}((A, W)^\perp) = (\mathrm{TG}(A, W))^\perp .$$

Proof. From the construction TG, because alternating sequences

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots$$

in $\mathcal{C}^\infty(A)$ correspond to alternating sequences

$$\varnothing \cdots \subset^- x_i \subset^+ x_{i+1} \subset^- \cdots$$

in $\mathcal{C}^\infty(A^\perp)$. □

Proposition 10.15. *Suppose (A, W) is a bounded-concurrent game. Maximal alternating sequences have one of two forms,*

(i) finite:

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots x_k ,$$

where x_i is finite for all $0 < i < k$ (where possibly x_k is infinite), or

(iii) infinite:

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots ,$$

where each x_i is finite.

Proof. Otherwise, taking the first infinite x_i, within configuration x_{i+1} there would be an event of $x_{i+1} \setminus x_i$ concurrent with infinitely many events of x_i of opposite polarity—contradicting the bounded-concurrency of A. □

10.4.2 Borel determinacy of concurrent games

Now assume that the concurrent game (A, W) is race-free and bounded-concurrent. Suppose that $str : T \to TA$ is a (winning) strategy in the tree game $TG(A, W)$. Note that T is necessarily tree-like. We construct $\sigma_0 : S \to A$, a (winning) strategy in the original concurrent game (A, W). We construct S indirectly, from a prime-algebraic domain $\mathcal{Q}$, built as follows. For technical reasons, in the construction of $\mathcal{Q}$ it is convenient to assume—as can easily be arranged—that

$$A \cap (A \times T) = \varnothing \,.$$

Via *str* a sub-branch

$$\vec{t} = (t_1, \cdots, t_i, \cdots)$$

of T determines a *tagged alternating sequence*

$$\varnothing \;\cdots\; \overset{t_{i-1}}{\subset^-} \; x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

where $str(t_i) = [\varnothing, \ldots, x_{i-1}, x_i]$. (Informally, the arc t_i is associated with a round extending x_{i-1} to x_i in the original concurrent game.)

Define $q(\vec{t})$ to be the partial order comprising events

$$\bigcup \{(x_i \smallsetminus x_{i-1}) \mid t_i \text{ is a } -\text{ve arc of } \vec{t}\} \,\cup$$
$$\bigcup \{(x_i \smallsetminus x_{i-1}) \times \{t_i\} \mid t_i \text{ is a } +\text{ve arc of } \vec{t}\}$$

—so a copy of the events $\bigcup_i x_i$ but with +ve events tagged by the +ve arc of T at which they occur[2]—with order a copy of that $\bigcup_i x_i$ inherits from A with additional causal dependencies pairs from

$$x_{i-1}^- \;\times\; ((x_i \smallsetminus x_{i-1}) \times \{t_i\})$$

—making the +ve events occur after the −ve events which precede them in the alternating sequence.

Define the partial order $\mathcal{Q}$ as follows. Its elements are partial orders q, not necessarily finite, for which there is a rigid inclusion

$$q \hookrightarrow q(t_1, t_2, \cdots, t_i, \cdots) \,,$$

for some sub-branch $(t_1, t_2, \cdots, t_i, \cdots)$ of T. The order on $\mathcal{Q}$ is that of rigid inclusion. Define the function $\sigma : \mathcal{Q} \to \mathcal{C}^\infty(A)$ by taking

$$\sigma q = \{a \in A \mid a \text{ is } -\text{ve } \& \; a \in q\} \cup \{a \in A \mid \exists t \in T.\; a \text{ is } +\text{ve } \& \; (a, t) \in q\}$$

for $q \in \mathcal{Q}$. We should check that σq is indeed a configuration of A. Clearly, $\sigma q(\vec{t}) = \bigcup_{i \in I} x_i$ where

$$\varnothing \;\cdots\; \overset{t_{i-1}}{\subset^-} \; x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

is the tagged alternating sequence determined by $\vec{t} =_{\text{def}} (t_1, \cdots, t_i, \cdots)$. Any q for which there is a rigid inclusion $q \hookrightarrow q(\vec{t})$ will be sent to a sub-configuration of $\bigcup_i x_i$.

[2] It is so that the two components remain disjoint under tagging that we make the technical assumption above.

Proposition 10.16. *Let* $(t_1, \cdots, t_i, \cdots)$ *be a sub-branch of* T*, so corresponding to a configuration* $\{t_1, \cdots, t_i, \cdots\} \in \mathcal{C}^\infty(T)$*. Then,*

$$str\{t_1, \cdots, t_i, \cdots\} \in TW \iff \sigma q(t_1, \cdots, t_i, \cdots) \in W\,.$$

Proof. Let $\vec{t} =_{\mathrm{def}} (t_1, \cdots, t_i, \cdots)$. We have $str(t_i) = [\varnothing, \ldots, x_{i-1}, x_i]$ for some

$$\varnothing\ \cdots\ \subset^- x_{i-1} \subset^+ x_i \subset^- \cdots,$$

an alternating sequence of (A, W). Directly from the definitions of TW, $q(\vec{t})$ and σ,

$$\begin{aligned} str\{\vec{t}\} \in TW &\iff \bigcup_i x_i \in W \\ &\iff \sigma q(\vec{t}) \in W\,. \end{aligned}$$

□

We shall make use of the following proposition.

Proposition 10.17. *For all* $q, q' \in \mathcal{Q}$*, whenever there is an inclusion of the events of* q *in the events of* q' *there is a rigid inclusion* $q \hookrightarrow q'$*.*

Proof. To see this, suppose the events of q are included in the events of q'. To establish the rigid inclusion $q \hookrightarrow q'$ we require that, for all $a \in q, b \in q'$,

$$b \rightarrowtriangle_q a \iff b \rightarrowtriangle_{q'} a\,. \qquad (\dagger)$$

However, in the construction of $q(t_1, t_2, \cdots, t_i, \cdots)$ the only immediate dependencies introduced beyond those of A are those of the form $b \rightarrowtriangle (a', t)$, of tagged +ve events on −ve rounds specified earlier in the branch on which the +ve arc t occurs. This property is inherited by q and q' in $\mathcal{Q}$. Thus in checking (†) we can restrict attention to the case where b is −ve and a is +ve and of the form (a', t) for some $a' \in A$ and arc t of T. The arc t determines a sub-branch $t_1, \cdots, t_k = t$ of T and a corresponding tagged alternating sequence

$$\varnothing\ \cdots\ \overset{t_{k-1}}{\subset^-} x_{k-1} \overset{t_k}{\subset^+} x_k\,.$$

So in this case,

$$\begin{aligned} b \rightarrowtriangle_q a &\iff b \text{ is } \le_A\text{-maximal in } x_{k-1}^- \ \&\ a' \text{ is } \le_A\text{-maximal in } x_k \setminus x_{k-1} \\ &\iff b \rightarrowtriangle_{q'} a\,, \end{aligned}$$

which ensures (†), and the proposition. □

Notation 10.18. Proposition 10.17, justifies us in writing $\subseteq$ for the order of $\mathcal{Q}$. We shall also write $q \subseteq^- q'$ when all the events in q' above those of q are −ve, and similarly $q \subseteq^+ q'$ when all the events in q' above those of q are +ve. □

The following lemma is crucial and depends critically on (A, W) being race-free and bounded-concurrent.

Lemma 10.19. *The order $(\mathcal{Q}, \subseteq)$ is a prime algebraic domain in which the primes are precisely those (necessarily finite) partial orders with a maximum.*

Proof. Any compatible finite subset X of $\mathcal{Q}$ has a least upper bound: if all the members of X include rigidly in a common q then taking the union of their images in q, with order inherited from q, provides their least upper bound. Provided $\mathcal{Q}$ has least upper bounds of directed subsets it will then be consistently complete with the additional property that every $q \in \mathcal{Q}$ is the least upper bound of the primes below it—this will make $\mathcal{Q}$ a prime algebraic domain.

To establish prime algebraicity it remains to show that $\mathcal{Q}$ has least upper bounds of directed sets.

Let S be a directed subset of $\mathcal{Q}$. The +ve events of orders $q \in S$ are tagged by +ve arcs of T. Because S is directed the +ve tags which appear throughout all $q \in S$ must determine a common sub-branch of T, *viz.*

$$\vec{t} =_{\mathrm{def}} (t_1, t_2, \cdots, t_i, \cdots)\,.$$

Every +ve arc of the sub-branch appears in some $q \in S$ and all −ve arcs are present only by virtue of preceding a +ve arc. The sub-branch $\vec{t}$ may be

(1) infinite and necessarily a full branch of T, if the elements of S together mention infinitely many tags;

(2) finite with $q(\vec{t})$ infinite, and necessarily finishing with a +ve arc;

(3) finite and non-empty with $q(\vec{t})$ finite, and necessarily finishing with a +ve arc; or

(4) empty with $\vec{t} = ()$.

(1) Consider the case where $\vec{t}$ forms an infinite branch of T. We shall argue that for all $q \in S$, there is a rigid inclusion

$$q \hookrightarrow q(\vec{t})\,.$$

Then, forming the partial order $\bigcup S$ comprising the union of the events of all $q \in S$ with order the restriction of that on $q(\vec{t})$ we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(\vec{t})\,,$$

so a least upper bound of S in $\mathcal{Q}$.

Let $q \in S$. By Proposition 10.17, to establish the rigid inclusion $q \hookrightarrow q(\vec{t})$ it suffices to show the events of q are included in those of $q(\vec{t})$. From the nature of the sub-branch determined by S, we must have that all the +ve events of q are included in those of $q(\vec{t})$—all +ve events of q are tagged by a +ve arc of $\vec{t}$. Suppose, to obtain a contradiction, that there is some −ve event a of q not in $q(\vec{t})$. For every +ve arc t_i in $\vec{t}$ there is $q_i \in S$ with a +ve tagged event (a_i, t_i). Let

$$I \subseteq_{\mathrm{fin}} \{i \mid t_i \text{ is a +ve arc of } \vec{t}\}\,.$$

As S is directed, there is an upper bound in S of

$$\{q\} \cup \{q_i \mid i \in I\} .$$

It follows that

$$\{a\} \cup \{a_i \mid i \in I\} \in \mathrm{Con}_A ,$$

Hence, forming the down-closure in A of $\{a\} \cup \{a_i \mid t_i \text{ is a +ve arc in } \vec{t}\}$, we obtain

$$[\{a\} \cup \{a_i \mid t_i \text{ is a +ve arc in } \vec{t}\}] \in \mathcal{C}^\infty(A) .$$

Moreover it is a configuration which violates the assumption of bounded-concurrency—the −ve event a is concurrent with infinitely many of the +ve events a_i. From this contradiction we deduce that the events of q are included in the events of $q(\vec{t})$.

(2) Consider the case where $\vec{t}$ is a finite branch $(t_1, \cdots, t_k)$, where necessarily t_k is a +ve arc, and where $q(\vec{t})$ is infinite. By bounded-concurrency, all $q(t_1, \cdots, t_i)$, for $0 \le i < k$, are finite with only $q(\vec{t}) = q(t_1, \cdots, t_k)$ infinite.

Let $q \in S$. By Proposition 10.17, we can show there is a rigid inclusion

$$q \hookrightarrow q(\vec{t})$$

by showing all the events of q are in $q(\vec{t})$. Again, all the +ve events of q are in $q(\vec{t})$. Suppose, to obtain a contradiction, that $b \in q$ with $b \notin q(\vec{t})$, so b has to be −ve. There is a member of S with an event tagged by t_k. Thus, using the directedness of S, there has to be $q_1 \in S$ with $q \subseteq q_1$ and where q_1 has an event tagged by t_k. Because of the extra dependencies introduced in the construction of $q(\vec{t})$, all the −ve events of $q(\vec{t})$ are included in q_1. Note in addition that

$$[q_1^+] \subseteq q(\vec{t})$$

because all the +ve events of q_1 are in $q(\vec{t})$. We deduce

$$[q_1^+] \subseteq^+ q(\vec{t}) . \tag{i}$$

Also,

$$[q_1^+] \subset^- q_1 , \tag{ii}$$

where the inclusion has to be strict because $b \in q_1 \setminus q(\vec{t})$. Consider the images of (i) and (ii) in $\mathcal{C}^\infty(A)$:

$$\sigma[q_1^+] \subseteq^+ \sigma q(\vec{t}) \quad \text{and} \quad \sigma[q_1^+] \subset^- \sigma q_1 .$$

As A is race-free, we obtain the configuration $x =_{\mathrm{def}} \sigma q(\vec{t}) \cup \sigma q_1 \in \mathcal{C}^\infty(A)$ and the strict inclusion

$$\sigma q(\vec{t}) \subset^- x ,$$

making x a configuration which contains the −ve event b concurrent with infinitely many +ve events—the images of those tagged by t_k. But this contradicts the bounded-concurrency of A. Hence all the events of q are in $q(\vec{t})$.

As in case (1) we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(\vec{t})\,,$$

and a least upper bound of S in $\mathcal{Q}$.

(3) The case where $\vec{t}$ is a non-empty finite branch $(t_1, \cdots, t_k)$ and $q(\vec{t})$ is finite. Again, t_k is necessarily a +ve arc. As S is directed, the set of events $\bigcup_{q \in S} \sigma q$ is a configuration in $\mathcal{C}^\infty(A)$. Again, all the +ve events of any $q \in S$ are in $q(\vec{t})$, from which it follows that as sets,

$$(\bigcup_{q \in S} \sigma q)^+ \subseteq \sigma q(\vec{t})\,.$$

Hence, the down-closure

$$[(\bigcup_{q \in S} \sigma q)^+]_A \subseteq \sigma q(\vec{t}) \ \text{ in } \mathcal{C}^\infty(A)\,. \tag{iii}$$

There is $q_1 \in S$ with an event tagged by t_k. Because of the extra dependencies introduced in the construction of $q(\vec{t})$, all the −ve events of $q(\vec{t})$ are included in q_1. Consequently, all the −ve events of $\sigma q(\vec{t})$ are included in $\bigcup_{q \in S} \sigma q$. From this and ($iii$) we deduce

$$[(\bigcup_{q \in S} \sigma q)^+] \subseteq^+ \sigma q(\vec{t}) \ \text{ in } \mathcal{C}^\infty(A)\,. \tag{iv}$$

Also, straightforwardly,

$$[(\bigcup_{q \in S} \sigma q)^+] \subseteq^- \bigcup_{q \in S} \sigma q \ \text{ in } \mathcal{C}^\infty(A)\,. \tag{v}$$

From (iv) and (v), because A is race-free, we obtain the configuration

$$y =_{\text{def}} (\sigma q(\vec{t}) \cup \bigcup_{q \in S} \sigma q) \in \mathcal{C}^\infty(A)$$

for which

$$\sigma q(\vec{t}) \subseteq^- y \in \mathcal{C}^\infty(A)\,.$$

But by receptivity of the original strategy $str : T \to TA$, there is a unique extension of the branch $\vec{t} = (t_1, \cdots, t_k)$ to $(t_1, \cdots, t_k, t_{k+1})$ in T such that

$$\sigma q(t_1, \cdots, t_k, t_{k+1}) = y\,.$$

W.r.t. this extension, forming the partial order $\bigcup S$ comprising the union of the events of all $q \in S$ with order the restriction of that on $q(t_1, \cdots, t_k, t_{k+1})$, we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(t_1, \cdots, t_k, t_{k+1})\,,$$

so a least upper bound of S in $\mathcal{Q}$.

(4) Finally, consider the case where $\vec{t} = ()$. Then all $q \in S$ consist purely of −ve events. As S is directed, $\bigcup_{q \in S} \sigma q \in \mathcal{C}^\infty(A)$. If $\bigcup_{q \in S} \sigma q = \varnothing$ we have $\bigcup S = q()$. Assume $\bigcup_{q \in S} \sigma q$ is non-empty.

Suppose first that $\varnothing \in W$. We can form the alternating sequence

$$\varnothing \subset^- \bigcup_{q \in S} \sigma q\,.$$

By the receptivity of $str : T \to TA$ there is a unique 1-arc branch (t_1) of T with $\bigcup_{q \in S} \sigma q = \sigma q(t_1)$. Then $\bigcup S = q(t_1)$.

Now suppose $\varnothing \notin W$. In this case all alternating sequences must begin $\varnothing \subset^+ x_1 \cdots$ and consequently all initial arcs of T must be +ve. We are assuming $\bigcup_{q \in S} \sigma q$ is non-empty so contains some non-empty q. There must therefore be a rigid inclusion $q \hookrightarrow q(\vec{u})$ for some non-empty sub-branch $\vec{u} = (u_1, \cdots)$. Via str the sub-branch $\vec{u}$ determines the alternating sequence $\varnothing \subset^+ x_1 \subset^- \cdots$. Noting $\varnothing \subset^- \bigcup_{q \in S} \sigma q$, because A is race-free there is $x_1 \cup \bigcup_{q \in S} \sigma q \in \mathcal{C}^\infty(A)$. Form the alternating sequence

$$\varnothing \subset^+ x_1 \subset^- x_1 \cup \bigcup_{q \in S} \sigma q\,.$$

From the receptivity of str there is a sub-branch (u_1, u_2') such that $x_1 \cup \bigcup_{q \in S} \sigma q = \sigma q(u_1, u_2')$. We obtain $\bigcup S \hookrightarrow q(u_1, u_2')$. $\square$

Definition 10.20. Define S to be the event structure with polarity, with events the primes of $\mathcal{Q}$; causal dependency the restriction of the order on $\mathcal{Q}$; with a finite subset of events consistent if they include rigidly in a common element of $\mathcal{Q}$. The polarity of event of S is the polarity in A of its top element (recall the event is a prime in $\mathcal{Q}$). Define $\sigma_0 : S \to A$ to be the function which takes a prime with top element an untagged event $a \in A$ to a and top element a tagged event (a, t) to a.

Lemma 10.21. *The function which takes $q \in \mathcal{Q}$ to the set of primes below q in $\mathcal{Q}$ gives an order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$. The function $\sigma_0 : S \to A$ is a strategy for which*

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, phantom, "\cong"] \arrow[d, "\sigma"'] & \mathcal{C}^\infty(S) \arrow[dl, "\sigma_0"] \\
\mathcal{C}^\infty(A) &
\end{tikzcd}$$

commutes.

Proof. The isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ is established in [1]. The diagram is easily seen to commute. Via the order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ we can carry out the argument that σ_0 is a strategy in terms of $\mathcal{Q}$ and σ. Innocence follows because the only additional causal dependencies introduced in $q(\vec{t})$ are of +ve events on −ve events. To show receptivity, suppose $q \in \mathcal{Q}$ is finite and $\sigma q \subset^- y$ in $\mathcal{C}(A)$.

There is a rigid inclusion $q \hookrightarrow q(\vec{t})$ for some $\vec{t} = (t_1, \cdots, t_i, \cdots)$, a sub-branch of T. Let

$$\varnothing \ \cdots \ \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

be the tagged sequence determined by $\vec{t}$.

First consider when $(\sigma q)^+ \neq \varnothing$. Suppose x_k is the earliest configuration at which $(\sigma q)^+ \subseteq x_k$. Then, t_k has to be +ve and

$$q^+ \cap ((x_k \smallsetminus x_{k-1}) \times \{t_k\}) \neq \varnothing\,.$$

The latter entails

$$x_k^- \subseteq \sigma q$$

because of the extra causal dependencies introduced in the definition of $q(\vec{t})$. It follows that

$$(\sigma q) \cap x_k \subseteq^+ x_k\,.$$

Moreover, as $(\sigma q)^+ \subseteq x_k$, we deduce

$$(\sigma q) \cap x_k \subseteq^- \sigma q \subseteq^- y\,.$$

By race-freeness, $x_k \cup y \in \mathcal{C}(A)$ with

$$x_k \subseteq^- x_k \cup y \text{ in } \mathcal{C}(A)\,.$$

In fact $x_k \subset^- x_k \cup y$ as $x_k^- \subseteq \sigma q \subset^- y$. Now

$$\varnothing \cdots \ \subset^+ x_k \subset^- x_k \cup y$$

is seen to form an alternating sequence, so a sub-branch of TA. From the receptivity of *str* there is a unique sub-branch $t_1, \ldots, t_k, t'_{k+1}$ of T which has this alternating sequence as image. Take q' to be the down-closure of y in $q(t_1, \ldots, t_k, t'_{k+1})$. This gives the unique q' such that $q \subseteq q'$ and $\sigma q' = y$.

Now consider when $(\sigma q)^+ = \varnothing$. Then $\varnothing \subseteq^- \sigma q \subset^- y$.

In the case where $\varnothing \in W$ we may form the alternating sequence

$$\varnothing \subset^- y\,.$$

The receptivity of *str* ensures there is a unique 1-arc branch (u_1) of T such that $\sigma q(u_1) = y$.

In the case where $\varnothing \notin W$ we also have $\varnothing \notin TW$. In this case all alternating sequences must begin $\varnothing \subset^+ x_1 \cdots$ and consequently all initial arcs of T must be +ve. Also, the empty configuration (or branch) of T cannot be +-maximal because its image under *str* is the empty configuration (or branch) of TW—impossible because *str* is a winning strategy. Thus there must be v_1, an initial, necessarily +ve arc of T. Via *str* the sub-branch (v_1) yields the alternating sequence $\varnothing \subset^+ x_1$, say. As A is race-free we obtain $x_1 \cup y \in \mathcal{C}^\infty(A)$ and the alternating sequence

$$\varnothing \subset^+ x_1 \subset^- x_1 \cup y\,.$$

From the receptivity of *str* there is a unique sub-branch (v_1, v_2) of T for which $\sigma q(v_1, v_2) = x_1 \cup y$. Take q' to be the down-closure of y in $q(v_1, v_2)$. This furnishes the unique q' such that $q \subseteq q'$ and $\sigma q' = y$.

We have shown the receptivity of σ, as required. □

Theorem 10.22. *Suppose that* $str : T \to TA$ *is a winning strategy in the tree game* $\mathrm{TG}(A, W)$. *Then* $\sigma_0 : S \to A$ *is a winning strategy in* (A, W).

Proof. For σ_0 to be winning we require that $\sigma_0 x \in W$ for any $+$-maximal $x \in \mathcal{C}^\infty(S)$. Via the order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ we can carry out the proof in $\mathcal{Q}$ rather than $\mathcal{C}^\infty(S)$. For any q which is $+$-maximal in $\mathcal{Q}$ (*i.e.* whenever $q \subseteq^+ q'$ in $\mathcal{Q}$ then $q = q'$) we require that $\sigma q \in W$.

Let q be $+$-maximal in $\mathcal{Q}$. We will show that $q = q(\vec{u})$ for some $+$-maximal branch $\vec{u}$ of T. Certainly there is a rigid inclusion $q \hookrightarrow q(\vec{t})$ for some sub-branch $\vec{t} = (t_1, \cdots, t_i, \cdots)$ of T. Let

$$\varnothing \ \cdots \stackrel{t_{i-1}}{\subset^-} x_{i-1} \stackrel{t_i}{\subset^+} x_i \stackrel{t_{i+1}}{\subset^-} \cdots$$

be the tagged sequence determined by $\vec{t}$.

Consider the case in which the set q^+ is infinite. There are two possibilities. Suppose first that

$$q^+ \cap ((x_i \smallsetminus x_{i-1}) \times \{t_i\}) \neq \varnothing\,.$$

for infinitely many $+$ve t_i. Because of the extra causal dependencies introduced in the definition of $q(\vec{t})$, the set of $-$ve events $q(\vec{t})^-$ is included in q. Hence $q \subseteq^+ q(\vec{t})$. But q is $+$-maximal, so $q = q(\vec{t})$. The second possibility is that $(\sigma q)^+ \subseteq x_k$ for some necessarily terminal configuration in the tagged alternating sequence, which now has to be of the form

$$\varnothing \ \cdots \stackrel{t_{i-1}}{\subset^-} x_{i-1} \stackrel{t_i}{\subset^+} x_i \stackrel{t_{i+1}}{\subset^-} \cdots \subset^+ x_k\,.$$

Because of the causal dependencies in $q(\vec{t})$, the set $q(\vec{t})^-$ is included in q. Hence $q \subseteq^+ q(\vec{t})$, so $q = q(\vec{t})$ because q is $+$-maximal.

Now consider the case where the set q^+ is finite. Then the set $(\sigma q)^+$, also finite, must be included in some x_k of the tagged alternating sequence, which we may assume is the earliest. Then t_k must be $+$ve. If $\sigma q \subseteq q(t_1, \cdots, t_k)$, then the set $q(t_1, \cdots, t_k)^-$ is included in q—again because of the causal dependencies there; and again $q \subseteq^+ q(t_1, \cdots, t_k)$ so $q = q(t_1, \cdots, t_k)$ because q is $+$-maximal. Otherwise, $x_k \subset^- x_k \cup (\sigma q)$ and we can extend the alternating sequence to

$$\varnothing \cdots \subset^+ x_k \subset^- x_k \cup (\sigma q)\,.$$

From the receptivity of *str* there is a sub-branch $t_1, \ldots, t_k, t'_{k+1}$ of T which has this alternating sequence as image. Now $q \subseteq^+ q(t_1, \ldots, t_k, t'_{k+1})$ so $q = q(t_1, \ldots, t_k, t'_{k+1})$ from the $+$-maximality of q.

Thus any $q \in \mathcal{Q}$ which is $+$-maximal has the form $q = q(\vec{u})$ for some sub-branch $\vec{u}$ of T. Any extension of $\vec{u}$ by a $+$-ve arc would yield a $+$-ve extension of $q(\vec{u})$, contradicting the $+$-maximality of q. Therefore $\vec{u}$ is $+$-maximal, so its image $str\{\vec{u}\}$ is in TW, as str is a winning strategy in $(TG(A,W),TW)$. But, by Proposition 10.16,

$$str\{\vec{u}\} \in TW \iff \sigma q(\vec{u}) \in W\,.$$

Hence, $\sigma q \in W$, as required. □

Corollary 10.23. *Let* (A,W) *be a race-free, bounded-concurrent game. If the tree game* $\mathrm{TG}(A,W)$ *has a winning strategy, then* (A,W) *has a winning strategy.*

Theorem 10.24. *Any race-free, concurrent-bounded game* (A,W)*, in which* W *is a Borel subset of* $\mathcal{C}^\infty(A)$*, is determined.*

Proof. Assuming (A,W) is race-free, concurrent-bounded and W is Borel, we obtain a tree game $\mathrm{TG}(A,W) = (\mathit{TA},\mathit{TW})$ in which TW is also Borel. To see that TW is Borel, recall that a configuration y of TA corresponds to an alternating sequence

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots\,,$$

so determines $f(y) =_{\mathrm{def}} \bigcup_i x_i \in \mathcal{C}^\infty(A)$. This yields a Scott-continuous function $f : \mathcal{C}^\infty(\mathit{TA}) \to \mathcal{C}^\infty(A)$. The set TW is the inverse image $f^{-1}W$, so Borel. As the tree game $\mathrm{TG}(A,W)$ is determined—Theorem 10.10—we obtain a winning strategy for Player or a winning strategy for Opponent in the tree game.

Suppose first that $\mathrm{TG}(A,W)$ has a winning strategy (for Player). By Corollary 10.23 we obtain a winning strategy for (A,W). Suppose, on the other hand, that $\mathrm{TG}(A,W)$ has a winning strategy for Opponent, *i.e.* there is a winning strategy in the dual game $(\mathrm{TG}(A,W))^\perp$. By Lemma 10.14, $\mathrm{TG}((A,W)^\perp) = \mathrm{TG}(A,W)^\perp$ has a winning strategy. By Corollary 10.23, $(A,W)^\perp$ has a winning strategy, *i.e.* there is a winning strategy for Opponent in (A,W). □

Chapter 11

Games with imperfect information

11.1 Motivation

Consider the game "rock, scissors, paper" in which the two participants Player and Opponent independently sign one of r ("rock"), s ("scissors") or p ("paper"). The participant with the dominant sign w.r.t. the relation

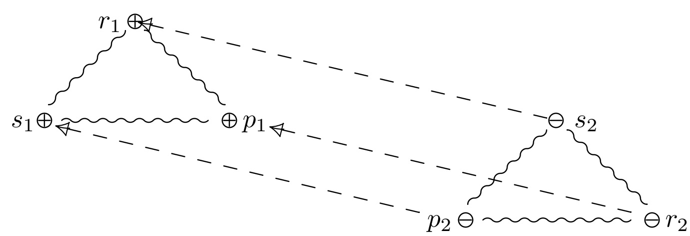
$$r \text{ beats } s, \; s \text{ beats } p \text{ and } p \text{ beats } r$$

wins. It seems sensible to represent this game by RSP, the event structure with polarity

comprising the three mutually inconsistent possible signings of Player in parallel with the three mutually inconsistent signings of Opponent. In the absence of neutral configurations, a reasonable choice is to take the *losing* configurations (for Player) to be

$$\{s_1, r_2\}, \; \{p_1, s_2\}, \; \{r_1, p_2\}$$

and all other configurations as winning for Player. In this case there is a winning strategy for Player, *viz.* await the move of Opponent and then beat it with a dominant move. Explicitly, the winning strategy $\sigma : S \to RSP$ is given as the obvious map from S, the following event structure with polarity:



But this strategy cheats. In "rock, scissors, paper" participants are intended to make their moves *independently*. The problem with the game RSP as it stands is that it is a game of *perfect information* in the sense that all moves are visible to both participants. This permits the winning strategy above with its unwanted dependencies on moves which should be unseen by Player. To adequately model "rock, scissors, paper" requires a game of *imperfect information* where some moves are masked, or inaccessible, and strategies with dependencies on unseen moves are ruled out.

11.2 Games with imperfect information

We extend concurrent games to games with imperfect information. To do so in way that respects the operations of the bicategory of games we suppose a fixed preorder of *levels* $(\Lambda, \le)$. The levels are to be thought of as levels of access, or permission. Moves in games and strategies are to respect levels: moves will be assigned levels in such a way that a move is only permitted to causally depend on moves at equal or lower levels; it is as if from a level only moves of equal or lower level can be seen.

An Λ-game (G, l) comprises a game $G = (A, W, L)$ with winning/losing conditions together with a *level function* $l : A \to \Lambda$ such that

$$a \le_A a' \implies l(a) \le l(a')$$

for all $a, a' \in A$. A Λ-strategy in the Λ-game (G, l) is a strategy $\sigma : S \to A$ for which

$$s \le_S s' \implies l\sigma(s) \le l\sigma(s')$$

for all $s, s' \in S$.

For example, for "rock, scissors, paper" we can take Λ to be the discrete preorder consisting of levels 1 and 2 unrelated to each other under $\le$. To make RSP into a suitable Λ-game the level function l takes +ve events in RSP to level 1 and −ve events to level 2. The strategy above, where Player awaits the move of Opponent then beats it with a dominant move, is now disallowed because it is not a Λ-strategy—it introduces causal dependencies which do not respect levels. If instead we took Λ to be the unique preorder on a single level the Λ-strategies would coincide with all the strategies.

11.2.1 The bicategory of Λ-games

The introduction of levels meshes smoothly with the bicategorical structure on games.

For a Λ-game (G, l_G), define its dual $(G, l_G)^\perp$ to be $(G^\perp, l_{G^\perp})$ where $l_{G^\perp}(\overline{a}) = l_G(a)$, for a an event of G.

For Λ-games (G, l_G) and (H, l_H), define their parallel composition $(G, l_G)\|(H, l_H)$ to be $(G\|H, l_{G\|H})$ where $l_{G\|H}((1,a)) = l_G(a)$, for a an event of G, and $l_{G\|H}((2,b)) = l_H(b)$, for b an event of H.

A strategy between Λ-games from (G, l_G) to (H, l_H) is a strategy in $(G, l_G)^\perp\|(H, l_H)$.

Proposition 11.1.
(i) Let (G, l_G) be a Λ-game where G satisfies **(Cwins)**. *The copy-cat strategy on G is a Λ-strategy.*
(ii) The composition of Λ-strategies is a Λ-strategy.

Proof. (i) The additional causal links introduced in the construction of the copy-cat strategy are between complementary events in $G^\perp$ and G, at the same level in Λ, and so respect ≤.

(ii) Let (G, l_G), (H, l_H) and (K, l_K) be Λ-games. Let $\sigma : G \mathrel{+\!\!\!\rightarrow} H$ and $\tau : H \mathrel{+\!\!\!\rightarrow} K$ be Λ-strategies. We show their composition $\tau\odot\sigma$ is a Λ-strategy.

It suffices to show $p \rightarrowtriangle p'$ in $T\odot S$ implies $l_{G^\perp\|K}\tau\odot\sigma(p) \le l_{G^\perp\|K}\tau\odot\sigma(p')$. Suppose $p \rightarrowtriangle p'$ in $T\odot S$ with $top(p) = e$ and $top(p') = e'$. Take $x \in \mathcal{C}(T\odot S)$ containing p' so p too. Then,

$$e \rightarrowtriangle_{\bigcup x} e_1 \rightarrowtriangle_{\bigcup x} \cdots \rightarrowtriangle_{\bigcup x} e_{n-1} \rightarrowtriangle_{\bigcup x} e'$$

where $e, e' \in V_0$ and $e_i \notin V_0$ for $1 \le i \le n-1$. (V_0 consists of 'visible' events of the stable family, those of the form $(s,*)$ with $\sigma_1(s)$ defined, or $(*,t)$, with $\tau_2(t)$ defined.) The events e_i have the form (s_i, t_i) where $\sigma_2(s_i) = \tau_1(t_i)$, for $1 \le i \le n-1$.

Any individual link in the chain above has one of the forms:

$$(s,t) \rightarrowtriangle_{\bigcup x} (s',t'),\ (s,*) \rightarrowtriangle_{\bigcup x} (s',t'),$$
$$(*,t) \rightarrowtriangle_{\bigcup x} (s',t'),\ (s,t) \rightarrowtriangle_{\bigcup x} (s',*),\ \text{ or }\ (s,t) \rightarrowtriangle_{\bigcup x} (*,t').$$

By Lemma 3.27, for any link either $s \rightarrowtriangle_S s'$ or $t \rightarrowtriangle_T t'$. As σ and τ are Λ-strategies, this entails

$$l_{G^\perp\|H}\sigma(s) \le l_{G^\perp\|H}\sigma(s') \ \text{ or }\ l_{H^\perp\|K}\tau(t) \le l_{H^\perp\|K}\tau(t')$$

for any link. Consequently ≤ is respected across the chain and $l_{G^\perp\|K}\tau\odot\sigma(p) \le l_{G^\perp\|K}\tau\odot\sigma(p')$, as required. □

W.r.t. a particular choice of access levels $(\Lambda, \le)$ we obtain a bicategory $\mathbf{WGames}_\Lambda$. Its objects are Λ-games (G, l) where G satisfies **(Cwins)** with arrows the Λ-strategies and 2-cells maps of spans. It restricts to a sub-bicategory of deterministic Λ-strategies, which as before is equivalent to an order-enriched category.

11.3 Hintikka's IF logic

We present a variant of Hintikka's Independence-Friendly (IF) logic and propose a semantics in terms of concurrent games with imperfect information. Assume a preorder $(\Lambda, \leq)$. The syntax for IF logic is essentially that of the predicate calculus, but with levels in Λ associated with quantifiers: formulae are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists^\lambda x.\ \phi \mid \forall^\lambda x.\ \phi$$

where $\lambda \in \Lambda$, R ranges over basic relation symbols of a fixed arity and $x, x_1, x_2, \cdots$ over variables.

Assume M, a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M; so M is a model for the predicate calculus in which the quantifier levels are stripped away. Again, an environment ρ is a function from variables to values; again, $\rho[v/x]$ means the environment ρ updated to value v at variable x. W.r.t. a model M and an environment ρ, we denote each closed formula ϕ of IF logic by a Λ-game, following very closely the definitions in Section 9.8. The differences are the assignment of levels to events and that the order on Λ has to be respected by the (modified) prefixed sums which quantified formulae denote.

The prefixed game $\oplus^\lambda.(A, W, l)$ comprises the event structure with polarity $\oplus.A$ in which all the events of $a \in A$ where $\lambda \leq l(a)$ are made to causally depend on a fresh +ve event $\oplus$, itself assigned level λ. Its winning conditions are those configurations $x \in \mathcal{C}^\infty(\oplus.A)$ of the form $\{\oplus\} \cup y$ for some $y \in W$. The game $\bigoplus^\lambda_{v\in V}(A_v, W_v, l_v)$ has underlying event structure with polarity the sum $\sum_{v\in V} \oplus^\lambda.A_v$, maintains the same levels as its components, with a configuration winning iff it is the image of a winning configuration in a component under the injection to the sum. The game $\bigominus^\lambda_{v\in V} G_v$ is defined dually, as $(\bigoplus^\lambda_{v\in V} G_v^\perp)^\perp$. In this game the empty configuration is winning but Opponent gets to make the first move.

True denotes the Λ-game the unit w.r.t. $\otimes$ and false denotes he unit w.r.t. $\parr$. Denotations of conjunctions and disjunctions are given by the operations of $\otimes$ and $\parr$ on Λ-games, while negations denote dual games. W.r.t. an environment ρ, universal and existential quantifiers denote the *prefixed sums* of games:

$$[\![\exists^\lambda x.\ \phi]\!]^\Lambda_M \rho = \bigoplus^{\lambda}_{v\in V_M} [\![\phi]\!]^\Lambda_M \rho[v/x]$$

$$[\![\forall^\lambda x.\ \phi]\!]^\Lambda_M \rho = \bigominus^{\lambda}_{v\in V_M} [\![\phi]\!]^\Lambda_M \rho[v/x]\,.$$

As a definition, an IF formula ϕ is satisfied w.r.t. an environment ρ, written

$$\rho \vDash^\Lambda_M \phi\,,$$

iff the Λ-game $[\![\phi]\!]^\Lambda_M \rho$ has a winning strategy.

Chapter 12

Linear strategies

It has recently become clear that concurrent strategies support several refinements. For example, define a *rigid* strategy to be a strategy σ in which both components σ_1 and σ_2 preserve causal dependency where defined. Copy-cat strategies are rigid, and the composition of rigid strategies is rigid, so rigid strategies form a sub-bicategory of **Strat**. We can refine rigid strategies further to *linear* strategies, where each +ve output event depends on a maximum +ve event of input, and dually, a −ve event of input depends on a maximum −ve event of output. By introducing this extra relevance, of input to output and output to input, we can recover coproducts and products lacking in **Strat**. Though doing so we lose monoidal closure.

12.1 Rigid strategies

Definition 12.1. A partial map of event structures which preserves causal dependency whenever it is defined, *i.e.* $e' \leq e$ implies $f(e') \leq f(e)$ whenever both $f(e')$ and $f(e)$ are defined, is called *partial rigid*.

A strategy $\sigma : S \to A$ in a game A is *rigid* iff the map σ is rigid. Rigidity subsumes innocence, so a rigid strategy in A amounts to a rigid map $\sigma : S \to A$ which is receptive.

A *rigid strategy from a game A to a game B* is a strategy $\sigma : S \to A^\perp \| B$ where σ_1 and σ_2 are partial-rigid maps.

Definition 12.2. *Let A and B be event structures with polarity. Define $A \parr_r B = \Pr(\mathcal{Q})$ and $\mathcal{Q}$ is the rigid family consisting of all partial orders*

$$(\{1\} \times x \cup \{2\} \times y, \leq)\,,$$

with $x \in \mathcal{C}(A)$, $y \in \mathcal{C}(B)$, in which

$$
\begin{aligned}
(1,a) \le (1,a') &\iff a \le_A a' \,, \\
(2,b) \le (1,b') &\iff b \le_B b' \,, \\
(1,a) \rightarrowtriangle (2,b) &\implies pol_A(a) = - \ \& \ pol_B(b) = + \,, \\
(2,b) \rightarrowtriangle (1,a) &\implies pol_A(a) = + \ \& \ pol_B(b) = - \,;
\end{aligned}
$$

in other words $\mathcal{Q}$ contains augmentations of the partial order induced by $A\|B$ on $\{1\}\times x \cup \{2\}\times y$ which maintain innocence of the inclusion map $\{1\}\times x \cup \{2\}\times y \hookrightarrow A\|B$. The total map $top : A \parr_r B \to A\|B$ of event structures with polarity takes a prime to its top element.

Proposition 12.3. *A rigid strategy from A to B corresponds to a rigid strategy in the game $A^\perp \parr_r B$.*

Proof. By specializing to rigid strategies the natural correspondence of the adjunction from the category of event structures with rigid maps to that with total maps [7]. □

12.1.1 The bicategory of rigid strategies

Proposition 12.4. *For any game A, the copy-cat strategy γ_A is rigid.*

The composition of rigid strategies is rigid.

Lemma 12.5. *Let $\sigma : S \to A^\perp\|B$ and $\tau : T \to B^\perp\|C$ be rigid strategies. Let $z \in \mathcal{C}(T) \circledast \mathcal{C}(S)$. If $(s,t) \rightarrowtriangle_z (s',t')$, then $s \rightarrowtriangle_S s'$ & $t \rightarrowtriangle_T t'$.*

Proof. By Lemma 3.27(iii), either $s \rightarrowtriangle_S s'$ or $t \rightarrowtriangle_T t'$. Suppose the case $s \rightarrowtriangle_S s'$. Then $\sigma_2(s) \rightarrowtriangle_B \sigma_2(s')$ by rigidity, so $\overline{\sigma_2(s)} \rightarrowtriangle_{B^\perp} \overline{\sigma_2(s')}$. Recall from the construction of $\mathcal{C}(T) \circledast \mathcal{C}(S)$ that $\tau_1(t) = \overline{\sigma_2(s)}$ and $\tau_1(t') = \overline{\sigma_2(s')}$. By Proposition 3.14 (taking $x = \pi_2 z$), we deduce that $t <_T t'$. However, by Lemma 3.27(iii), either $t \rightarrowtriangle_T t'$ or $t \, co \, t'$, whence we must have $t \rightarrowtriangle_T t'$. The case $t \rightarrowtriangle_T t'$ similarly entails $s \rightarrowtriangle_S s'$. □

Lemma 12.6. *Let $\sigma : S \to A^\perp\|B$ and $\tau : T \to B^\perp\|C$ be rigid strategies. Let $z \in \mathcal{C}(T) \circledast \mathcal{C}(S)$. If $e \le_z e'$, then*

(i) if $\pi_1(e)$ and $\pi_1(e')$ are defined, then $\pi_1(e) \le_S \pi_1(e')$, and

(ii) if $\pi_2(e)$ and $\pi_2(e')$ are defined, then $\pi_2(e) \le_T \pi_2(e')$.

Proof. We show for all $\rightarrowtriangle_z$-chains

$$e \rightarrowtriangle_z e_1 \rightarrowtriangle_z \cdots \rightarrowtriangle_z e_m = e'$$

from e to e' that (i) and (ii), by induction on the length m.

The basis when $m = 1$, where $e \rightarrowtriangle_z e'$, follows by Lemmas 3.27 and 12.5.

Suppose $m > 1$. We show (i)—the proof of (ii) is analogous. Assume $\pi_1(e)$ and $\pi_1(e')$ are defined, with $\pi_1(e) = s$ and $\pi_1(e') = s'$.

If for some i with $0 < i < m$ we have $\pi_1(e_i) = s_i$, for some $s_i \in S$, then $s \leq_S s_i$ and $s_i \leq_S s'$ from the induction hypothesis. Hence $\pi_1(e) = s \leq_S s' = \pi_1(e')$.

Suppose otherwise, that for all i with $0 < i < m$ we have $\pi_1(e_i)$ undefined so $e_i = (*, t_i)$, for some $t_i \in T$. In particular,

$$e \rightarrowtriangle_z (*, t_1) \quad \text{and} \quad (*, t_{m-1}) \rightarrowtriangle_z e' \,.$$

By Lemma 3.27, e and e' must have the forms $e = (s, t)$ and $e' = (s', t')$ with $t \rightarrowtriangle_T t_1$ and $t_{m-1} \rightarrowtriangle_T t'$, for some $t, t' \in T$. From the induction hypothesis $t_1 \leq_T t_{m-1}$, so $t \leq_T t'$. As τ_1 is partial rigid, $\tau_1(t) \leq_{B^\perp} \tau_1(t')$. Hence from the definition of $\mathcal{C}(T) \circledast \mathcal{C}(S)$ we obtain $\sigma_2(s) = \overline{\tau_1(t)} \leq_B \overline{\tau_1(t')} = \sigma_2(s')$. By Proposition 3.14, we deduce $s \leq_S s'$, *i.e.* $\pi_1(e) \leq_S \pi_1(e')$, as required. $\square$

Corollary 12.7. *The composition $\tau \odot \sigma$ of rigid strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ is rigid.*

12.2 Nondeterministic linear strategies

Formally, a (nondeterministic) *linear* strategy is a strategy

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ A^\perp & & B\,, \end{tikzcd}$$

where σ_1 and σ_2 are partial rigid maps such that

$$\begin{aligned} &\forall s \in S.\ pol_S(s) = + \ \&\ \sigma_2(s) \text{ is defined} \\ &\Longrightarrow \\ &\exists s_0 \in S.\ pol_S(s_0) = - \ \&\ \sigma_1(s_0) \text{ is defined } \&\ s_0 \leq_S s\ \& \\ &\forall s_1 \in S.\ pol_S(s_1) = - \ \&\ \sigma_1(s_1) \text{ is defined } \&\ s_1 \leq_S s \implies s_1 \leq_S s_0 \end{aligned}$$

and

$$\begin{aligned} &\forall s \in S.\ pol_S(s) = + \ \&\ \sigma_1(s) \text{ is defined} \\ &\Longrightarrow \\ &\exists s_0 \in S.\ pol_S(s_0) = - \ \&\ \sigma_2(s_0) \text{ is defined } \&\ s_0 \leq_S s\ \& \\ &\forall s_1 \in S.\ pol_S(s_1) = - \ \&\ \sigma_2(s_1) \text{ is defined } \&\ s_1 \leq_S s \implies s_1 \leq_S s_0\,. \end{aligned}$$

More informally, this says

- every +ve event of S over B depends on a $\leq_S$-maximum −ve event over $A^\perp$, and symmetrically

- every +ve event of S over $A^\perp$ depends on a $\leq_S$-maximum −ve event over B.

We now demonstrate that copy-cat strategies are linear and linear strategies are closed under composition, so that linear strategies form a sub-bicategory **Strat**.

Lemma 12.8. *For all games A the copy-cat strategy γ_A is linear. Let $\sigma : A \rightarrow\!\!\!\!\!\!+\; B$ and $\tau : B \rightarrow\!\!\!\!\!\!+\; C$ be linear strategies. Then their composition $\tau \odot \sigma : A \rightarrow\!\!\!\!\!\!+\; C$ is linear.*

Proof. Consider the copy-cat strategy

$$\begin{tikzcd} & \mathbb{C}_A \arrow[dl, "\gamma_{A1}"'] \arrow[dr, "\gamma_{A2}"] & \\ A^\perp & & A, \end{tikzcd}$$

defined in Proposition 4.1. Let $c \in \mathbb{C}_A$ where $pol_{\mathbb{C}_A}(c) = +$ and $\gamma_{A2}(c)$ is defined. From the proof of Proposition 4.1,

$$\begin{aligned} c' \leq_{\mathbb{C}_A} c \text{ iff } & (i)\ c' \leq_{A^\perp \| A} c \text{ or} \\ & (ii)\ \exists c_0 \in A^\perp \| A.\ pol_{A^\perp \| A}(c_0) = + \ \& \\ & \qquad c' \leq_{A^\perp \| A} \overline{c_0} \ \&\ c_0 \leq_{A^\perp \| A} c\,. \end{aligned}$$

In particular for $c' \in \mathbb{C}_A$ with $\gamma_{A1}(c')$ defined,

$$\begin{aligned} c' \leq_{\mathbb{C}_A} c \text{ iff } & \exists c_0 \in A^\perp \| A.\ pol_{A^\perp \| A}(c_0) = + \ \& \\ & \qquad c' \leq_{A^\perp \| A} \overline{c_0} \ \&\ c_0 \leq_{A^\perp \| A} c\,. \end{aligned}$$

It follows that $c' \leq_{\mathbb{C}_A} \overline{c}$. This ensures that $\overline{c}$ is the $\leq_{\mathbb{C}_A}$-maximum −ve event for which $\gamma_{A1}(\overline{c})$ is defined and $\overline{c} \leq_{\mathbb{C}_A} c$. Similarly, if $pol_{\mathbb{C}_A}(c) = +$ and $\gamma_{A1}(c)$ is defined, $\overline{c}$ is the maximum −ve event for which $\gamma_{A2}(\overline{c})$ is defined and $\overline{c} \leq_{\mathbb{C}_A} c$.

Suppose

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ A^\perp & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & T \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ B^\perp & & C \end{tikzcd}$$

are linear strategies. Recall the construction of their composition from Section 4.3.2. Consider any chain of immediate dependencies

$$(s, *) \rightarrowtriangle_z \cdots \rightarrowtriangle_z (*, t)\,,$$

where $s \in S$ is −ve and $t \in T$ is +ve, within a configuration z of $\mathcal{C}(T) \circledast \mathcal{C}(S)$. The chain must contain an element (s_j, t_j) where $\sigma_2(s_j) \in B$ and $\tau_1(t_j) \in B^\perp$ with $\sigma_2(s_j) = \overline{\tau_1(t_j)}$; otherwise there would have to be a link $(s_i, *) \rightarrowtriangle_z (*, t_{i+1})$, which is impossible by Lemma 3.27(i). Consider the earliest stage along the chain at which such an element appears, say

$$(s,*) \rightarrowtriangle_z \cdots \rightarrowtriangle_z (s_{n-1},*) \rightarrowtriangle_z (s_n,t_n) \rightarrowtriangle_z \cdots \rightarrowtriangle_z (*,t)\,.$$

From Lemma 12.6, parts (i) and (ii), respectively,

$$s \le_S s_n \text{ and } t_n \le_T t\,.$$

By Lemma 3.27(i), $s_{n-1} \rightarrowtriangle_{\pi_1 z} s_n$ where $\sigma_1(s_{n-1}) \in A^\perp$ and $\sigma_2(s_n) \in B$. As σ is innocent, we must have $pol_S(s_{n-1}) = -$ and $pol_S(s_n) = +$. Consequently, $pol_T(t_n) = -$.

Now, exploiting the linearity of τ, let t' be the maximum −ve event in T over $B^\perp$ on which t depends. As $t' \le_T t$ there must be (a unique) $s' \in S$ such that $(s',t') \in z$; this is because $\pi_2 z \in \mathcal{C}(T)$ so is down-closed. Let s'' be the maximum −ve event in S over $A^\perp$ on which s' depends. We will show $s \le_S s''$.

As $t_n \le_T t$ and t_n is −ve,

$$t_n \le_T t'\,.$$

From the rigidity of τ,

$$\tau_1(t_n) \le_{B^\perp} \tau_1(t')\,.$$

From the definition of $\mathcal{C}(T) \circledast \mathcal{C}(S)$, we know $\sigma_2(s_n) = \overline{\tau_1(t_n)}$ and $\sigma_2(s') = \overline{\tau_1(t')}$ and hence that $\sigma_2(s_n) \le_B \sigma_2(s')$. Via Proposition 3.14, $s_n \le_S s'$. Combined with the established $s \le_S s_n$, this entails $s \le_S s'$. From the linearity of σ, as s is −ve,

$$s \le_S s''\,.$$

Whenever $p \le_{T\odot S} q$ with p −ve over $A^\perp$, q +ve over C defined, there is $z \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ such that $p = [(s,*)]_z$ and $q = [(*,t)]_z$ with $(s,*) \rightarrowtriangle_z \cdots \rightarrowtriangle_z (*,t)$, as above. The description of s'' given above furnishes $[(s'',*)]_z$, the $\le_{T\odot S}$-maximum −ve event over $A^\perp$ on which $[(*,t)]_z$ depends.

The remaining, symmetric, condition for the linearity of $\tau\odot\sigma$ is proved analogously. □

12.3 Deterministic linear strategies

Deterministic linear strategies are, of course, linear strategies

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ A^\perp & & B\,, \end{tikzcd}$$

where S is deterministic. They determine a sub-bicategory of **DGames** maintaining duality.

Proposition 12.9. *The full sub-bicategory of deterministic linear strategies in which objects are games in which all polarities are +ve is equivalent to Girard's (order-enriched) category of coherence spaces and linear maps.*

Its sub-bicategory **Lin** of deterministic subcategories **DLin** has products and coproducts constructed as follows.

The coproduct $A \oplus B$ comprises the parallel composition $A \| B$ with additional conflict (lack of consistency) between all pairs of +ve events of A and +ve events of B. In other words

$$X \in \mathrm{Con}_{A \oplus B} \iff X \in \mathrm{Con}_{A \| B} \ \& \\ X_1 \cap A^+ \neq \emptyset \implies X_2 \cap B^+ = \emptyset\,.$$

Recall the operations $X_1 =_{\mathrm{def}} \{a \mid (1,a) \in X\}$ and $X_2 =_{\mathrm{def}} \{b \mid (2,b) \in X\}$ project X to its set of events in A and B respectively.

Dually, the product $A \& B$ comprises the parallel composition $A \| B$ with additional conflict between all pairs of −ve events of A and −ve events of B. In other words

$$X \in \mathrm{Con}_{A \& B} \iff X \in \mathrm{Con}_{A \| B} \ \& \\ X_1 \cap A^- \neq \emptyset \implies X_2 \cap B^- = \emptyset\,.$$

But **Lin** and **DLin** are not monoidal closed!

12.4 Linear strategies as pairs of relations

A linear strategy from $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\!\rightarrow B$ is associated with a pair of dependency relations, one from A^+ to B^+ and another from B^- to A^-.

Deterministic linear strategies can be characterised in terms of Girard's linear maps extended to event structures. A *G-linear* map $F : A \to_G B$ from and event structure A to an event structure B is a function

$$F : \mathcal{C}^\infty(A) \to \mathcal{C}^\infty(B)$$

which preserves unions and is stable. Such maps can be described as certain relations between A and B. We will write

$$aFb \iff b \in F([a])\,,$$

where $a \in A, b \in B$.

A deterministic linear strategy $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\!\rightarrow B$ corresponds to a pair of G-linear maps $F_+ : A^+ \to_G B^+$ and $F_- : B^- \to_G A^-$ such that

$$a \leq_A a' \ \&\ pol_A(a) = + \ \&\ pol_A(a') = - \ \&\ \ \&\ a' F_+ b' \ \&\ bF_- a \implies b \leq_B b'$$

and

$$b \leq_B b' \ \&\ pol_A(b) = + \ \&\ pol_A(b') = - \ \&\ \ \&\ aF_+ b \ \&\ b' F_- a' \implies a \leq_A a'$$

for all $a, a' \in A, b, b' \in B$.

To be completed.

Chapter 13

Strategies with neutral events

NOT UP TO DATE*NEEDS TO CATCH UP WITH MFPS 14 SUBMISSION + UPDATE OF DEFN OF PARTIAL STRATEGY ****

Neutral events occur through the synchronization of moves of opposing polarities in the composition of strategies. Here we consider strategies with neutral events in order to

1. deal more accurately with deadlocks which can occur in the composition of strategies, and in particular support 'may' and 'must' equivalences;

2. provide a structural operational semantics for strategies;

3. give a more accurate treatment of winning strategies, through a true account of those configurations which may be the end result of a strategy—these need not be +-maximal.

13.1 Deadlocks

Composition of strategies can introduce deadlock which is presently undetected:

Example 13.1. ****deadlock through imposing incompatible causal dependencies between events in B****

Example 13.2. $B = \oplus \| \oplus$ *****
strategy σ_1 nondeterministically chooses right or left move in B
strategy σ_2 chooses just right move in B
strategy τ yields output in C if gets right event of B as input
**** the two strategy compositions $\tau \odot \sigma_1$ and $\tau \odot \sigma_2$ are indistinguishable*

If we are to detect the possibility of deadlock we should take some account of the hidden neutral moves a strategy can perform.

We extend event structures with polarity with neutral events. An event structure with polarity is an event structure E with a polarity function pol : $E\{+,-,0\}$; events tagged by 0 are *neutral* events. Neutral events are drawn as $\circledcirc$. Maps are maps of event structures which preserve polarity when defined.

13.2 Strategies with neutral moves

We continue to assume games only possess events of +ve or −ve polarity. (Later we may consider 'reactive games' which also contain neutral events.)

To treat such phenomena explicitly and in order to obtain a transition semantics we extend strategies with neutral events. Extend event structures with polarity to include a neutral polarity 0; as before, maps preserve polarities when defined.

Definition 13.3. A *partial strategy* in a game A (in which all events have +ve or −ve polarity) comprises a total map $\sigma : S \to N\|A$ of event structures with polarity (in which S may also have neutral events)

where

(i) N is an event structure consisting solely of neutral events;

(ii) σ is *receptive*,$\forall x \in \mathcal{C}(S).\ \sigma x \stackrel{a}{\longrightarrow\!\!\subset}\ \&\ pol_A(a) = - \implies \exists! s.\ x \stackrel{s}{\longrightarrow\!\!\subset}\ \&\ \sigma(s) = a$;

(iii) σ is innocent in that it is both +-innocent and −-innocent:

+-innocent: if $s \rightarrowtriangle s'\ \&\ pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$;

−-innocent: if $s \rightarrowtriangle s'\ \&\ pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

(Note that s' in +-innocence and s in −-innocence may be neutral events, so this generalizes the condition of innocence of before. This definition of innocence appears in the work of Faggian and Piccolo*****.)

Conditions (i), (ii) and (iii) imply:

(iv) in the partial-total factorization of the composition of $S \xrightarrow{\sigma} N\|A$ with the projection $N\|A \to A$

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\sigma"'] & S_0 \arrow[d, "\sigma_0"] \\ N\|A \arrow[r] & A \end{tikzcd}$$

the defined part σ_0 is a strategy, as formerly understood.

(The old definition of partial strategy given in [?] is a little weaker in that it doesn't entail +-innocence in its sense extended to neutral events—see Lemma 13.6.)

Note that strategies are those partial strategies in which N is the empty event structure.

It may seem odd that partial strategies are total as functions. The following proposition should make the choice of name more understandable. Firstly, as earlier in Definition 4.6, it is useful to define innocence and receptivity on partial maps of event structures with polarity including now neutral polarities.

Definition 13.4. Let $f : S \to A$ be a partial map of event structures with polarity with neutral polarities.. Say f is *receptive* when

$$f(x) \overset{a}{\longrightarrow}\!\!\subset \ \& \ pol_A(a) = - \implies \exists! s \in S.\ x \overset{s}{\longrightarrow}\!\!\subset \ \& \ f(s) = a$$

for all $x \in \mathcal{C}(S)$, $a \in A$.

Say f is *innocent* when it is both +-innocent and −-innocent, *i.e.*

$$\begin{aligned} s \rightarrowtriangle s' \ \& \ pol(s) = + \ \& \ f(s) \text{ is defined } &\implies \\ & f(s') \text{ is defined } \ \& \ f(s) \rightarrowtriangle f(s')\,, \\ s \rightarrowtriangle s' \ \& \ pol(s') = - \ \& \ f(s') \text{ is defined } &\implies \\ & f(s) \text{ is defined } \ \& \ f(s) \rightarrowtriangle f(s')\,. \end{aligned}$$

Proposition 13.5. *Let A be an event structure with polarity in which all events have +ve or −ve polarity. Let $\sigma : S \to A$ be a (partial) map of event structures with polarity (in which S may have neutral events) which is receptive and innocent and has domain of definition the non-neutral events of S. Define N to be the event structure obtained as the projection of S to its neutral events, in which all events are considered neutral. Then, its defined part σ_0 is a strategy and the function $\sigma' : S \to N \| A$ which acts as the identity function on neutral events and as σ on non-neutral events is a partial strategy.*

Why have we not taken the partial maps of Proposition 13.5 as our definition of partial strategies? Because the partial maps of the proposition do not behave well under pullback, and this would complicate the definition of composition and spoil later results such as that the pullback of a partial strategy is a partial strategy. Very roughly, with our choice of definition we are able to localise neutral events to the games over which they occur—with the definition Proposition 13.5 suggests, different forms of undefined would become conflated.

Lemma 13.6. *Let A be a game (with no neutral events) and N an event structure consisting solely of neutral events. Let S be an event structure with polarity, possibly with neutral events. Let $\sigma : S \to N \| A$ be a total map of event structures preserving polarities. Then, σ is a partial strategy iff σ is receptive, there is no incidence of a +ve event immediately preceding a neutral event in S (i.e. no $\oplus \rightarrowtriangle \circledcirc$) and axiom (iv), viz. in the partial-total factorization of the composition of $S \xrightarrow{\sigma} N \| A$ with the projection $N \| A \to A$*

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\sigma"'] & S_0 \arrow[d, "\sigma_0"] \\ N \| A \arrow[r] & A \end{tikzcd}$$

the defined part σ_0 is a strategy.

Proof. *"If":* Assume σ is receptive, no incidence of $\oplus \rightarrowtriangle \circledcirc$ in S and that the defined part σ_0 is a strategy. For σ to be a partial strategy we require in addition that σ is innocent. Suppose $s \rightarrowtriangle s'$ in S where s is +ve. By assumption, s'

cannot be neutral. It follows that $s \rightarrowtriangle s'$ in S_0 so $\sigma(s) = \sigma_0(s) \rightarrowtriangle \sigma_0(s') = \sigma(s')$ by the innocence of σ_0. Similarly if $s \rightarrowtriangle s'$ in S where s' is −ve and s is not neutral we obtain $s \rightarrowtriangle s'$ in S_0 so inherit $\sigma(s) \rightarrowtriangle \sigma(s')$ from the innocence of σ_0. It remains to show the impossibility of $s \rightarrowtriangle s'$ in S where s' is −ve and s is neutral. Then s would be a $\leq$-maximal element of $[s')$ ensuring that $x =_{\mathrm{def}} [s') \setminus \{s\}$ is a configuration. We must have $\sigma x \stackrel{\sigma(s')}{\longrightarrow\!\!\subset}$ in $N \| A$ as $\sigma(s')$ cannot causally depend on $\sigma(s)$. By the receptivity of σ we get $s'' \neq s'$ such that $\sigma(s'') = \sigma(s')$; we have $s'' \neq s'$ as s'' does not share with s' its causal dependency on s. But now, letting $x_0 =_{\mathrm{def}} x \cap S_0$, we obtain a configuration of S_0 for which $x_0 \stackrel{s'}{\longrightarrow\!\!\subset}$ and $x_0 \stackrel{s''}{\longrightarrow\!\!\subset}$ with $\sigma_0(s') = \sigma_0(s'')$, contradicting the receptivity of σ_0.

"Only if": Suppose σ is a partial strategy. Certainly σ is receptive and from its innocence there is no incidence of $\oplus \rightarrowtriangle \circledcirc$. We require that its defined part σ_0 is receptive and innocent. For receptivity, suppose $\sigma_0 x_0 \stackrel{a}{\longrightarrow\!\!\subset}$ with a −ve and x_0 a finite configuration of S_0. Taking $x =_{\mathrm{def}} [x_0]_S$ we obtain $\sigma x \stackrel{a}{\longrightarrow\!\!\subset}$. From the receptivity of σ there is (a unique) s such that $x \stackrel{s}{\longrightarrow\!\!\subset}$ with $\sigma(s) = a$. But $s \in S_0$, being −ve, with $\sigma_0(s) = a$. Its uniqueness follows from the uniqueness part of the receptivity of σ once we remember that from the innocence of σ no −ve event of S can immediately causally depend on a neutral event; so that $x_0 \stackrel{s'}{\longrightarrow\!\!\subset}$ in S_0 implies $[x_0]_S \stackrel{s'}{\longrightarrow\!\!\subset}$ in S. Because, in addition, no neutral event can immediately causally depend on a +ve event, whenever $s \rightarrowtriangle s'$ in S_0 we also have $s \rightarrowtriangle s'$ in S. It follows that σ_0 inherits innocence from σ. □

Recall we assume that in games all events have +ve or −ve polarity.

Definition 13.7. A *partial strategy from* a game A *to* a game B comprises a total map $\sigma : S \to A^\perp \| N \| B$ of event structures with polarity (in which S may also have neutral events) where
(i) N is an event structure consisting solely of neutral events;
(ii) σ is *receptive*, $\forall x \in \mathcal{C}(S).\ \sigma x \stackrel{a}{\longrightarrow\!\!\subset}\ \&\ pol_A(a) = - \implies \exists! s.\ x \stackrel{s}{\longrightarrow\!\!\subset}\ \&\ \sigma(s) = a$;
(iii) σ is innocent in that it is both +-innocent and −-innocent:
+-innocent: if $s \rightarrowtriangle s'\ \&\ pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$;
−-innocent: if $s \rightarrowtriangle s'\ \&\ pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.
(Note again that s' in +-innocence and s in −-innocence may be neutral events.)

Again, conditions (i), (ii) and (iii) imply:
(iv) in the partial-total factorization of the composition of σ with the projection $A^\perp \| N \| B \to A^\perp \| B$,

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\sigma"'] & S_0 \arrow[d, "\sigma_0"] \\ A^\perp \| N \| B \arrow[r] & A^\perp \| B \end{tikzcd}$$

the defined part σ_0 is a strategy.

Note that partial strategies in a game A correspond to partial strategies from the empty game to A, and that strategies between games in A correspond to those partial strategies in which the neutral events N are the empty event structure.

We can compose two partial strategies

$$\sigma : S \to A^{\perp} \| N_S \| B \quad \text{and} \quad \tau : T \to B^{\perp} \| N_T \| C$$

by pullback. Ignoring polarities temporarily, and padding with identity maps, we obtain $\tau \circledast \sigma$ via the pullback

$$\begin{tikzcd}
& T \circledast S \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
S \| N_T \| C \arrow[dr, "\sigma \| N_T \| C"'] & \, & A \| N_S \| T \arrow[dl, "A \| N_S \| \tau"] \\
& A \| N_S \| B \| N_T \| C &
\end{tikzcd}$$

as the ensuing map

$$\tau \circledast \sigma : T \circledast S \to A^{\perp} \| (N_S \| B \| N_T) \| C$$

once we reinstate polarities and make the events of B neutral. Receptivity of $\tau \circledast \sigma$ follows directly from that of σ and τ. That the defined part of $\tau \circledast \sigma$ is a strategy follows once we have shown that the defined part of the composite

$$T \circledast S \xrightarrow{\tau \circledast \sigma} A^{\perp} \| (N_S \| B \| N_T) \| C \longrightarrow A^{\perp} \| C$$

is isomorphic to $\tau_0 \odot \sigma_0$, the composition of the defined parts of σ and τ. This relies on the fact that, under certain conditions, the defined part of a pullback is the pullback of its defined parts. Recall a map of event structures is a projection if its defined part is an isomorphism. ***PROOFS BELOW NEEDS REVISION IN LINE WITH REVN OF DEFN OF PARTL STRAT***

Lemma 13.8. *In the category of event structures, let $\sigma : S \to C$ and $\tau : T \to C$ be total maps, and $p : C \to C_0$ be a projection. Let $\sigma_0 : S_0 \to C_0$ be the defined part of the composite $p\sigma$ and $\tau_0 : T_0 \to C_0$ be the defined part of $p\tau$. Let P, π_S, π_T be the pullback of σ, τ. Let $P_0, \pi_{S_0}, \pi_{T_0}$ be the pullback of σ_0, τ_0. Then,*

$\sigma_0\pi_{S_0} = \tau_0, \pi_{T_0} : P_0 \to C_0$ *is the defined part of* $p\,\sigma\pi_S = p\,\tau\pi_T : P \to C_0$.

$$\begin{tikzcd}
P_0 \arrow[rrr, "\pi_{T_0}"] \arrow[ddd, "\pi_{S_0}"'] & & & T_0 \arrow[ddd, "\tau_0"] \\
& P \arrow[ul] \arrow[r, "\pi_T"] \arrow[d, "\pi_S"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[ur] \arrow[d, "\tau"] & \\
& S \arrow[dl] \arrow[r, "\sigma"'] & C \arrow[dr, "p"] & \\
S_0 \arrow[rrr, "\sigma_0"'] & & & C_0
\end{tikzcd}$$

Proof. ********** □

****EARLIER***

Proposition 13.9. *Let* $f : S \to A$ *and* $p : A \to B$ *be partial maps of event structures. Let* $f_0 : S_0 \to A$ *be the defined part of* f. *Then, the defined part of* pf_0 *is the defined part of* pf.

Proof. Directly from the definition of 'defined part' of a partial map of event structures. □

*****

Lemma 13.10. *The composition* $\tau \circledast \sigma$ *is a partial strategy.*

Proof. From Lemma 13.8 it follows immediately that $\tau_0 \circledast \sigma_0$ is the defined part of the composite

$$T \circledast S \stackrel{\tau\circledast\sigma}{\longrightarrow} A^\perp \| (N_S \| B \| N_T) \| C \longrightarrow A^\perp \| B \| C\,.$$

By definition, $\tau_0 \odot \sigma_0$ is the defined part of the composite

$$T_0 \circledast S_0 \stackrel{\tau_0\circledast\sigma_0}{\longrightarrow} A^\perp \| B \| C \longrightarrow A^\perp \| C\,.$$

By Proposition 13.9, it follows that $\tau_0 \odot \sigma_0$ is the defined part of

$$T \circledast S \stackrel{\tau\circledast\sigma}{\longrightarrow} A^\perp \| (N_S \| B \| N_T) \| C \longrightarrow A^\perp \| C\,.$$

ensuring that $\tau \circledast \sigma$ is a partial strategy. □

With partial strategies we no longer generally have that composition with copy-cat yields the same strategy up to isomorphism—there will generally be extra neutral events introduced through synchronizations.

Lemma 13.11. *A configuration* $z \in \mathcal{C}^\infty(T \circledast S)$ *is* $+/0$*-maximal configuration iff* $\Pi_1 z$ *is* $+/0$*-maximal in* $\mathcal{C}^\infty(S)$ *and* $\Pi_2 z$ *is* $+/0$*-maximal in* $\mathcal{C}^\infty(T)$.

Proof. Very similar to the proofs of Lemma 9.2 and Corollary 9.3. □

Remark. In a partial strategy $\sigma : S \to A^\perp \| N \| B$ the purely neutral event structure N is reminiscent of a *global channel type* [**?**].

13.2.1 As synchronized composition

A partial strategy $\sigma : S \to A^\perp \| N \| B$ from game A to game B determines three partial maps to the three components $A^\perp$, N and B. As before, we write $\sigma_1 : S \to A^\perp$ and $\sigma_2 : S \to B$ for left and right components. Write $\sigma_n : S \to N$ for the component into neutral events.

Proposition 13.12. *Let A, B be event structures with polarity in which no events are neutral. Let N be an event structure with polarity in which all events are neutral. Partial strategies $\sigma : S \to A^\perp \| N \| B$ are in 1-1 correspondence with triples of maps $\sigma_1 : S \to A^\perp$, $\sigma_2 : S \to B$ and $\sigma_n : S \to N$ s.t. ******

Assume partial strategies $\sigma : S \to A^\perp \| N_S \| B$ and $\tau : T \to B^\perp \| N_T \| C$. We can define their composition via a synchronized composition (without hiding). We only synchronize events of S and T when they are over complementary events the game B, yielding the synchronized composition

$$S \times T \upharpoonright \mathit{top}^{-1} R$$

where

$$\begin{aligned} R \;=\; & \{(s,*) \mid s \in S \;\&\; \sigma_1(s) \text{ is defined or } pol_S(s) = 0\} \cup \\ & \{(s,t) \mid s \in S \;\&\; t \in T \;\&\; \sigma_2(s) = \overline{\tau_1(t)} \text{ with both defined}\} \cup \\ & \{(*,t) \mid t \in T \;\&\; \tau_2(t) \text{ is defined or } pol_T(t) = 0\}\,. \end{aligned}$$

Modifying B so all its events are neutral, we obtain a partial strategy

$$\upsilon : S \times T \upharpoonright \mathit{top}^{-1} R \to A^\perp \| (N_S \| B \| N_T) \| C$$

in which
υ_1 takes an event p to $\sigma_1(s)$ if $\mathit{top}(p) = (s,*)$, and is undefined otherwise;
υ_2 takes an event p to $\tau_2(s)$ if $\mathit{top}(p) = (*,t)$, and is undefined otherwise;
υ_n takes an event p to an event in a component of $N_S \| B \| N_T$, to $\sigma_2(s) = \overline{\tau_1(t)}$ if $\mathit{top}(p) = (s,t)$, to $\sigma_n(s)$ if $\mathit{top}(p) = (s,*)$ and $pol_S(s) = 0$ and to $\tau_n(s)$ if $\mathit{top}(p) = (*,t)$ and $pol_T(t) = 0$, and is undefined otherwise.

Proposition 13.13. *The construction is isomorphic to composition of partial strategies given earlier via pullbacks.*

13.3 2-cells for partial strategies

$f : \sigma \Rightarrow \sigma'$ where $\sigma : S \to A^\perp \| N \| B$ and $\sigma' : S' \to A^\perp \| N \| B$

***** Given $f : \sigma \Rightarrow \sigma'$ and $g : \tau \Rightarrow \tau'$ from universality of pullback obtain $g \circledast f : \tau \circledast \sigma \Rightarrow \tau' \circledast \sigma'$***

Lemma 13.14. *Let $f : \sigma \Rightarrow \sigma'$ and $g : \tau \Rightarrow \tau'$ be 2-cells between composable partial strategies. Then, $g \circledast f$ is a 2-cell of partial strategies. It is rigid if f and g are rigid.*

13.4 May and must tests

NOTATION **** partial operations $ysncircx$, $y \odot x$ on configurations, ALSO infinite configs ****EARLIER***

Consider the following three strategies in the game A comprising a single +ve event. Recall neutral events are drawn as $\circledcirc$.

$$\begin{tikzcd} S_1 \arrow[d, "\sigma_1"'] & & \oplus \arrow[d, dotted, |->] \\ A & & \oplus \end{tikzcd}$$

$$\begin{tikzcd} S_2 \arrow[d, "\sigma_2"'] & \circledcirc \arrow[r, -Triangle] & \oplus \arrow[d, dotted, |->] \\ A & & \oplus \end{tikzcd}$$

$$\begin{tikzcd} & \circledcirc \arrow[d, squiggly, no head] & \\ S_3 \arrow[d, "\sigma_3"'] & \circledcirc \arrow[r, -Triangle] & \oplus \arrow[d, dotted, |->] \\ A & & \oplus \end{tikzcd}$$

From the point of view of observing the move over the game A the first two strategies, σ_1 and σ_2, differ from the the third, σ_3. In a maximal play both σ_1 and σ_2 will result in the observation of the single move of A. However, in σ_3 one maximal play is that in which the topmost neutral event of S_3 has occurred, in conflict with the only way of observing the single move of A.

We follow [?] in making these ideas precise. For configurations x, y of an event structure with polarity which may have neutral events write $x \subseteq^p y$ to mean $x \subseteq y$ and all events of $y \setminus x$ have polarity + or 0. We write $\subseteq^0$ to mean the inclusion involves only neutral events

Definition 13.15. Let σ be a partial strategy in a game A. Let $\tau : T \to A^\perp \| N \| \oplus$ be a 'test' partial strategy from A to a the game consisting of a single Player move $\oplus$. Write $\checkmark =_{\mathrm{def}} (3, \oplus)$.

Say σ *may pass* τ iff there exists $y \circledast x \in \mathcal{C}^\infty(T \circledast S)$, where $x \in \mathcal{C}^\infty(S)$ and $y \in \mathcal{C}^\infty()$, with the image τy containing $\checkmark$. (Note that we may w.l.o.g. assume that the configuration $y \circledast x$ is finite.)

Say σ *must pass* τ iff for all $y \circledast x \in \mathcal{C}^\infty(T \circledast S)$, where $x \in \mathcal{C}^\infty(S)$ and $y \in \mathcal{C}^\infty()$, which are $\subseteq^p$-maximal the image τy contains $\checkmark$.

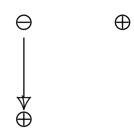
Say two partial strategies are *'may' ('must') equivalent* iff the tests they may (respectively, must) pass are the same.

The definitions extend in the obvious fashion to partial strategies of type $A^{\perp}\|N\|B$.

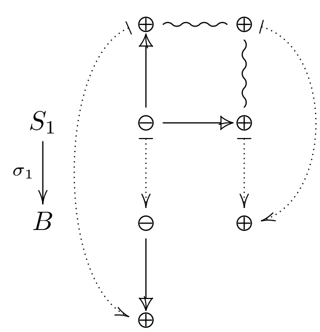
A partial strategy is 'may' equivalent, but need not be 'must' equivalent, to the strategy which is its defined part; 'must' inequivalence is lost in moving from partial strategies to strategies.

Example 13.16. This example shows that strategies σ_1 and σ_2 in a game B may have the same configurations in B as images and yet not be equivalent w.r.t. 'may equivalence.' The game B takes the form:

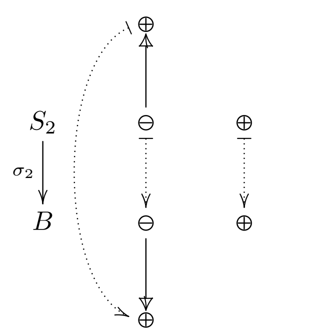
$B :$



The first (nondeterministic) strategy σ_1 is:



The second (deterministic) strategy is:

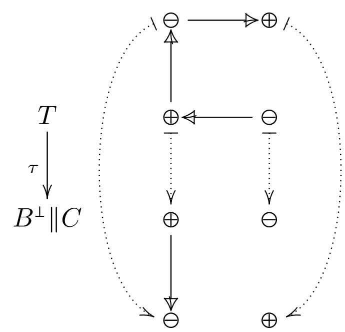


The test comprises $\tau : T \to B^{\perp}\|C$ where C is consists of a single $\oplus$ event.

Observe that $B^\perp \| C$ takes the form

$$\begin{tikzcd} B^\perp \| C : & \oplus \arrow[d] & \ominus \\ & \ominus & \oplus \end{tikzcd}$$

with the event of C being the $+$-event to the right. The test strategy is:



Note that $\sigma_1 \mathcal{C}(S_1) = \sigma_2 \mathcal{C}(S_2) = \mathcal{C}(B)$. The composition $\tau \odot \sigma_2$ can perform the event over C—its causal constraints on events over B are consistent with those of the test. However, the other composition $\tau \odot \sigma_1$ cannot perform the event over C—its causal constraints on events over B are inconsistent with those of the test. □

13.5 Strategies with stopping configurations—the race-free case

Partial strategies lack identities w.r.t. composition, so they do not form a bicategory. Fortunately, for 'may' and 'must' tests it is not necessary to use partial strategies; it is sufficient to carry with a strategy the extra structure of 'stopping' configurations which are to be thought of as images of +/0-maximal configurations in an underlying partial strategy. Composition and copy-cat on strategies extend to composition and copy-cat on strategies with stopping configurations, while maintaining a bicategory, in the following way. We tackle the simpler case in which games are assumed to be race-free. (The extension to games which are not race-free is outlined in [**?**].)

Let $\sigma : S \to A^\perp \| N \| B$ be a partial strategy between race-free games, from a game A to a game B. Recall its associated partial-total factorization

$$\begin{tikzcd} S \arrow[r, "d"] \arrow[d, "\sigma"'] & S_0 \arrow[d, "\sigma_0"] \\ A^\perp \| N_S \| B \arrow[r] & A^\perp \| B \end{tikzcd}$$

Its defined part is a strategy σ_0. Define the (possibly) *stopping* configurations in $\mathcal{C}^\infty(S_0)$ to be

$$\mathrm{Stop}(\sigma) =_{\mathrm{def}} \{d\,x \mid x \in \mathcal{C}^\infty(S) \text{ is } +/0\text{-maximal}\}\,.$$

In other words, the stopping configurations are the images of configurations which are maximal w.r.t. neutral or Player moves. Note that $\mathrm{Stop}(\sigma)$ will include all the +-maximal configurations of S_0: any +-maximal configuration y of S_0 is the image under p of its down-closure $[y]$ in S, and by Zorn's lemma this extends (necessarily by neutral events) to a maximal configuration x of S with image y under d; by maximality, if $x \overset{s}{\longrightarrow}\!\subset$ then s cannot be neutral, nor can it be +ve as this would violate the +-maximality of y.

Note that if σ is in fact a strategy, *i.e.* it has no neutral events, then $\mathrm{Stop}(\sigma)$ is the set consisting of all +-maximal configurations of S. We can identify strategies between race-free games with strategies with stopping configurations the +-maximal configurations.

A *strategy with stopping configurations* in a game A comprises a strategy $S \to A$ together with a subset $M_S \subseteq \mathcal{C}^\infty(S)$. As usual, a *strategy with stopping configurations* from a game A to game B is a strategy with stopping configurations in the game $A^\perp \| B$.

There is an issue of axioms on stopping configurations. We do not insist that stopping conigurations include all +-maximal configurations as this property will not be preserved in taking the rigid image of a strategy with stopping configurations. This is because not all infinite configurations in the rigid image are direct images of a configuration in the original strategy—see Example 13.26.

Question. If $\sigma_0 : S_0 \to A$ is a strategy and $M_0 \subseteq \mathcal{C}^\infty(S_0)$ and includes all the +-maximal configurations in $\mathcal{C}^\infty(S_0)$, is there a partial strategy $\sigma : S \to N\|A$ with defined part σ_0 and stopping configurations M_0? **** ****NOT SURE *****To see this extend the event structure S_0 with neutral events to an event structure S in the following way. For all +ve events $s \in S_0$ for which there is $x \in M_0$ with $x \overset{s}{\longrightarrow}\!\subset$ adjoin a neutral 'shadow' event s' with $[s') = [s)$ and take a finite subset of events Y in the extended set to be consistent iff Y doesn't both contain an event of S_0 and its shadow, and its down-closure $[Y]$ satisfies $[Y] \cap S_0 \in \mathrm{Con}_{S_0}$. *****?

The operation $\mathrm{St} : \sigma \mapsto (\sigma_0, \mathrm{Stop}(\sigma))$ above, from partial strategies to strategies with stopping configurations, preserves composition w.r.t. the following definition.

Given two strategies with stopping configurations $\sigma : S \to A^\perp \| B$, M_S and $\tau : T \to B^\perp \| C$, M_T we define their composition by

$$(\tau, M_T) \odot (\sigma, M_S) =_{\mathrm{def}} (\tau \odot \sigma, M_T \odot M_S)$$

where

$$x \in M_T \odot M_S \text{ iff } \exists z \in \mathcal{C}^\infty(T \circledast S).\ [x]_{T \circledast S} \subseteq^0 z \;\&\; \Pi_1 z \in M_S \;\&\; \Pi_2 z \in M_T\,.$$

Above we write $\subseteq^0$ to mean the inclusion only involves neutral events. Recall, $T \circledast S$ is the result of composition before hiding neutral synchronizations. In other words, if we define the stopping configurations of $T \circledast S$ by

$$z \in M_T \circledast M_S \text{ iff } z \in \mathcal{C}^\infty(T \circledast S) \ \&\ \Pi_1 z \in M_S \ \&\ \Pi_2 z \in M_T$$

—sensible because of Lemma 13.11—we have

$$x \in M_T \odot M_S \text{ iff } \exists z \in M_T \circledast M_S.\ [x]_{T \circledast S} \subseteq^0 z\,.$$

We should also extend copy-cat $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ to a strategy with stopping configurations. Assuming A is race-free, we do this by taking

$$M_{\mathbb{C}\mathbb{C}_A} =_{\text{def}} \{(\overline{x} \| x) \mid x \in \mathcal{C}(A)\}.$$

Because A is race-free, $M_{\mathbb{C}\mathbb{C}_A}$ comprises all the $+$-maximal configurations of $\mathbb{C}\mathbb{C}_A$. Then, $\gamma_A, M_{\mathbb{C}\mathbb{C}_A}$ is an identity w.r.t. the extended composition.

Proposition 13.17. *When A is race-free, $\gamma_A, M_{\mathbb{C}\mathbb{C}_A}$ is identity w.r.t. composition.*

Proof. ***** By definition,

$$x \in M_{\mathbb{C}\mathbb{C}_B} \odot M_S \text{ iff } \exists z \in \mathcal{C}^\infty(\mathbb{C}\mathbb{C}_B \circledast S).\ [x]_{\mathbb{C}\mathbb{C}_B \circledast S} \subseteq^0 z \ \&\ \Pi_1 z \in M_S \ \&\ \Pi_2 z \in M_{\mathbb{C}\mathbb{C}_B}\,.$$

****** □

Lemma 13.18. *Let σ be a partial strategy from A to B and τ a partial strategy from B to C. Then*

$$\mathrm{Stop}(\tau \circledast \sigma) = \mathrm{Stop}(\tau) \odot \mathrm{Stop}(\sigma)\,.$$

Proof. We can describe the partial-total factorizations associated with the partial strategies $\sigma : S \to A^\perp \| N_S \| B$ and $\tau : T \to B^\perp \| N_T \| C$ as

$$\begin{tikzcd} S \arrow[r, "d_1"] \arrow[d, "\sigma"'] & S_0 \arrow[d, "\sigma_0"] \\ A^\perp \| N_S \| B \arrow[r] & A^\perp \| B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "d_2"] \arrow[d, "\tau"'] & T_0 \arrow[d, "\tau_0"] \\ B^\perp \| N_T \| C \arrow[r] & B^\perp \| C\,. \end{tikzcd}$$

As preparation, in the diagram

$$\begin{tikzcd} S \arrow[d, "d_1"'] & T \circledast S \arrow[l, "\Pi_1"'] \arrow[r, "\Pi_2"] \arrow[d, "d_2 \circledast d_1"] & T \arrow[d, "d_2"] \\ S_0 & T_0 \circledast S_0 \arrow[l, "\Pi_1"] \arrow[r, "\Pi_2"'] \arrow[d, "d'"] & T_0 \\ & T_0 \odot S_0 & \end{tikzcd}$$

the two squares commute, and $T_0 \circledast S_0 \xrightarrow{d'} T_0 \odot S_0 \xrightarrow{\tau_0 \odot \sigma_0} A^\perp \| C$ gives the partial-total factorization associated with the definition of $\tau_0 \odot \sigma_0$. By Proposition 13.9,

$$T \circledast S \xrightarrow{d} T_0 \odot S_0 \xrightarrow{\tau_0 \odot \sigma_0} A^\perp \| C$$

is a partial-total factorization, where we write $d =_{\text{def}} d' \circ (d_2 \circledast d_1)$.

"$\text{Stop}(\tau \circledast \sigma) \subseteq \text{Stop}(\tau) \odot \text{Stop}(\sigma)$": Let $x \in \text{Stop}(\tau \circledast \sigma)$. We have $x = d\,w$ for some $+/0$-maximal configuration of $T \circledast S$. Then, $\Pi_1 w$ is $+/0$-maximal in S and $\Pi_2 w$ is $+/0$-maximal in T, by Lemma 13.11. Hence $d_1 \Pi_1 w \in \text{Stop}(\sigma)$ and $d_2 \Pi_2 w \in \text{Stop}(\tau)$. Take $z =_{\text{def}} (d_2 \circledast d_1) w$. As

$$d' z = d'(d_2 \circledast d_1) w = d\,w = x$$

we have $[x]_{T_0 \circledast S_0} \subseteq^0 z$. Moreover, by the commuting squares above,

$$\Pi_1 z = \Pi_1 (d_2 \circledast d_1) w = d_1 \Pi_1 w \in \text{Stop}(\sigma)$$

and similarly $\Pi_2 z \in \text{Stop}(\tau)$. Therefore $x \in \text{Stop}(\tau) \odot \text{Stop}(\sigma)$, as required.

"$\text{Stop}(\tau \circledast \sigma) \supseteq \text{Stop}(\tau) \odot \text{Stop}(\sigma)$": Let $x \in \text{Stop}(\tau) \odot \text{Stop}(\sigma)$. Then,

$$[x]_{T \circledast S} \subseteq^0 z \ \&\ \Pi_1 z \in \text{Stop}(S) \ \&\ \Pi_2 z \in \text{Stop}(T)\,,$$

for some $z \in \mathcal{C}^\infty(T_0 \circledast S_0)$. Now, $\Pi_1 z \in \text{Stop}(S)$ implies $\Pi_1 z = d_1 w_1$ for some $+/0$-maximal $w_1 \in \mathcal{C}^\infty(S)$, so $[\Pi_1 z]_S \subseteq^0 w_1$. Similarly, $[\Pi_2 z]_T \subseteq^0 w_2$ for some $+/0$-maximal $w_2 \in \mathcal{C}^\infty(T)$. Construct

$$w =_{\text{def}} [z]_{T \circledast S} \quad \cup \quad (w_1 \setminus [\Pi_1 z]_S) \times \{*\} \quad \cup \quad \{*\} \times (w_2 \setminus [\Pi_2 z]_T)\,.$$

(It's convenient to use the description of $T \circledast S$ as a form of synchronized composition in Section 13.2.1.) Then, $w \in \mathcal{C}^\infty(T \circledast S)$ and $(d_2 \circledast d_1) w = z$. By Lemma 13.11, w is $+/0$-maximal as $\Pi_1 w = w_1$ and $\Pi_2 w = w_2$ are $+/0$-maximal. Noting $d' z = x$, as it is equivalent to $[x]_{T \circledast S} \subseteq^0 z$, we deduce

$$d'(d_2 \circledast d_1) w = d' z = x$$

ensuring $x \in \text{Stop}(\tau \circledast \sigma)$, as required. $\square$

Definition 13.19. Let σ be a strategy with stopping configurations M_S in a game A. Let $\tau : T \to A^\perp \| N \| \oplus$ be a 'test' partial strategy from A to a the game consisting of a single Player move $\oplus$. Write $St(\tau)$ as (τ_0, M_0) where $\tau_0 : T_0 \to A \| \oplus$ is the defined part of τ and M_0 are its stopping configurations, obtained as images of the p-maximal configurations of T. Write $\checkmark =_{\text{def}} (2, \oplus)$.

Say (σ, M_S) *may pass* τ iff there exists $y \circledast x \in \mathcal{C}^\infty(T_0 \circledast S)$, where $x \in \mathcal{C}^\infty(S)$ and $y \in \mathcal{C}^\infty(T_0)$, with the image τy containing $\checkmark$. (Note again, we may w.l.o.g. assume that the configurations x and y are finite.)

Say (σ, M_S) *must pass* τ iff for all $y \circledast x \in M_0 \circledast M_S$, where $x \in \mathcal{C}^\infty(S)$ and $y \in \mathcal{C}^\infty(T_0)$, the image $\tau_0 y$ contains $\checkmark$.

Say two strategies with stopping configurations are *'may' ('must') equivalent* iff the tests they may (respectively, must) pass are the same.

Proposition 13.20. *With the notation above,*

(σ, M_S) may pass τ *iff there exists* $y \circledast x \in \mathcal{C}^\infty(T_0 \odot S)$, *where* $x \in \mathcal{C}^\infty(S)$ *and* $y \in \mathcal{C}^\infty(T_0)$, *with the image* τy *containing* $\checkmark$ *—the configurations* x, y *may be assumed finite; and*

(σ, M_S) must pass τ *iff for all* $y \odot x \in M_0 \odot M_S$, *where* $x \in \mathcal{C}^\infty(S)$ *and* $y \in \mathcal{C}^\infty(T_0)$, *the image* $\tau_0\, y$ *contains* $\checkmark$.

Lemma 13.21. *Let* A *be a race-free game. Let* σ *be a partial strategy in* A. *Then,*

σ may pass a test τ *iff* $St(\sigma)$ may pass τ*;*

σ must pass a test τ *iff* $St(\sigma)$ must pass τ.

Proof. Directly from the definitions, for the 'if' of the 'must' case, using Lemma 13.11. □

Example 13.22. It is tempting to think of neutral events as behaving like the internal "tau" events of CCS [**?**]. However, in the context of strategies they behave rather differently. Consider three partial strategies, over a game comprising of just two concurrent +ve events, say a and b. The partial strategies have the following event structures in which we have named events by the moves they correspond to in the game:

$$
\begin{tikzcd}
S_1 & a \arrow[d, no head, squiggly] & & S_2 & \circledcirc \arrow[r] \arrow[d, no head, squiggly] & a & & S_3 & \circledcirc \arrow[r] \arrow[d, no head, squiggly] & a \\
 & b & & & \circledcirc \arrow[r] & b & & & b &
\end{tikzcd}
$$

All three become isomorphic under St so are 'may' and 'must' equivalent to each other. □

In making strategies with stopping configurations a bicategory we must settle on an appropriate notion of 2-cell. The following choice of definition seems most useful.

A 2-cell $f : (\sigma, M_S) \Rightarrow (\sigma', M_{S'})$ between strategies with stopping configurations is a 2-cell of strategies $f : \sigma \Rightarrow \sigma'$ such that $f M_S \subseteq M_{S'}$. With this choice of 2-cell, strategies with stopping configurations inherit the structure of a bicategory from strategies; its objects are restricted to race-free games.

The 2-cells between strategies with stopping configurations respect 'may' and 'must' behaviour in the sense of the following lemma.

Lemma 13.23. *Let* $f : (\sigma, M_S) \Rightarrow (\sigma', M_{S'})$ *be a 2-cell between strategies with stopping configurations. Then for any test* τ,

(σ, M_S) *may pass* τ *implies* $(\sigma', M_{S'})$ *may pass* τ*; and*

$(\sigma', M_{S'})$ *must pass* τ *implies* (σ, M_S) *must pass* τ.

Moreover, if f *is rigid epi and* $f M_S = M_{S'}$, *then* (σ, M_S) *and* $(\sigma', M_{S'})$ *are both 'may' and 'must' equivalent.*

Proof. In this proof, we shall identify the partial-strategy test τ with its associated strategy with stopping configurations $\mathrm{St}(\tau)$, writing M_T for its stopping configurations.

Let $f : (\sigma, M_S) \Rightarrow (\sigma', M_{S'})$ be a 2-cell. Assume $\sigma : S \to A$ and $\sigma' : S' \to A$. Let $\tau : T \to A^\perp \| \oplus$.

Suppose (σ, M_S) may pass τ. Then there is a (finite) configuration which we write $y \circledast x$ of $T \circledast S$, built as a pairing of $y \in \mathcal{C}(T)$ and $x \in \mathcal{C}(S)$, which contains $\checkmark$. (We are using the term 'pairing' so as to remain neutral between the two equivalent ways of defining configurations of $T \circledast S$, via pullbacks when the 'pairing' is a secured bijection, or as a synchronised composition.) The pairing induces a pairing $y \circledast fx$, containing $\checkmark$, of $y \in \mathcal{C}(T)$ and $fx \in \mathcal{C}(S)$. (The secured bijection built from y and x induces a secured bijection built from y and fx; this is because fx has no more causal dependency than x with which it is in bijection.)

Suppose $(\sigma', M_{S'})$ must pass τ. Any $y \circledast x \in M_T \circledast M_S$ images under $\tau \circledast f$ to $y \circledast fx \in M_T \circledast M_{S'}$. As $(\sigma', M_{S'})$ must pass τ, the configuration $y \circledast fx$ contains $\checkmark$, ensuring that $y \circledast x$ does too. *** REQUIRES GENERALISATION OF $y \circledast x$ TO INFINITE CONFIGS***

Finally suppose that f is rigid epi and $fM_S = M_{S'}$. We have just shown that f preserves the passing of 'may' tests and reflects the passing of 'must' tests. Because f is rigid epi it also reflects the passing of 'may' tests. Because f is rigid and $fM_S = M_{S'}$ it preserves the passing of 'must' tests: any pairing $y \circledast fx$ in $M_T \circledast M_{S'}$ ensures by the rigidity of f a pairing $y \circledast x$ in $M_T \circledast M_S$; as (σ, M_S) must τ we have $y \circledast x$ contains $\checkmark$ ensuring $y \circledast fx$ does too. $\square$

As a corollary of Lemma 13.23, with an appropriate construction of the rigid image of a strategy with stopping configurations we are assured not to lose any 'may' and 'must' behaviour.

Definition 13.24. Let (σ, M_S) be a strategy with stopping configurations. Let σ_1 be the rigid image of σ with accompany 2-cell $f : \sigma \Rightarrow \sigma_1$ where f is rigid epi. We define the *rigid image* of (σ, M_S) to be (σ_1, fM_S).

A *rigid-image* strategy with stopping configurations is one in which the strategy is rigid-image.

Corollary 13.25. *A strategy with stopping configurations is both 'may' and 'must' equivalent to its rigid image.*

Proof. A direct consequence of the last part of Lemma 13.23. $\square$

Thus w.r.t. 'may' and 'must' behaviour we can choose to work in the category of rigid-image strategies with stopping configurations.

Example 13.26. In forming the rigid image $\sigma_1 : S_! \to A$ of a strategy $\sigma : S \to A$, related by rigid epi 2-cell $f : \sigma \Rightarrow \sigma_1$, it is possible to have an infinite configuration of S_1 which is not in the direct image under f of any configuration of S; in particular it is possible to have a +-maximal configuration of S_1 which is not a direct image of any +-maximal configuration S. For example, let A comprise an infinite chain of Player events. Take S to be the sum of all finite subchains. The rigid image of S is A itself which has +-maximal configuration all the events in the infinite chain, not the image of any configuration of S_1.

Thus, in forming the rigid image of strategy with stopping configurations, we cannot assume that all the $+$-maximal configurations of the rigid image are stopping. $\square$

As far as 'may' and 'must' behaviour is concerned it is sensible to regard two strategies with stopping configurations to be equivalent if they share a common rigid image. The equivalence transfers to an equivalence between partial strategies: two partial strategies are equivalent if under St we obtain equvalent strategies with stopping configurations.

Example 13.27. Tests based on partial strategies are more discriminating than tests based on (pure) strategies. Let a game comprise a single Player move. Consider two strategies with stopping configurations:

σ_1, the empty strategy with the empty configuration $\varnothing$ as its single stopping configuration;

σ_2, the strategy performing the single Player move $\oplus$ with stopping configurations $\varnothing$ and $\{\oplus\}$.

By Lemma 13.23, we have $(\sigma_2, \{\varnothing, \{\oplus\}\})$ must pass τ implies $(\sigma_1, \{\varnothing\})$ must pass τ, for any test τ. (The above would not hold if we had not included $\varnothing$ in the stopping configurations of σ_2.) Using the fact that we need only consider rigid images of tests, a little argument by cases establishes the converse too.

The strategies with stopping configurations would be must equivalent w.r.t. tests based just on strategies. However with tests based on partial strategies we can distinguish them. Consider the test τ comprising three events, one of them neutral, with only nontrivial causal dependency $\ominus \rightarrowtriangle \circledcirc$ and $\circledcirc$ in conflict with the 'tick' event $\oplus$. Then, it is not the case that $(\sigma_2, \{\varnothing, \{\oplus\}\})$ must pass τ —the occurrence of the neutral event blocks success in a maximal execution—while $(\sigma_1, \{\varnothing\})$ must pass τ. $\square$

We exploit the discriminating power of tests based on partial strategies in the following proposition.***NO** ***SEEMS CAN ONLY DISCRIMINATE UP TO TRACES OF***

Lemma 13.28. *****NEED IT'S A RIG-IM STRAT SEE BELOW *****AH, EVEN THEN NOT ENOUGH Let $\sigma : S \to A$ be rigid-image strategy with stopping configurations M. For all $x \in \mathcal{C}^\infty(S)$ there is a test τ_x such that*

$$x \in M \text{ iff it is not the case that } (\sigma, M) \text{ must pass } \tau_x\,.$$

NO!

Proof. For $x \in \mathcal{C}^\infty(S)$, construct the test $\tau_x : T \to A^\perp \| N \| \oplus$ as follows.

Let T_1' comprise the set $T_1 =_{\rm def} \sigma_1 x_1$ saturated with all accessible Opponent moves within $A^\perp$, *i.e.* events

$$T_1' = \{a \in A \mid pol_{A^\perp}([a] \smallsetminus T_1) \subseteq \{-\}\}$$

with order that of $A^\perp$. The set event structure with polarity T is built from events $T_1' \cup N \cup T_2$, the union assumed disjoint, where:

in the set $T_1' = \sigma x$, we attribute to events their polarity in $A^\perp$;

N is a copy of all the −ve events in $\mathbb{T}_1$, regarded as of neutral polarity;

the set T_2 is a copy of T_1, but in which all events are assigned +ve polarity;

for causal dependency we order T_1' by its order in $A^\perp$ and adjoin causal dependencies from any −ve event of T_1 to its respective copy in N;

the consistency relation of T is that minimal relation which ensures that any two distinct events of T_2 are in conflict; a +ve event of T_1 conflicts with its corresponding copy in T_2; and a neutral event in N conflicts with its corresponding copy in T_2;

the map $\tau_x : T \to A^\perp \| N \| \oplus$ takes any event of $T_!'$ to its copy in $A^\perp$, any neutral event of N to its copy, and all events of T_2 to $\checkmark =_{\text{def}} (3, \oplus)$.

All the events over $\checkmark$, which together comprise the set T_2, can occur initially but can become blocked as moves are made in T_1. In particular, the set $T_1 \cup N$ is a p-maximal configuration of T with image in $A^\perp \| N \| \oplus$ not containing any event over $\checkmark$. On the other hand any p-maximal configuration of T not including all the events T_1 will contain an event over $\checkmark$. Hence $\mathrm{St}(\tau_x)$ has an unsuccessful stopping configuration consisting of precisely all the events of T_1—it does not have an event over $\checkmark$—while all stopping configurations of $\mathrm{St}(\tau_x)$ which do not contain all the events of T_1 are successful—they contain an event over $\checkmark$.

Any stopping configuration of $\tau_x \circledast \sigma$ comprises $w \circledast y$ where w is a stopping configuration of $\mathrm{St}(\tau_x)$ and $y \in M$. Consequently, if $x \in M$ then $w \circledast x$ is an unsuccessful stopping configuration of $\tau_x \circledast \sigma$ showing that it is not the case that (σ, M) must pass τ_x. For the converse we rely on σ being rigid-image. Under this assumption, if $x \notin M$, then all stopping configurations $w \circledast y$ *****

NEED σ, M to be a rigid-image strat w stopping configs; otherwise have $x \notin M$, $x' \in M$ with $\sigma x = \sigma x'$ and x, x' ordered the same way, whereupon $x \notin M$ and σ, M unsuccessful **

NOT ENOUGH: MAY HAVE $y \in M$ stopping $\sigma y = \sigma x$ with x a proper augmentation of y so $T_1 \circledast y$ defd so unsuccessful

□

We can interpret the metalanguage directly in terms of strategies with stopping configurations in such a way that the denotation of a term as a strategy with stopping configurations is the image under St of its denotation as a partial strategy. To achieve this, we specify the stopping configurations of both the sum and pullback of strategies.

For the sum of strategies $\coprod_{i \in I} \sigma_i$ with stopping configurations σ_i, a configuration of the sum is stopping iff it is the image of a stopping configuration under the injection from a component.

Consider strategies $\sigma : S \to A$ and $\tau : T \to A$ with stopping configurations M_S and M_T respectively. Let their pullback be denoted by $\sigma \wedge \tau : P \to A$ with projection morphisms $\pi_1 : P \to S$ and $\pi_2 : P \to T$. A configuration of P is defined to be stopping iff there exist x_1, x_2 such that $\pi_1 x \subseteq^+ x_1$ and $\pi_2 x \subseteq^+ x_2$ and $x_1 \in M_S$ and $x_2 \in M_T$, and furthermore there exists a partition $x^+ = Y_1 \cup Y_2$ satisfying $x_i \cap Y_i = \varnothing$. The set of stopping configurations of P coincides with the stopping configurations obtained via St from the pullback of partial strategies.

The treatment of winning strategies of Chapter 9 generalises straightforwardly, with the role of +-maximal configurations replaced by that of stopping configurations.

13.6 May and Must behaviour characterised

******REDO THE INTRODUCTORY BIT: countable enumeration of a set - regard it as a total order; serialisation of a configuration; trace of a configuration in a strategy ****

Let $\sigma : S \to A$ be a strategy in a game A. A *trace* in σ is a possibly infinite sequence

$$\alpha = (\sigma(s_1), \sigma(s_2), \cdots, \sigma(s_n), \cdots)$$

of events in A in which $\{s_1, \cdots, s_i\} \in \mathcal{C}(S)$, for all i at which the sequence is defined. The set

$$x = \{s_1, s_2, \cdots, s_n, \cdots\}$$

is necessarily a configuration of S, as is any intial subsequence—they are clearly down-closed and consistent. The sequence

$$s_1, s_2, \cdots, s_n, \cdots$$

constitutes a *serialisation* of the configuration x. Note that we can regard a serialisation as an elementary event structure in which causal dependency takes the form of a total order; a serialisation of a configuration is associated with a map to S whose image is the configuration. From the local injectivity of σ, the configuration x will be finite/infinite according as the trace is finite/infinite. We sometimes say that α is a trace of the configuration x or that x has trace α. Of course traces determine serialisations of (necessarily countable) configurations, and *vice versa*.

Proposition 13.29. *(i) Any countable configuration of a strategy has a trace. (ii) W.r.t. a strategy $\sigma : S \to A$, let $x \in \mathcal{C}^\infty(S)$ and α is an enumeration of σx. Then, x has trace α in σ iff for all $s, s' \in x$ if $s \rightarrowtriangle s'$ then $\sigma(s)$ precedes or equals $\sigma(s')$ in the enumeration α.*

Proof. (i) Let x be a countable configuration of S in a strategy $\sigma : S \to A$. This follows because there is a serialisation $x = \{s_1, s_2, \cdots, s_n, \cdots\}$, in which $\{s_1, \cdots, s_i\}$ is down-closed in S at all i in the enumeration. To see this, from its countability we may assume a countable enumeration of x, which need not be a serialisation. Define $s_1 \in x$ to be the earliest event of the enumeration for which $[s_1) = \emptyset$; such an s_1 is ensured to exist by the well-foundedness of causal dependency provided $x \neq \emptyset$. Inductively, define s_n to be the earliest event of the enumeration which is in $x \setminus \{s_1, \cdots, s_{n-1}\}$ and for which $[s_n) \subseteq \{s_1, \cdots, s_{n-1}\}$; again the well-foundedness of causal dependency ensures such an s_n exists provided $x \setminus \{s_1, \cdots, s_{n-1}\} \neq \emptyset$. It is elementary to check this provides a serialisation of x.

(ii) Obvious.

□

Lemma 13.30. *Let $\sigma : S \to A$ be a strategy in a game A. Let $x \in \mathcal{C}^\infty(S)$. Let α be a serialisation of σx and not a trace of $x \in \mathcal{C}^\infty(S)$. Then, there are $s, s' \in x$ with $pol(s) = -$ and $pol(s') = +$ and $s \rightarrowtriangle_S s'$ and (note the order reversal) $\sigma(s') \leq_\alpha \sigma(s)$ in α (regarded as a total order).*

Proof. By assumption, any trace of x differs from α. We deduce there is $s \rightarrowtriangle s'$ in x with $\sigma(s) \nleq \sigma(s')$ in the total order of α; otherwise we could serialise x to obtain the trace α. Now, $\sigma(s) \nleq_A \sigma(s')$ in A as any serialisation must respect the order $\leq_A$. Hence, by the innocence of σ, we must have $pol(s) = -$ and $pol(s') = +$. Because α is totally ordered, $\sigma(s') \leq \sigma(s)$ in α. □

****

For strategies with stopping configurations (games assumed race-free) we have:

Lemma 13.31. *Let (σ_1, M_1) and (σ_2, M_2) be strategies with stopping configurations in a common game.*

(σ_1, M_1) may pass τ implies (σ_2, M_2) may pass τ, for all tests τ,

iff

all finite traces of σ_1 are traces of σ_2.

Proof. Assume strategies $\sigma_1 : S_1 \to A$ and $\sigma_2 : S_2 \to A$. *"if":* Assume all finite traces of σ_1 are traces of σ_2. Suppose (σ_1, M_1) may pass test τ with event structure T. Then there is a successful configuration $w \circledast x_1 \in \mathcal{C}(T \circledast S_1)$, where $x_1 \in \mathcal{C}(S_1)$ and $w \in \mathcal{C}(T)$; it is successful in the sense that its image contains the success event $\checkmark$. Take a serialisation of $w \circledast x_1$; this induces a serialisation of x_1 to yield a trace. Then, by assumption, σ_2 has a configuration $x_2 \in \mathcal{C}(S_2)$ with the same trace, so a matching serialisation. Consequently the pairing $w \circledast x_2$ is defined with $w \circledast x_2 \in \mathcal{C}(T \circledast S_2)$; sharing the same image as $w \circledast x_1$ it is also successful.

"only if": We show the contraposition: assuming not all traces of σ_1 are traces of σ_2, we produce a test τ for which σ_1 may pass τ while it is not the case that σ_2 may pass τ.

Assume a trace α_1 of $x_1 \in \mathcal{C}(S_1)$ is not a trace of any $x_2 \in \mathcal{C}(S_2)$. Note that the trace α_1, and correspondingly x_1, must have at least one +ve event as otherwise, by receptivity, σ_2 could match the trace α_1. Any trace of x_2, with $\sigma_2 x_2 = \sigma_1 x_1$, differs from α_1. We deduce there is $s \rightarrowtriangle_2 s'$ in x_2 with $\sigma_2(s) \nleq_1 \sigma_2(s')$ w.r.t. the total order of α_1; otherwise we could serialise x_2 to obtain the trace α_1. ***THIS ARGUMENT REPEATS IN THE NEXT LEMMA - MAKE A PROP OR LEMMA? **** Now, $\sigma_2(s) \nleq_A \sigma_2(s')$ in A as any serialisation must respect the order $\leq_A$. Hence, by the innocence of σ_2, we must have $pol(s) = -$ and $pol(s') = +$. Because α_1 is totally ordered, $\sigma_2(s') \leq_1 \sigma_2(s)$ in α_1.

Thus for each $x_2 \in \mathcal{C}(S_2)$ with $\sigma_2 x_2 = \sigma_1 x_1$ we can choose $\theta(x_2) = (s, s')$ so that $s \rightarrowtriangle_2 s'$ in x_2 with $pol(s) = -$ and $pol(s') = +$ and $\sigma_2(s') \leq_1 \sigma_2(s)$ in α_1.

We now describe a test $\tau : T \to A^\perp \| \oplus$ which will discriminate between σ_1 and σ_2. Let T_1' be the elementary event structure comprising events $T_1 =_{\rm def} \sigma_1 x_1$ saturated with all accessible Opponent moves (note, in $A^\perp$), *i.e.* events

$$T_1' = \{a \in A \mid pol_{A^\perp}([a] \smallsetminus T_1) \subseteq \{-\}\}$$

with order that of $A^\perp$ augmented with $\sigma_2(s') \leq_1 \sigma_2(s)$ for every choice $\theta(x_2) = (s, s')$ where $x_2 \in M_2$ and $\sigma_2 x_2 = \sigma_1 x_1$; the ensuing relation on T_1 is included in the total order α_1 so forms a partial order in which every element has only finitely many elements below it. (By design, T_1' "disagrees" with the causal dependency of each $x_2 \in \mathcal{C}(S_2)$ for which $\sigma_2 x_2 = \sigma_1 x_1$.) The polarities of events of T_1' are those of its events in $A^\perp$. On T_1' the map τ takes an event to its same event in $A^\perp$.

Let T be the event structure with polarity obtained from T_1' by adjoining a fresh 'success' event $\oplus$ with additional causal dependency so $t_1 \leq_T \oplus$ iff t_1 is $-$ve; as noted above there has to be at least one $+$ve event in x_1 and thus, by the reversal of polarity, at least one $t_1 \in T_1$ of $-$ve polarity. Then the obvious map $\tau : T \to A^\perp \| \oplus$ is a strategy, and a suitable test for σ_1 and σ_2.

We have (i) σ_1 may pass τ, while (ii) it is not the case that σ_2 may pass τ.

To see (i), remark that the relation of causal dependency on T_1 is included in the the total order of the trace α_1 of x_1. Hence $\tau \circledast \sigma_1$ has a successful configuration $(T_1 \cup \{\oplus\}) \circledast x_1$.

To show (ii), consider any finite configuration of $\tau \circledast \sigma_2$. It has the form $w \circledast x_2$ where $w \in \mathcal{C}(T)$ and $x_2 \in \mathcal{C}(S_2)$. The configuration $w \circledast x_2$ is unsuccessful because $\oplus \notin w$, as we now show. By design, τ and σ_2 enforce opposing causal dependencies on a pair of synchronisations needed for $T_1 \circledast x_2$ to be defined whenever $x_2 \in \mathcal{C}(S_2)$ with $\sigma_2 x_2 = T_1$. At least two events of opposing polarity in T_1 are excluded from any pairing $w \circledast x_2$; one must be a $-$ve event of T_1 on which $\oplus$ causally depends; hence $\oplus \notin w$. $\square$

Clearly he proof above does not rely on stopping configurations or tests being partial rather than pure strategies; the test used in the proof patently has no neutral events. The extra discriminating power of tests based on partial strategies, illustrated in Example 13.27, does play an essential role in the analgous result in the 'must' case, to be considered shortly.

Recall an event structure $E = (E, \leq, \mathrm{Con})$ is *consistent-countable* iff there is a function $\chi : E \to \omega$ from the events such that

$$\{e_1, e_2\} \in \mathrm{Con} \ \&\ \chi(e_1) = \chi(e_2) \implies e_1 = e_2\,.$$

Any configuration $x \in \mathcal{C}^\infty(E)$ of a consistent-countable event structure E is countable and so may be serialised as

$$x = \{e_1, e_2, \cdots, e_n, \cdots\}$$

so that $\{e_1, \cdots, e_n\} \in \mathcal{C}(E)$ for any finite subsequence. For the must case we assume that games are consistent-countable. It follows that strategies $\sigma : S \to A$ in consistent-countable games A have S consistent-countable. W.r.t. such a strategy σ, we have traces of all configurations.

Lemma 13.32. *Assume game A is consistent-countable. Let (σ_1, M_1) and (σ_2, M_2) be strategies in A with stopping configurations. Then,*

(σ_2, M_2) must pass τ implies (σ_1, M_1) must pass τ, for all tests τ,

iff

all traces of stopping configurations M_1 are traces of stopping configurations M_2.

Proof. "if": Assume all traces of stopping configurations M_1 are traces of stopping configurations M_2. A stopping configuration of $\tau \circledast \sigma_1$ has the form $w \circledast x_1$ where w and x_1 are stopping configurations of τ and σ_1, respectively. A serialisation of $w \circledast x_1$ into a (possibly infinite) sequence induces a serialisation of $x_1 \in M_1$. By assumption, there is $x_2 \in M_2$ with the same trace in A as x_1. Consequently, $w \circledast x_2$ is a configuration of $\tau \circledast \sigma_2$ with the same image in $A \| \oplus$. Moreover, $w \circledast x_2$ is a stopping configuration of $\tau \circledast \sigma_2$. Supposing (σ_2, M_2) must pass a test τ, the image of $w \circledast x_2$ contains $\checkmark$ whence the image of $w \circledast x_1$ contains $\checkmark$ ensuring (σ_1, M_1) must pass a test τ.

"only if": We show the contraposition: assuming not all traces of stopping configurations M_1 are traces of stopping configurations M_2, we produce a test τ for which (σ_2, M_2) must pass τ while it is not the case that (σ_1, M_1) must pass τ.

Assume a trace α_1 of $x_1 \in M_1$ is not a trace of any $x_2 \in M_2$.

In particular, consider any $x_2 \in M_2$ with $\sigma_2 x_2 = \sigma_1 x_1$. Then, any trace of x_2 differs from α_1. We deduce there is $s \rightarrowtriangle_2 s'$ in x_2 with $\sigma_2(s) \nleq_1 \sigma_2(s')$ in the total order of α_1; otherwise we could serialise x_2 to obtain the trace α_1. Now, $\sigma_2(s) \nleq_A \sigma_2(s')$ in A as any serialisation must respect the order $\leq_A$. Hence, by the innocence of σ_2, we must have $pol(s) = -$ and $pol(s') = +$. Because α_1 is totally ordered, $\sigma_2(s') \leq_1 \sigma_2(s)$ in α_1.

Thus for each $x_2 \in M_2$ with $\sigma_2 x_2 = \sigma_1 x_1$ we can choose $\theta(x_2) = (s, s')$ so that $s \rightarrowtriangle_2 s'$ in x_2 with $pol(s) = -$ and $pol(s') = +$ and $\sigma_2(s') \leq_1 \sigma_2(s)$ in α_1.

We build an event structure with polarity T and a test as partial strategy $\tau : T \to A^\perp \| N \| \oplus$. We build the events of T as $T_1' \cup N \cup T_2$, a union of sets of events, assumed disjoint, described as follows.

- Let T_1' be the elementary event structure comprising events $T_1 =_{\text{def}} \sigma_1 x_1$ saturated with all accessible Opponent moves, *i.e.* events

  $$T_1' = \{a \in A \mid pol_{A^\perp}([a] \setminus T_1) \subseteq \{-\}\}$$

  with order that of A augmented with $\sigma_2(s') \leq_1 \sigma_2(s)$ for every choice $\theta(x_2) = (s, s')$ where $x_2 \in M_2$ and $\sigma_2 x_2 = \sigma_1 x_1$; the ensuing relation on T_1 is included in the total order α_1 so forms a partial order in which every element has only finitely many elements below it. (By design, T_1' "disagrees" with the causal dependency of each $x_2 \in M_2$ for which $\sigma_2 x_2 = \sigma_1 x_1$.) The polarities of events of T_1' are those of its events in $A^\perp$. On T_1' the map τ takes an event to its same event in $A^\perp$.

- N comprises a copy of the set of events of −ve polarity in T_1; all the events of N have neutral polarity; an event of N is sent by τ to its copy.

- T_2 comprises a copy of the set of events T_1; all the events of T_2 have +ve polarity; they are all sent by τ to $\checkmark =_{\text{def}} (3, \oplus)$.

- Causal dependency on T is that of T_1' augmented with dependencies from events of T_1 of −ve polarity to their corresponding copies in N.

- The consistency relation of T is that minimal relation which ensures that any two distinct events of T_2 are in conflict; a +ve event of T_1 conflicts with its corresponding copy in T_2; and a neutral event in N conflicts with its corresponding copy in T_2. More formally,

$$
\begin{aligned}
&X \in \mathrm{Con}_T \text{ iff } X \subseteq_{\text{fin}} T_1 \cup N \cup T_2 \;\&\; |X \cap T_2| \le 1 \;\&\\
&\quad (\forall t_1 \in X \cap T_1^+, t_2 \in X \cap T_2.\ t_1, t_2 \text{ are not copies of a common event}) \;\&\\
&\quad (\forall n \in X \cap N, t_2 \in X \cap T_2.\ n, t_2 \text{ are not copies of a common event}).
\end{aligned}
$$

Note that all the events over $\checkmark$, which together comprise the set T_2, can occur initially but can become blocked as moves are made in T_1. In particular, the set $T_1 \cup N$ is a p-maximal configuration of T with image in $A^\perp \| N \| \oplus$ not containing any event over $\checkmark$. On the other hand any p-maximal configuration of T not including all the events T_1 will contain an event over $\checkmark$. Hence $\mathrm{St}(\tau)$ has an unsuccessful stopping configuration consisting of precisely all the events of T_1—it does not have an event over $\checkmark$—while all stopping configurations of $\mathrm{St}(\tau)$ which do not contain all the events of T_1 are successful—they contain an event over $\checkmark$.

Consequently, (i) it is not the case that (σ_1, M_1) must τ, while (ii) (σ_2, M_2) must τ. To see (i), remark that the relation of causal dependency on T_1 is included in the the total order of the trace α_1 of x_1. Hence $\mathrm{St}(\tau) \circledast \sigma_1$ has a stopping configuration $T_1 \circledast x_1$ which is unsuccessful and thus (σ_1, M_1) fails the must test τ. To show (ii), consider any stopping configuration of $\mathrm{St}(\tau) \circledast \sigma_2$. It comprises $w \circledast x_2$ where w is a stopping configuration of $\mathrm{St}(\tau)$ and $x_2 \in M_2$, a stopping configuration of σ_2. Now $w \not\supseteq T_1$, as by design τ and σ_2 enforce opposing causal dependencies on a pair of synchronisations needed for $T_1 \circledast x_2$ to be defined whenever $x_2 \in M_2$ with $\sigma_2 x_2 = T_1$. Thus w is successful in that it contains an event over $\checkmark$. Hence (σ_2, M_2) must pass τ. This completes the proof. □

Remark. By Example 13.27, the result above would not hold if tests were based solely on pure strategies.

Example 13.33. ***over game $\ominus_1 \rightarrowtriangle \oplus_1 \| \ominus_2 \rightarrowtriangle \oplus_2$ the id strat and one where make $\ominus_1 \rightarrowtriangle (copyof)\oplus_2$ and $\ominus_2 \rightarrowtriangle (copyof)\oplus_1$, stopping configs +-maximal configs, are ‘must ’equiv ****

13.6.1 Sum decomposition

It is straightforward to decompose an arbitrary strategy $\sigma : S \to A$ into a sum of deterministic sub-strategies $\sum_{i\in I}\sigma_i$ with the same rigid image. Any configuration $x \in \mathcal{C}^\infty(S)$ determines a deterministic strategy σ_x: its events are those of x together with those Opponent events enabled from x to ensure receptivity, *viz.*

$$x \cup \{s \in S \mid [s)^+ \subseteq x\}$$

with causal dependency and consistency inherited from S. It is easy to see that the obvious map

$$f : \sum_{x\in\mathcal{C}(S)} \sigma_x \to \sigma$$

sending an event to its original is rigid epi. This esures that σ and $\sum_{x\in\mathcal{C}(S)}\sigma_x$ have the same rigid image, so are 'may' equivalent.

With stopping configurations, we can perform a similar decomposition respecting 'may' and 'must' behaviour. Firstly, say a strategy $\sigma : S \to A$ with stopping configurations M is deterministic iff σ is deterministic and M consists precisely of the $+$-maximal configurations of S. Now given an arbirtrary strategy $\sigma : S \to A$ with stopping configurations M for which

(i) $\forall x \in \mathcal{C}(S) \exists y \in M.\ x \subseteq y$ and
(ii) $\forall y \in M, x \in \mathcal{C}^\infty(S).\ x \subseteq y \ \&\ x$ is $+$-maximal $\implies x \in M$,

we can decompose σ, M into a sum of deterministic strategies with stopping configurations.[1] Under the above assumptions, we can decompose σ, M into a sum of deterministic strategies with stopping configurations, *viz.*

$$\sum_{y\in M}(\sigma_y, M_y)\,,$$

in which each component is a deterministic strategy with stopping configurations

$$M_y =_{\text{def}} \{x \in \mathcal{C}^\infty(S) \mid x \subseteq y \ \&\ x \text{ is } +\text{-maximal}\}\,.$$

The obvious map

$$f : \sum_{y\in M}(\sigma_y, M_y) \to (\sigma, M)$$

is rigid and epi, by (i). Moreover, because $\bigcup_{y\in M} M_y = M$, by construction and (ii), the map f sends stopping configurations onto M. By Lemma 13.23, the strategy σ and its decomposition $\sum_{y\in M}(\sigma_y, M_y)$ are 'may' and 'must' equivalent.

[1] Example 13.26 shows why we cannot assume all $+$-maximal configurations are stopping. That property is not preserved by taking the rigid image. However the axioms above are, and would seem a reasonable weakening to impose generally on stopping configurations. The axioms hold for $\mathrm{St}(\sigma')$ of a partial strategy σ'.

13.7 A language for partial strategies

The earlier language of strategies extends to a language for partial strategies, reading the operations on strategies as the corresponding operations on partial strategies.

13.8 Operational semantics—an early attempt

Let A be a game with configuration x. Write A/x for the game after x. If $f : A \to B$ is a map between games A and B and $x \in \mathcal{C}^\infty(A)$ write $f/x : A/x \to B/fx$ for the restriction of f between subsequent games.

Say a configuration x of a game A is *+-pure* if $polx \subseteq \{+\}$, *−-pure* if $polx \subseteq \{-\}$ and pure if either. We identify configurations of $A\|B$ with pairs x, y where $x \in \mathcal{C}^\infty(A)$ and $y \in \mathcal{C}^\infty(B)$.
Composition

$$\frac{A, B : \sigma \xrightarrow{x,y} \sigma' : A/x, B/y \quad B^\perp, C : \tau \xrightarrow{\overline{y},z} \tau' : B^\perp/\overline{y}, C/z}{A, C : \tau \circledast \sigma \xrightarrow{x,z} \tau' \circledast \sigma' : A/x, C/z}$$

Without typing,

$$\frac{\sigma \xrightarrow{x,y} \sigma' \quad \tau \xrightarrow{\overline{y},z} \tau'}{\tau \circledast \sigma \xrightarrow{x,z} \tau' \circledast \sigma'}$$

Relabelling

$$\frac{A : \sigma \xrightarrow{x} \sigma' : A/x}{B : f_* \sigma \xrightarrow{fx} (f/x)_* \sigma' : B/fx} \quad x \in \mathcal{C}(A)$$

Without typing,

$$\frac{\sigma \xrightarrow{x} \sigma'}{f_* \sigma \xrightarrow{fx} (f/x)_* \sigma'} \quad x \in \mathcal{C}(A)$$

Pullback

$$\frac{B : \sigma \xrightarrow{fx} \sigma' : B/fx}{A : f^* \sigma \xrightarrow{x} (f/x)^* \sigma' : A/x} \quad x \in \mathcal{C}(A) \text{ is pure}$$

Without typing,

$$\frac{\sigma \xrightarrow{fx} \sigma'}{f^* \sigma \xrightarrow{x} (f/x)^* \sigma'} \quad x \in \mathcal{C}(A) \text{ is pure}$$

Sum of strategies, without typing,

$$\frac{\sigma_i \xrightarrow{x} \sigma_i', \ i \in I}{\Box_{i \in I} \sigma_i \xrightarrow{x} \Box_{i \in I} \sigma_i'} \quad x \in \mathcal{C}(A) \text{ is } -\text{-pure}$$

$$\frac{\sigma_j \xrightarrow{x} \sigma_j}{\prod_{i \in I} \sigma_i \xrightarrow{x} \sigma'_j} \quad j \in I \ \& \ x \in \mathcal{C}(A) \ \& \ + \in pol x$$

We assume certain primitive strategies $\sigma_0 : A$, so as a map $\sigma_0 : S \to A$, for which we assume a rule

$$\frac{}{A : \sigma_0 \xrightarrow{x} \sigma'_0 : A/x} \quad y \in \mathcal{C}(S) \ \& \ \sigma_0 y = x$$

Proposition 13.34. *Derivations in the operational semantics of*

$$A : \sigma \xrightarrow{x} \sigma' : A/x \ ,$$

in which σ denotes the map $\sigma : S \to A$, are in 1-1 correspondence with configurations $y \in \mathcal{C}(S)$ such that $\sigma y = x$.

13.9 Transition semantics

A transition semantics is presented for partial strategies. Transitions are associated with three kinds of actions: an action o associated with a hidden neutral action,

$$\begin{array}{ccc} \Gamma \ \vdash & t & \dashv \ \Delta \\ & \Big\downarrow o & \\ \Gamma \ \vdash & t' & \dashv \ \Delta \, ; \end{array}$$

an initial event located in the left environment and an initial event located in the right environment,

$$\begin{array}{ccc} \Gamma, x : A \ \vdash & t & \dashv \ \Delta \\ & \Big\downarrow x : a : x' & \\ \Gamma, x' : A/a \ \vdash & t' & \dashv \ \Delta \end{array} \qquad \begin{array}{ccc} \Gamma \ \vdash & t & \dashv \ y : B, \Delta \\ & \Big\downarrow y : b : y' & \\ \Gamma \ \vdash & t' & \dashv \ y' : B/b, \Delta \, . \end{array}$$

Notice that a neutral action leaves the types unchanged but may affect the term. An action $x : a : x'$ is associated with an initial event $ev(x : a : x') =_{\text{def}} x : a$ at the x-component of the environment. On its occurrence the component of the environment $x : A$ is updated to $x' : A/a$ in which x', a fresh resumption variable, stands for the configuration remaining in the remaining game A/a. Say an action $y : b : y'$ on the right is +ve/−ve according as b is +ve/−ve. Dually, say an action $x : a : x'$ on the left is +ve/−ve according as a is −ve/+ve.

Rules for composition:

$$\dfrac{\begin{array}{lcr} \Gamma \vdash & t & \dashv\ y:B,\Delta \\ & \Big\downarrow{\scriptstyle y:b:y'} & \\ \Gamma \vdash & t' & \dashv\ y':B/b,\Delta \end{array} \qquad \begin{array}{lcr} \Delta^{\perp}, y:B^{\perp} \vdash & u & \dashv\ \mathrm{H} \\ & \Big\downarrow{\scriptstyle y:\bar{b}:y'} & \\ \Gamma, y':B^{\perp}/\bar{b} \vdash & u' & \dashv\ \mathrm{H} \end{array}}{\begin{array}{lcr} \Gamma \vdash & \exists y:B,\Delta.[\,t \parallel u\,] & \dashv\ \mathrm{H} \\ & \Big\downarrow{\scriptstyle o} & \\ \Gamma \vdash & \exists y':B/b,\Delta.[\,t' \parallel u'\,] & \dashv\ \mathrm{H} \end{array}}$$

Below we use α for o or an action on the left of the form $x:a:x'$, and β for o or an action on the right of the form $y:b:y'$.

$$\dfrac{\begin{array}{lcr} \Gamma \vdash & t & \dashv\ \Delta \\ & \Big\downarrow{\scriptstyle \alpha} & \\ \Gamma' \vdash & t' & \dashv\ \Delta \end{array}}{\begin{array}{lcr} \Gamma \vdash & \exists\Delta.[\,t \parallel u\,] & \dashv\ \mathrm{H} \\ & \Big\downarrow{\scriptstyle \alpha} & \\ \Gamma' \vdash & \exists\Delta.[\,t' \parallel u\,] & \dashv\ \mathrm{H} \end{array}} \qquad \dfrac{\begin{array}{lcr} \Delta \vdash & u & \dashv\ \mathrm{H} \\ & \Big\downarrow{\scriptstyle \beta} & \\ \Delta \vdash & u' & \dashv\ \mathrm{H}' \end{array}}{\begin{array}{lcr} \Gamma \vdash & \exists\Delta.[\,t \parallel u\,] & \dashv\ \mathrm{H} \\ & \Big\downarrow{\scriptstyle \beta} & \\ \Gamma \vdash & \exists\Delta.[\,t \parallel u'\,] & \dashv\ \mathrm{H}' \end{array}}$$

Rules for hom-sets:

Assuming a is an initial event of A for which $p[\{a\}/x][\varnothing] \sqsubseteq_C p'[\{a\}/x][\varnothing]$,

$$\begin{array}{lcr} \Gamma, x:A \vdash & p \sqsubseteq_C p' & \dashv\ \Delta \\ & \Big\downarrow{\scriptstyle x:a:x'} & \\ \Gamma, x':A/a \vdash & p[\{a\}\cup x'/x] \sqsubseteq_C p'[\{a\}\cup x'/x] & \dashv\ \Delta\,. \end{array}$$

Above, the variable x will in fact only appear in one of p and p', though because of duality in forming terms we cannot *prima facie* be sure which.

Dually, assuming b is an initial event of B for which $p[\{b\}/y][\varnothing] \sqsubseteq_C p'[\{b\}/y][\varnothing]$,

$$\begin{array}{lcr} \Gamma \vdash & p \sqsubseteq_C p' & \dashv\ y:B,\Delta \\ & \Big\downarrow{\scriptstyle y:b:y'} & \\ \Gamma \vdash & p[\{b\}\cup y'/y] \sqsubseteq_C p'[\{b\}\cup y'/y] & \dashv\ y':B/b,\Delta\,. \end{array}$$

Rules for sum of partial strategies:

$$\frac{\begin{array}{ccc}\Gamma \vdash & t_i & \dashv \Delta\\ & \big\downarrow{\scriptstyle\epsilon} & \\ \Gamma' \vdash & t'_i & \dashv \Delta',\ i\in I\end{array}}{\begin{array}{ccc}\Gamma \vdash & \Box_{i\in I}\, t_i & \dashv \Delta \\ & \big\downarrow{\scriptstyle\epsilon} & \\ \Gamma'\vdash & \Box_{i\in I}\, t'_i & \dashv \Delta'\end{array}}\ \epsilon \text{ is } -\text{ve}
\qquad
\frac{\begin{array}{ccc}\Gamma \vdash & t_j & \dashv \Delta\\ & \big\downarrow{\scriptstyle o} & \\ \Gamma' \vdash & t'_j & \dashv \Delta'\end{array}}{\begin{array}{ccc}\Gamma \vdash & \Box_{i\in I}\, t_i & \dashv \Delta \\ & \big\downarrow{\scriptstyle o} & \\ \Gamma'\vdash & (\Box_{i\in I}\, t_i)[t'_j/j] & \dashv \Delta'\end{array}}\ j\in I$$

$$\frac{\begin{array}{ccc}\Gamma \vdash & t_j & \dashv \Delta\\ & \big\downarrow{\scriptstyle\epsilon} & \\ \Gamma' \vdash & t'_j & \dashv \Delta'\end{array}}{\begin{array}{ccc}\Gamma \vdash & \Box_{i\in I}\, t_i & \dashv \Delta \\ & \big\downarrow{\scriptstyle\epsilon} & \\ \Gamma'\vdash & t'_j & \dashv \Delta'\end{array}}\ j\in I \;\&\; \epsilon \text{ is } +\text{ve}$$

Rules for pullback of partial strategies:

$$\frac{\begin{array}{ccc}\Gamma \vdash & t_1 & \dashv \Delta\\ & \big\downarrow{\scriptstyle z:c:z'} & \\ \Gamma' \vdash & t'_1 & \dashv \Delta'\end{array}\qquad\begin{array}{ccc}\Gamma \vdash & t_2 & \dashv \Delta\\ & \big\downarrow{\scriptstyle z:c:z'} & \\ \Gamma' \vdash & t'_2 & \dashv \Delta'\end{array}}{\begin{array}{ccc}\Gamma \vdash & t_1\wedge t_2 & \dashv \Delta \\ & \big\downarrow{\scriptstyle z:c:z'} & \\ \Gamma'\vdash & t'_1\wedge t_2 & \dashv \Delta'\end{array}}$$

$$\frac{\begin{array}{ccc}\Gamma \vdash & t_1 & \dashv \Delta\\ & \big\downarrow{\scriptstyle o} & \\ \Gamma \vdash & t'_1 & \dashv \Delta\end{array}}{\begin{array}{ccc}\Gamma \vdash & t_1\wedge t_2 & \dashv \Delta \\ & \big\downarrow{\scriptstyle o} & \\ \Gamma\vdash & t'_1\wedge t_2 & \dashv \Delta\end{array}}
\qquad
\frac{\begin{array}{ccc}\Gamma \vdash & t_2 & \dashv \Delta\\ & \big\downarrow{\scriptstyle o} & \\ \Gamma \vdash & t'_2 & \dashv \Delta\end{array}}{\begin{array}{ccc}\Gamma \vdash & t_1\wedge t_2 & \dashv \Delta \\ & \big\downarrow{\scriptstyle o} & \\ \Gamma\vdash & t_1\wedge t'_2 & \dashv \Delta\end{array}}$$

Rules for δ:

Provided b is an initial −ve event of B,

$$\begin{array}{lcl} \Gamma \vdash & \delta_C(p,q_1,q_2) & \dashv y:B,\Delta \\ & \Big\downarrow {\scriptstyle y:b:y'} & \\ \Gamma \vdash & \delta_C(p,q_1,q_2)[\{b\}\cup y'/y] & \dashv y':B/b,\Delta\,. \end{array}$$

Dually, provided a is an initial +ve event of A,

$$\begin{array}{lcl} \Gamma, x:A \vdash & \delta_C(p,q_1,q_2) & \dashv \Delta \\ & \Big\downarrow {\scriptstyle x:a:x'} & \\ \Gamma, x':A/a \vdash & \delta_C(p,q_1,q_2)[\{a\}\cup x'/x] & \dashv \Delta\,. \end{array}$$

In typed judgements of $\delta_C(p,q_1,q_2)$ a variable can appear free in at most one of p,q_1,q_2. Write, for example, $y \in \mathrm{fv}(p)$ for y is a free variable of p, and $q_1(y:b) \in p[\varnothing]$ to mean the image of b under the map q_1 denotes is in the configuration denoted by $p[\varnothing]$.

Provided b is an initial +ve event of B, $y \in \mathrm{fv}(q_1)$ and $q_1(y:b) \in p[\varnothing]$,

$$\begin{array}{lcl} \Gamma \vdash & \delta_C(p,q_1,q_2) & \dashv y:B,\Delta \\ & \Big\downarrow {\scriptstyle y:b:y'} & \\ \Gamma \vdash & \delta_C(p,q_1,q_2)[\{b\}\cup y'/y] & \dashv y':B/b,\Delta\,. \end{array}$$

Similarly for q_2. And dually.

Provided b is an initial +ve event of B, $y \in \mathrm{fv}(p)$ and $p(y:b) \in q_1[\varnothing]$,

$$\begin{array}{lcl} \Gamma \vdash & \delta_C(p,q_1,q_2) & \dashv y:B,\Delta \\ & \Big\downarrow {\scriptstyle y:b:y'} & \\ \Gamma \vdash & \delta_C(p,q_1,q_2)[\{b\}\cup y'/y] & \dashv y':B/b,\Delta\,. \end{array}$$

Similarly for q_2. And dually.

13.9.1 Duality

Above, as is to be expected from duality, we can derive a transition

$$\begin{array}{lcl} \Gamma, x:A \vdash & t & \dashv \Delta \\ & \Big\downarrow {\scriptstyle x:a:x'} & \\ \Gamma, x':A/a \vdash & t' & \dashv \Delta \end{array}$$

iff we can derive a transition

$$\begin{array}{ccc} \Gamma \vdash & t & \dashv x : A^{\perp}, \Delta \\ & \Big\downarrow{\scriptstyle x:a:x'} & \\ \Gamma \vdash & t' & \dashv x' : (A/a)^{\perp}, \Delta\,. \end{array}$$

13.10 Derivations and events

Assume certain primitive strategies $\Gamma \vdash \sigma_0 \dashv \Delta$, so as a map, $\sigma_0 : S \to \Gamma^{\perp} \| \Delta$, for which we assume rules,

$$\frac{}{\begin{array}{ccc} \Gamma \vdash & \sigma_0 & \dashv \Delta \\ & \Big\downarrow{\scriptstyle \epsilon} & \\ \Gamma' \vdash & \sigma_0' & \dashv \Delta' \end{array}} \quad s \text{ is initial in } S \ \& \ \sigma_0(s) = ev(\epsilon)\,.$$

Then, *derivations* in the operational semantics

$$\frac{\vdots}{\begin{array}{ccc} \Gamma \vdash & t & \dashv \Delta \\ & \Big\downarrow{\scriptstyle \epsilon} & \\ \Gamma' \vdash & t' & \dashv \Delta'\,, \end{array}}$$

up to α-equivalence, in which t denotes the partial strategy $\sigma : S \to \Gamma^{\perp} \| \Delta$, are in 1-1 correspondence with initial events s in S such that $\sigma(s) = ev(\epsilon)$ when $ev(\epsilon) \neq o$ or s is neutral when $ev(\epsilon) = o$.

Chapter 14

Probabilistic strategies

The chapter provides a new definition of probabilistic event structures, extending existing definitions, and characterised as event structures together with a continuous valuation on their domain of configurations. Probabilistic event structures possess a probabilistic measure on their domain of configurations. This prepares the ground for a very general definition of a probabilistic strategies, which are shown to compose, with probabilistic copy-cat strategies as identities. The result of the play-off of a probabilistic strategy and counter-strategy in a game is a probabilistic event structure so that a measurable pay-off function from the configurations of a game is a random variable, for which the expectation (the expected pay-off) is obtained as the standard Legesgue integral.

14.1 Probabilistic event structures

A probabilistic event structure comprises an event structure $(E, \leq, \mathrm{Con})$ together with a continuous valuation on its open sets of configurations, *i.e.* a function w from the open subsets of configurations $\mathcal{C}^\infty(E)$ to $[0,1]$ which is:

(normalized) $w(\mathcal{C}^\infty(E)) = 1$ *(strict)* $w(\varnothing) = 0$;

(monotone) $U \subseteq V \implies w(U) \leq w(V)$;

(modular) $w(U \cup V) + w(U \cap V) = w(U) + w(V)$;

(continuous) $w(\bigcup_{i \in I} U_i) = \sup_{i \in I} w(U_i)$ for *directed* unions $\bigcup_{i \in I} U_i$.

Continuous valuations play a central role in probabilistic powerdomains [27]. Continuous valuations are determined by their restrictions to basic open sets $\widehat{x} =_{\mathrm{def}} \{y \in \mathcal{C}^\infty(E) \mid x \subseteq y\}$, for x a finite configuration. The intuition: $w(U)$ is the probability of the resulting configuration being in the open set U. Indeed, continuous valuations extend to unique probabilistic measures on the Borel sets.

This description of a probabilistic event structure extends the definitions in [21]. It turns out to be equivalent to a more workable definition, which relates more directly to the configurations of E, that we develop now.

14.1.1 Preliminaries

Notation 14.1. *Let $\mathcal{F}$ be a stable family. Extend $\mathcal{F}$ to a lattice $\mathcal{F}^\top$ by adjoining an extra top element $\top$. Write its order as $x \sqsubseteq y$ and its join and meet operations as $x \vee y$ and $x \wedge y$ respectively.*

Definition 14.2. Let $\mathcal{F}$ be a stable family. Assume a function $v : \mathcal{F} \to \mathbb{R}$. Extend v to $v^\top : \mathcal{F}^\top \to \mathbb{R}$ by taking $v^\top(T) = 0$.

W.r.t. $v : \mathcal{F} \to \mathbb{R}$, for $n \in \omega$, define the *drop functions* $d_v^{(n)}[y; x_1, \cdots, x_n] \in \mathbb{R}$ for $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$ in $\mathcal{F}^\top$ as follows:

$$d_v^{(0)}[y;] =_{\text{def}} v^\top(y)$$
$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\text{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] .$$

Throughout this section assume $\mathcal{F}$ is a stable family and $v : \mathcal{F} \to \mathbb{R}$.

Proposition 14.3. *Let $n \in \omega$. For $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$,*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_{\varnothing \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) .$$

For $y, x_1, \cdots, x_n \in \mathcal{F}$ with $y \subseteq x_1, \cdots, x_n$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) ,$$

where the index I ranges over sets satisfying $\varnothing \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$.

Proof. We prove the first statement by induction on n. For the basis, when $n = 0$, $d_v^{(n)}[y;] = v(y)$, as required. For the induction step, with $n > 0$, we reason

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &=_{\text{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= v(y) - \sum_{\varnothing \neq I \subseteq \{1, \cdots, n-1\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \\ &\quad - v(x_n) + \sum_{\varnothing \neq J \subseteq \{1, \cdots, n-1\}} (-1)^{|I|+1} v(\bigvee_{j \in J} x_i \vee x_n) , \end{aligned}$$

making use of the induction hypothesis. Consider subsets K for which $\varnothing \neq K \subseteq \{1, \cdots, n\}$. Either $n \notin K$, in which case $\varnothing \neq K \subseteq \{1, \cdots, n-1\}$, or $n \in K$, in which case $K = \{n\}$ or $J =_{\text{def}} K \setminus \{n\}$ satisfies $\varnothing \neq J \subseteq \{1, \cdots, n-1\}$. From this observation, the sum above amounts to

$$v(y) - \sum_{\varnothing \neq K \subseteq \{1, \cdots, n\}} (-1)^{|K|+1} v(\bigvee_{k \in K} x_k) ,$$

as required to maintain the induction hypothesis.

The second expression of the proposition is got by discarding all terms $v(\bigvee_{i \in I} x_i)$ for which $\bigvee_{i \in I} x_i = \top$ which leaves the sum unaffected as they contribute 0. $\square$

Corollary 14.4. *Let $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. For ρ an n-permutation,*

$$d_v^{(n)}[y; x_{\rho(1)}, \cdots, x_{\rho(n)}] = d_v^{(n)}[y; x_1, \cdots, x_n] .$$

Proof. As by Proposition 14.3, the value of $d_v^{(n)}[y; x_1, \cdots, x_n]$ is insensitive to permutations of its arguments. □

Proposition 14.5. *Assume $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If $y = x_i$ for some i with $1 \leq i \leq n$ then $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$.*

Proof. By Corollary 14.4, it suffices to show $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$ when $y = x_n$. In this case,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] \\ &= 0 . \end{aligned}$$

□

Corollary 14.6. *Assume $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If $x_i \sqsubseteq x_j$ for distinct i, j with $1 \leq i, j \leq n$ then*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n] .$$

Proof. By Corollary 14.4, it suffices to show

$$d_v^{(n)}[y; x_1, \cdots, x_{n-1}, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}]$$

when $x_{n-1} \sqsubseteq x_n$. Then,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-2}, x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - 0 , \end{aligned}$$

by Proposition 14.5. □

Proposition 14.7. *Assume $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. Then, $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$ if $y = \top$ and $d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$ if $x_i = \top$ with $1 \leq i \leq n$.*

Proof. When $n = 0$, $d_v^{(0)}[\top;] = v^\top(\top) = 0$. When $n \geq 1$, $d_v^{(n)}[\top; x_1, \cdots, x_n] = 0$ by Proposition 14.5 as *e.g.* $x_n = \top$. For the remaining statement, w.l.og. we may assume $i = n$ and that $x_n = \top$, yielding

$$d_v^{(n)}[y; x_1, \cdots, \top] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[\top; x_1 \vee \top, \cdots, x_{n-1} \vee \top] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] .$$

□

Lemma 14.8. *Let $n \geq 1$. Let $y, x_1, \cdots, x_n, x'_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. Assume $x_n \sqsubseteq x'_n$. Then,*

$$d_v^{(n)}[y; x_1, \cdots, x'_n] = d_v^{(n)}[y; x_1, \cdots, x_n] + d_v^{(n)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n, x'_n].$$

Proof. By definition,

$$\begin{aligned}
\text{the r.h.s.} &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\
&\quad + d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] - d_v^{(n-1)}[x'_n; x_1 \vee x'_n, \cdots, x_{n-1} \vee x'_n] \\
&= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x'_n; x_1 \vee x'_n, \cdots, x_{n-1} \vee x'_n] \\
&= d_v^{(n)}[y; x_1, \cdots, x_{n-1}, x'_n] \\
&= \text{the l.h.s..}
\end{aligned}$$

□

14.1.2 The definition

Definition 14.9. Let $\mathcal{F}$ be a stable family. A *configuration-valuation* is function $v : \mathcal{F} \to [0, 1]$ such that $v(\varnothing) = 1$ and which satisfies the "drop condition:"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{F}$ with $y \subseteq x_1, \cdots, x_n$.

A *probabilistic stable family* comprises a stable family $\mathcal{F}$ together with a configuration-valuation $v : \mathcal{F} \to [0, 1]$.

A *probabilistic event structure* comprises an event structure E together with a configuration-valuation $v : \mathcal{C}(E) \to [0, 1]$.

Proposition 14.10. *Let $v : \mathcal{F} \to [0, 1]$. Then, v is a configuration-valuation iff $v^\top(\varnothing) = 1$ and $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ for all $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If v is a configuration-valuation, then*

$$y \sqsubseteq x \implies v^\top(y) \geq v^\top(x),$$

for all $x, y \in \mathcal{F}^\top$.

Proof. By Proposition 14.7 and as $d_v^{(1)}[y; x] = v^\top(y) - v^\top(x)$. □

In showing we have a probabilistic event structure or stable family it suffices to verify the "drop condition" only for covering intervals.

Lemma 14.11. *Let $\mathcal{F}$ be a stable family and $v : \mathcal{F} \to [0, 1]$.*

(i) Let $y \subseteq x_1, \cdots, x_n$ in $\mathcal{F}$. Then, $d_v^{(n)}[y; x_1, \cdots, x_n]$ is expressible as a sum of terms

$$d_v^{(k)}[u; w_1, \cdots, w_k]$$

where $y \subseteq u \multimap\!\!\subset w_i$ *in* $\mathcal{F}$ *and* $w_i \subseteq x_1 \cup \cdots \cup x_n$*, for all* i *with* $1 \leq i \leq k$*. [The set* $x_1 \cup \cdots \cup x_n$ *need not be in* $\mathcal{F}$*.]*

(ii) A fortiori, v *is a configuration-valuation iff* $v(\varnothing) = 1$ *and*

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \geq 1$ *and* $y \multimap\!\!\subset x_1, \cdots, x_n$ *in* $\mathcal{F}$*.*

Proof. Define the *weight* of a term $d_v^{(n)}[y; x_1, \cdots, x_n]$, where $y \subseteq x_1, \cdots, x_n$ in $\mathcal{F}$, to be the product $|x_1 \setminus y| \times \cdots \times |x_n \setminus y|$.

Assume $y \subseteq x_1, \cdots, x'_n$ in $\mathcal{F}$. By Proposition 14.5, if y equals x'_n or some x_i, then $d_v^{(n)}[y; x_1, \cdots, x'_n] = 0$, so may be deleted as a contribution to a sum. Otherwise, if $y \subsetneq x_n \subsetneq x'_n$, by Lemma 14.8 we can rewrite $d_v^{(n)}[y; x_1, \cdots, x'_n]$ to the sum

$$d_v^{(n)}[y; x_1, \cdots, x_n] + d_v^{(n)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n, x'_n]\,,$$

where we further observe

$$|x_n \setminus y| < |x'_n \setminus y|\,, \qquad |x'_n \setminus x_n| < |x'_n \setminus y|$$

and

$$|(x_i \cup x_n) \setminus x_n| \leq |x_i \setminus y|\,,$$

whenever $x_i \vee x_n \neq \top$. Using Proposition 14.7 we may tidy away any mentions of $\top$. This reduces $d_v^{(n)}[y; x_1, \cdots, x'_n]$ to the sum of at most two terms, each of lesser weight. For notational simplicity we have concentrated on the nth argument in $d_v^{(n)}[y; x_1, \cdots, x'_n]$, but by Corollary 14.4 an analogous reduction is possible w.r.t. any argument.

Repeated use of the reduction, rewrites $d_v^{(n)}[y; x_1, \cdots, x_n]$ to a sum of terms of the form

$$d_v^{(k)}[u; w_1, \cdots, w_k]$$

where $k \leq n$ and $u \multimap\!\!\subset w_1, \cdots, w_k \subseteq x_1 \cup \cdots \cup x_n$. This justifies the claims of the lemma. □

14.1.3 The characterisation

Our goal is to prove that probabilistic event structures correspond to event structures with a continuous valuation. It is clear that a continuous valuation w on the Scott-open subsets of an event structure E gives rise to a configuration-valuation v on E: take $v(x) =_{\mathrm{def}} w(\widehat{x})$, for $x \in \mathcal{C}(E)$. We will show that this construction has an inverse, that a configuration-valuation determines a continuous valuation.

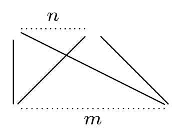
For this we need a combinatorial lemma:[1]

[1] The proof of the combinatorial lemma below is due to the author. It appears with acknowledgement as Lemma 6.App.1 in [28], the PhD thesis of my former student Daniele Varacca, whom I thank, both for the collaboration and the latex.

Lemma 14.12. *For all finite sets I, J,*

$$\sum_{\substack{\varnothing \neq K \subseteq I \times J \\ \pi_1(K) = I, \pi_2(K) = J}} (-1)^{|K|} = (-1)^{|I|+|J|-1}\,.$$

Proof. Without loss of generality we can take $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$. Also observe that a subset $K \subseteq I \times J$ such that $\pi_1(K) = I, \pi_2(K) = J$ is in fact a surjective and total relation between the two sets.



Let

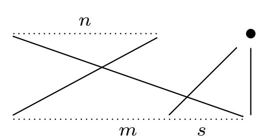
$$t_{n,m} =_{\text{def}} \sum_{\substack{\varnothing \neq K \subseteq I \times J \\ \pi_1(K) = I, \pi_2(K) = J}} (-1)^{|K|};$$

$$t^o_{n,m} =_{\text{def}} |\{\varnothing \neq K \subseteq I \times J \mid |K| \text{ odd}, \pi_1(K) = I, \pi_2(K) = J\}|;$$

$$t^e_{n,m} := |\{\varnothing \neq K \subseteq I \times J \mid |K| \text{ even}, \pi_1(K) = I, \pi_2(K) = J\}|.$$

Clearly $t_{n,m} = t^e_{n,m} - t^o_{n,m}$. We want to prove that $t_{n,m} = (-1)^{n+m+1}$. We do this by induction on n. It is easy to check that this is true for $n = 1$. In this case, if m is even then $t^e_{1,m} = 1$ and $t^o_{1,m} = 0$, so that $t^e_{1,m} - t^o_{1,m} = (-1)^{1+m+1}$. Similarly if m is odd.

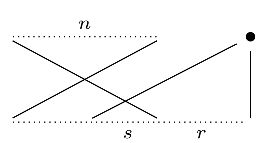
Now assume that for every p, $t_{n,p} = (-1)^{n+p+1}$ and compute $t_{n+1,m}$. To evaluate $t_{n+1,m}$ we count all surjective and total relations K between I and J together with their"sign." Consider the pairs in K of the form $(n+1, h)$ for $h \in J$. The result of removing them is a a total surjective relation between $\{1, \ldots, n\}$ and a subset J_K of $\{1, \ldots, m\}$.



Consider first the case where $J_K = \{1, \ldots, m\}$. Consider the contribution of such K's to $t_{n+1,m}$. There are $\binom{m}{s}$ ways of choosing s pairs of the form $(n+1, h)$. For every such choice there are $t_{n,m}$ (signed) relations. Adding the pairs $(n+1, h)$ possibly modifies the sign of such relations. All in all the contribution amounts to

$$\sum_{1 \leq s \leq m} \binom{m}{s} (-1)^s t_{n,m}\,.$$

Suppose now that J_K is a proper subset of $\{1, \ldots, m\}$ leaving out r elements.



Since K is surjective, all such elements h must be in a pair of the form $(n+1,h)$. Moreover there can be s pairs of the form $(n+1,h')$ with $h' \in J_K$. What is the contribution of such K's to $t_{n,m}$? There are $\binom{m}{r}$ ways of choosing the elements that are left out. For every such choice and for every s such that $0 \le s \le m-r$ there are $\binom{m-r}{s}$ ways of choosing the $h' \in J_K$. And for every such choice there are $t_{n,m-r}$ (signed) relations. Adding the pairs $(n+1,h)$ and $(n+1,h')$ possibly modifies the sign of such relations. All in all, for every r such that $1 \le r \le m-1$, the contribution amounts to

$$\binom{m}{r} \sum_{1 \le s \le m-r} \binom{m}{s} (-1)^{s+r} t_{n,m-n} \,.$$

The (signed) sum of all these contribution will give us $t_{n+1,m}$. Now we use the induction hypothesis and we write $(-1)^{n+p+1}$ for $t_{n,p}$.

Thus,

$$\begin{aligned}
t_{n+1,m} &= \sum_{1\le s\le m} \binom{m}{s} (-1)^s t_{n,m} \\
&\quad + \sum_{1\le r\le m-1} \binom{m}{r} \sum_{0\le s\le m-r} \binom{m-r}{s} (-1)^{s+r} t_{n,m-r} \\
&= \sum_{1\le s\le m} \binom{m}{s} (-1)^{s+n+m+1} \\
&\quad + \sum_{1\le r\le m-1} \binom{m}{r} \sum_{0\le s\le m-r} \binom{m-r}{s} (-1)^{s+n+m+1} \\
&= (-1)^{n+m+1} \left(\sum_{1\le s\le m} \binom{m}{s} (-1)^s \right. \\
&\quad \left. + \sum_{1\le r\le m-1} \binom{m}{r} \sum_{0\le s\le m-r} \binom{m-r}{s} (-1)^s \right) .
\end{aligned}$$

By the binomial formula, for $1 \le r \le m-1$ we have

$$0 = (1-1)^{m-r} = \sum_{0\le s\le m-r} \binom{m-r}{s} (-1)^s \,.$$

So we are left with

$$\begin{aligned}
t_{n+1,m} &= (-1)^{n+m+1} \left(\sum_{1\le s\le m} \binom{m}{s} (-1)^s \right) \\
&= (-1)^{n+m+1} \left(\sum_{0\le s\le m} \binom{m}{s} (-1)^s - \binom{m}{0} (-1)^0 \right) \\
&= (-1)^{n+m+1} \, (0-1) \\
&= (-1)^{n+1+m+1} \,,
\end{aligned}$$

as required. □

Theorem 14.13. *A configuration-valuation v on an event structure E extends to a unique continuous valuation w_v on the open sets of $\mathcal{C}^\infty(E)$, so that $w_v(\widehat{x}) = v(x)$, for all $x \in \mathcal{C}(E)$.*

Conversely, a continuous valuation w on the open sets of $\mathcal{C}^\infty(E)$ restricts to a configuration-valuation v_w on E, assigning $v_w(x) = w(\widehat{x})$, for all $x \in \mathcal{C}(E)$.

Proof. The proof is inspired by the proofs in the appendix of [21] and the thesis [28].

First, a continuous valuation w on the open sets of $\mathcal{C}^\infty(E)$ restricts to a configuration-valuation v defined as $v(x) =_{\mathrm{def}} w(\widehat{x})$ for $x \in \mathcal{C}(E)$. Note that any extension of a configuration-valuation to a continuous valuation is bound to be unique by continuity.

To show the converse we first define a function w from the basic open sets $Bs =_{\mathrm{def}} \{\widehat{x_1} \cup \cdots \cup \widehat{x_n} \mid x_1, \cdots, x_n \in \mathcal{C}(E)\}$ to $[0,1]$ and show that it is normalised, strict, monotone and modular. Define

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) =_{\mathrm{def}} &\ 1 - d_v^{(n)}[\varnothing; x_1, \cdots, x_n] \\ = &\sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \end{aligned}$$

—this can be shown to be well-defined using Corollaries 14.4 and 14.6.

Clearly, w is normalised in the sense that $w(\mathcal{C}^\infty(E)) = w(\widehat{\varnothing}) = 1$ and strict in that $w(\varnothing) = 1 - v(\varnothing) = 0$.

To see that it is monotone, first observe that

$$w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \leq w(\widehat{x_1} \cup \cdots \cup \widehat{x_{n+1}})$$

as

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_{n+1}}) - w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) &= d_v^{(n)}[\varnothing; x_1, \cdots, x_n] - d_v^{(n+1)}[\varnothing; x_1, \cdots, x_{n+1}] \\ &= d_v^{(n)}[x_{n+1}; x_1 \vee x_{n+1}, \cdots, x_n \vee x_{n+1}] \geq 0\,. \end{aligned}$$

By a simple induction (on m),

$$w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \leq w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m})\,.$$

Suppose that $\widehat{x_1} \cup \cdots \cup \widehat{x_n} \subseteq \widehat{y_1} \cup \cdots \cup \widehat{y_m}$. Then $\widehat{y_1} \cup \cdots \cup \widehat{y_m} = \widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}$. By the above,

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) &\leq w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ &= w(\widehat{y_1} \cup \cdots \cup \widehat{y_m})\,, \end{aligned}$$

as required to show w is monotone.

To show modularity we require

$$\begin{aligned} & w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) + w(\widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ =& w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) + w((\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \cap (\widehat{y_1} \cup \cdots \cup \widehat{y_m}))\,. \end{aligned}$$

Note

$$(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \cap (\widehat{y_1} \cup \cdots \cup \widehat{y_m}) = (\widehat{x_1} \cap \widehat{y_1}) \cup \cdots \cup (\widehat{x_i} \cap \widehat{y_j}) \cdots \cup (\widehat{x_n} \cap \widehat{y_m})$$
$$= \widehat{x_1 \vee y_1} \cup \cdots \cup \widehat{x_i \vee y_j} \cdots \cup \widehat{x_n \vee y_m} \,.$$

From the definition of w we require

$$\begin{aligned} & w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ = & \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) + \sum_{\emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|J|+1} v(\bigvee_{j \in J} y_j) \\ & - \sum_{\emptyset \neq R \subseteq \{1,\cdots,n\} \times \{1,\cdots,m\}} (-1)^{|R|+1} v(\bigvee_{(i,j) \in R} x_i \vee y_j) \,. \qquad (1) \end{aligned}$$

Consider the definition of $w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m})$ as a sum. Its components are associated with indices which either lie entirely within $\{1, \cdots, n\}$, entirely within $\{1, \cdots, m\}$, or overlap both. Hence

$$\begin{aligned} & w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ = & \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) + \sum_{\emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|J|+1} v(\bigvee_{j \in J} y_j) \\ & + \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}, \emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|I|+|J|+1} v(\bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j) \,. \qquad (2) \end{aligned}$$

Comparing (1) and (2), we require

$$\begin{aligned} & - \sum_{\emptyset \neq R \subseteq \{1,\cdots,n\} \times \{1,\cdots,m\}} (-1)^{|R|+1} v(\bigvee_{(i,j) \in R} x_i \vee y_j) \\ = & \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}, \emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|I|+|J|+1} v(\bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j) \,. \qquad (3) \end{aligned}$$

Observe that

$$\bigvee_{(i,j) \in R} x_i \vee y_j = \bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j$$

when $I = R_1 =_{\mathrm{def}} \{i \in I \mid \exists j \in J.\ (i,j) \in R\}$ and $J = R_2 =_{\mathrm{def}} \{j \in J \mid \exists i \in I.\ (i,j) \in R\}$ for a relation $R \subseteq \{1, \cdots, n\} \times \{1, \cdots, m\}$. With this observation we see that equality (3) follows from the combinatorial lemma, Lemma 14.12 above. This shows modularity.

Finally, we can extend w to all open sets by taking an open set U to $\sup_{b \in Bs \,\&\, b \subseteq U} w(b)$. The verification that w is indeed a continuous valuation extending v is now straightforward. □

The above theorem also holds (with the same proof) for Scott domains. Now, by [29], Corollary 4.3:

Theorem 14.14. *For a configuration-valuation v on E there is a unique probability measure μ_v on the Borel subsets of $\mathcal{C}^\infty(E)$ extending w_v.*

Example 14.15. Consider the event structure comprising two concurrent events e_1, e_2 with configuration-valuation v for which $v(\varnothing) = 1, v(\{e_1\}) = 1/3, v(\{e_2\}) = 1/2$ and $v(\{e_1, e_2\}) = 1/12$. This means in particular that there is a probability of 1/3 of a result within the Scott open set consisting of both the configuration $\{e_1\}$ and the configuration $\{e_1, e_2\}$. In other words, there is a probability of 1/3 of observing e_1 (possibly with or possibly without e_2). The induced probability measure p assigns a probability to any Borel set, in this simple case any subset of configurations, and is determined by its value on single configurations: $p(\varnothing) = 1 - 4/12 - 6/12 + 1/12 = 3/12$, $p(\{e_1\}) = 4/12 - 1/12 = 3/12$, $p(\{e_2\}) = 6/12 - 1/12 = 5/12$ and $p(\{e_1, e_2\}) = 1/12$. Thus there is a probability of 3/12 of observing neither e_1 nor e_2, and a probability of 5/12 of observing just the event e_2 (and not e_1). There is a drop $d_v^{(0)}[\varnothing; \{e_1\}, \{e_2\}] = 1-4/12-6/12+1/12 = 3/12$ corresponding to the probability of remaining at the empty configuration and not observing any event. Sometimes it's said that probability "leaks" at the empty configuration, but it's more accurate to think of this leak in probability as associated with a non-zero chance that the initial observation of no events will not improve.

Example 14.16. Consider the event structure with events $\mathbb{N}^+$ with causal dependency $n \leq n+1$, with all finite subsets consistent. It is not hard to check that all subsets of $\mathcal{C}^\infty(\mathbb{N}^+)$ are Borel sets. Consider the ensuing probability distributions w.r.t. the following configuration-valuations:
(i) $v_0(x) = 1$ for all $x \in \mathcal{C}(\mathbb{N}^+)$. The resulting probability distribution assigns probability 1 to the singleton set $\{\mathbb{N}^+\}$, comprising the single infinite configuration $\mathbb{N}^+$, and 0 to $\varnothing$ and all other singleton sets of configurations.
(ii) $v_1(\varnothing) = v_1(\{1\}) = 1$ and $v_1(x) = 0$ for all other $x \in \mathcal{C}(\mathbb{N}^+)$. The resulting probability distribution assigns probability 0 to $\varnothing$ and probability 1 to the singleton set $\{1\}$, and 0 to all other singleton sets of configurations.
(iii) $v_2(\varnothing) = 1$ and $v_2(\{1, \cdots, n\}) = (1/2)^n$ for all $n \in \mathbb{N}^+$. The resulting probability distribution assigns probability 1/2 to $\varnothing$ and $(1/2)^{n+1}$ to each singleton $\{\{1, \cdots, n\}\}$ and 0 to the singleton set $\{\mathbb{N}^+\}$, comprising the single infinite configuration $\mathbb{N}^+$.

When x a finite configuration has $v(x) > 0$ and $\mu_v(\{x\}) = 0$ we can understand x as being a transient configuration on the way to a final with probability $v(x)$. In general, there is a simple expression for the probability of terminating at a finite configuration.

Proposition 14.17. *Let E, v be a probabilistic event structure. For any finite configuration $y \in \mathcal{C}(E)$, the singleton set $\{y\}$ is a Borel subset with probability measure*

$$\mu_v(\{y\}) = \inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \& \ y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}.$$

Proof. Let $y \in \mathcal{C}(E)$. Then $\{y\} = \widehat{y} \setminus U_y$ is clearly Borel as $U_y =_{\mathrm{def}} \{x \in \mathcal{C}^\infty(E) \mid y \subsetneq x\}$ is open. Let w be the continuous valuation extending v. Then

$$w(U_y) = \sup\{w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}$$

as U_y is the directed union $\bigcup\{\widehat{x}_1 \cup \cdots \cup \widehat{x}_n \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}$. Hence

$$\begin{aligned}
\mu_v(\{y\}) = v(y) - w(U_y) &= v(y) - \sup\{w(\widehat{x}_1 \cup \cdots \cup \widehat{x}_n) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\} \\
&= \inf\{v(y) - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\} \\
&= \inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \&\ y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}\,.
\end{aligned}$$

□

Example 14.18. A non-example: ***CORRECT SO ALL OPEN SETS RECEIVED A VALUE***MEET OF OPEN SETS WITH 'HORIZON' ISN'T ALWAYS MEASURABLE**** Consider the event structure comprising events $[0,1]$ where the only non-empty consistent sets are singletons. Its configurations comprise the emptyset and all singletons of reals in $[0,1]$. Its maximal configurations thus inherit a measure from the Lebesgue measure on $[0,1]$. Given an open set U define $w(U)$ to be this measure if $\emptyset \notin U$ and 1 otherwise. Then w is a valuation on the open subsets of configurations, in the sense that it is normalized, strict, monotone and modular. However it is not continuous: any open set consisting of a singleton configuration will have value 0 which in the presence of continuity would force any open set not containing the empty configuration to have value 0 too. So this example lies outside the present definition of probabilistic event structure.

Example 14.19. It might be thought that probabilistic event structures could only capture discrete distributions. However consider the event structure representing streams of 0's and 1's. We saw this earlier in Example 2.1. Its finite configurations comprise the empty set and downwards-closures $[s]$ of single event occurrences s given by a finite sequence of 0's and 1's. Assign value 1 to the empty configuration and $1/2^n$ to a sequence $s = (s_1, s_2, \cdots, s_n)$. Then all finite configurations $[s]$ are transient it the sense that the probability of ending up at precisely the finite stream $[s]$ is zero; all the probabilistic measure is concentrated on the maximal configurations, the infinite streams. On the maximal configurations the probabilistic measure gives a continuous distribution with zero probability of the result being any particular infinite stream.

Remark. There is perhaps some redundancy in the definition of purely probabilistic event structures, in that there are two different ways to say, for example, that events e_1 and e_2 do not occur together at a finite configuration y where $y \xrightarrow{e_1}\!\subset x_1$ and $y \xrightarrow{e_2}\!\subset x_2$: either through $\{e_1, e_2\} \notin \mathrm{Con}$; or via the configuration-valuation v through $v(x_1 \cup x_2) = 0$. However, when we mix probability with nondeterminism, as we do in the next section, we shall make use of both order-consistency and the valuation.

14.2 Probability with an Opponent

Assume now that the events of the stable family or event structure carry a polarity, + or −. Imagine the event structure or stable family represents a strategy for Player. The Player cannot foresee what probabilities Opponent will ascribe to moves under Opponent's control. Nor, without information about the stochastic rates of Player and Opponent can we hope to ascribe probabilities to play outcomes in the presence of races. For this reason we shall restrict probabilistic event structures with polarity to those which are race-free.

It will be convenient, more generally, to define a probabilistic stable family in which some events are distinguished as Opponent events (where the other events may be Player events or "neutral" events due to synchronizations between Player and Opponent). Events which are not Opponent events we shall call p-events. For configurations x, y we shall write $x \subseteq^p y$ if $x \subseteq y$ and $y \setminus x$ contains no Opponent events; we write $x \longrightarrow\!\!\!\!\subset^p y$ when $x \longrightarrow\!\!\!\!\subset y$ and $x \subseteq^p y$; we continue to write $x \subseteq^- y$ if $x \subseteq y$ and $y \setminus x$ comprises solely Opponent events.

Definition 14.20. We extend the notion of configuration-valuation to the situation where events carry polarities. Let $\mathcal{F}$ be a stable family $\mathcal{F}$ together with a specified subset of its events which are Opponent events. A *configuration-valuation* is a function $v : \mathcal{F} \to [0,1]$ for which $v(\varnothing) = 1$,

$$x \subseteq^- y \implies v(x) = v(y) \tag{1}$$

for all $x, y \in \mathcal{F}$, and satisfies the "drop condition"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0 \tag{2}$$

for all $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}$ with $y \subseteq^p x_1, \cdots, x_n$.

The notion of *probabilistic stable family* thus extends to a stable family $\mathcal{F}$ together with a specified subset of Opponent events and a configuration-valuation $v : \mathcal{F} \to [0,1]$. The notion specialises to event structures with a distinguished subset of Opponent events.

In particular, a *probabilistic event structure with polarity* comprises E an event structure with polarity together with a configuration-valuation $v : \mathcal{C}(E) \to [0,1]$.

Remark There is an equivalent way of presenting a configuration-valuation for an event structure with polarity S as a family of conditional probabilities. Define a familiy of conditional probabilities over S to comprise $\mathrm{Prob}(x \mid y)$, indexed by $y \subseteq^+ x$ in $\mathcal{C}(S)$, such that

(i) $\mathrm{Prob}(y \mid y) = 1$ and $x \mapsto \mathrm{Prob}(x \mid y)$ satisfies the drop condition for x s.t. $y \subseteq^+ x$ in $\mathcal{C}(S)$;

(ii) $\mathrm{Prob}(w \mid y) = \mathrm{Prob}(w \mid x)\mathrm{Prob}(x \mid y)$ if $y \subseteq^+ x \subseteq^+ w$ in $\mathcal{C}(S)$;

(iii) $\mathrm{Prob}(x \mid y) = \mathrm{Prob}(x' \mid y')$ if $y \subseteq^+ x$, $y \subseteq^- y'$ and $x \cup y' = x'$.

Given a configuration-valuation v we define $\mathrm{Prob}(x \mid y) = v(x)/v(y)$ if $v(y) \neq 0$ and to be 0 otherwise. Conversely, given a family of conditional probabilities, as described above, first extend it by taking $\mathrm{Prob}(x \mid y) = 1$ for $y \subseteq^- x$. We then obtain a configuration-valuation by defining

$$v(x) =_{\mathrm{def}} \mathrm{Prob}(x_1 \mid x_0)\mathrm{Prob}(x_2 \mid x_1) \cdots \mathrm{Prob}(x_n \mid x_{n-1})$$

w.r.t. a covering chain

$$\varnothing = x_0 \longrightarrow\!\!\!\!\!\subset x_1 \longrightarrow\!\!\!\!\!\subset x_2 \longrightarrow\!\!\!\!\!\subset \cdots \longrightarrow\!\!\!\!\!\subset x_{n-1} \longrightarrow\!\!\!\!\!\subset x_n = x\,;$$

by (ii) and (iii) the choice of covering chain does not affect the value assigned to x. The two operations provide mutual inverses between configuration-valuations and families of conditional probabilities provided they in addition satisfy

$$\mathrm{Prob}(y \mid \varnothing) = 0 \ \&\ y \subseteq^{+} x \implies \mathrm{Prob}(x \mid y) = 0\,,$$

or, equivalently,

$$\mathrm{Prob}(x_1 \mid y_1) = 0 \ \&\ y_1 \subseteq^{+} x_1 \subseteq y \subseteq^{+} x \implies \mathrm{Prob}(x \mid y) = 0\,.$$

There is an analogous result for configuration-valuations for a stable family $\mathcal{F}$ together with a specified subset of Opponent events.

As indicated above, the extra generality in the definition of a probabilistic stable family with polarity is to cater for a situation later in which we shall ascribe probabilities not only to results of Player moves but also to events arising as synchronizations between Player and Opponent moves. As earlier, by Lemma 14.11(i), it suffices to verify the "drop condition" for p-covering intervals.

Definition 14.21. Let A be a race-free event structure with polarity. A *probabilistic strategy* in A comprises a probabilistic event structure S, v and a strategy $\sigma : S \to A$. [By Lemma 5.6, S will also be race-free.]

Let A and B be a race-free event structures with polarity. A *probabilistic strategy* from A to B comprises a probabilistic event structure S, v and a strategy $\sigma : S \to A^{\perp} \| B$.

We extend the usual composition of strategies to probabilistic strategies. Assume probabilistic strategies $\sigma : S \to A^{\perp} \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^{\perp} \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$. We first tentatively define their composition on stable families, taking $v : \mathcal{C}(T) \circledast \mathcal{C}(S) \to [0,1]$ to be

$$v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$$

for $x \in \mathcal{C}(T) \circledast \mathcal{C}(S)$.

Proposition 14.22. *Let $v : \mathcal{C}(T) \circledast \mathcal{C}(S) \to [0,1]$ be defined as above. Then, $v(\varnothing) = 0$. If $x \subseteq^{-} y$ in $\mathcal{C}(T) \circledast \mathcal{C}(S)$ then $v(x) = v(y)$.*

Proof. Clearly,

$$v(\varnothing) = v_S(\pi_1 \varnothing) \times v_T(\pi_2 \varnothing) = 1 \times 1 = 1\,.$$

Assuming $x \longrightarrow\!\!\!\!\!\subset^{-} y$ in $\mathcal{C}(T) \circledast \mathcal{C}(S)$, then either $x \overset{(s,*)}{\longrightarrow\!\!\!\!\!\subset} y$, with s a $-$ve event of S, or $x \overset{(*,t)}{\longrightarrow\!\!\!\!\!\subset} y$, with t a $-$ve event of T. Suppose $x \overset{(s,*)}{\longrightarrow\!\!\!\!\!\subset} y$, with s $-$ve. Then $\pi_1 x \overset{s}{\longrightarrow\!\!\!\!\!\subset} \pi_1 y$, where as s is −ve, $v_S(\pi_1 x) = v_S(\pi_1 y)$. In addition, $\pi_2 x = \pi_2 y$ so certainly $v_T(\pi_2 x) = v_T(\pi_2 y)$. Combined these two facts yield $v(x) = v(y)$. Similarly, $x \overset{(*,t)}{\longrightarrow\!\!\subset} y$, with t −ve, implies $v(x) = v(y)$. As $x \subseteq^- y$ is obtained via the reflexive transitive closure of $-\!\!\subset^-$ it entails $v(x) = v(y)$, as required. □

But of course we need to check that v is indeed a configuration-valuation. For this it remains to show that v satisfies the "drop condition." For this we need only consider covering intervals, by Lemma 14.11(i).

Lemma 14.23. *Let $y, x_1, \cdots, x_n \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ with $y -\!\!\subset^p x_1, \cdots, x_n$. Assume that $\pi_1 y -\!\!\subset^+ \pi_1 x_i$ when $1 \le i \le m$ and $\pi_2 y -\!\!\subset^+ \pi_2 x_i$ when $m+1 \le i \le n$. Then in $\mathcal{C}(T) \circledast \mathcal{C}(S), v$,*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_{v_S}^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_{v_T}^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] .$$

Proof. Under the assumptions of the lemma, by proposition 14.3,

$$d_{v_S}^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] = v_S(\pi_1 y) - \sum_{I_1} (-1)^{|I_1|+1} v_S(\bigcup_{i \in I_1} \pi_1 x_i) ,$$

where I_1 ranges over sets satisfying $\varnothing \ne I_1 \subseteq \{1, \cdots, m\}$ s.t. $\{\pi_1 x_i \mid i \in I_1\}\uparrow$. Similarly,

$$d_{v_T}^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] = v_T(\pi_2 y) - \sum_{I_2} (-1)^{|I_2|+1} v_T(\bigcup_{i \in I_2} \pi_2 x_i) ,$$

where I_2 ranges over sets satisfying $\varnothing \ne I_2 \subseteq \{m+1, \cdots, n\}$ s.t. $\{\pi_2 x_i \mid i \in I_2\}\uparrow$.

Note, by strong receptivity of τ, that when $\varnothing \ne I_1 \subseteq \{1, \cdots, m\}$,

$$\{\pi_1 x_i \mid i \in I_1\}\uparrow \text{ in } \mathcal{C}(S) \text{ iff } \{x_i \mid i \in I_1\}\uparrow \text{ in } \mathcal{C}(T) \circledast \mathcal{C}(S)$$

and, similarly by strong receptivity of σ, when $\varnothing \ne I_2 \subseteq \{m+1, \cdots, n\}$,

$$\{\pi_2 x_i \mid i \in I_2\}\uparrow \text{ in } \mathcal{C}(T) \text{ iff } \{x_i \mid i \in I_2\}\uparrow \text{ in } \mathcal{C}(T) \circledast \mathcal{C}(S) .$$

Hence

$$\bigcup_{i \in I_1} \pi_1 x_i = \pi_1 \bigcup_{i \in I_1} x_i \quad \text{and} \quad \bigcup_{i \in I_2} \pi_2 x_i = \pi_2 \bigcup_{i \in I_2} x_i .$$

Making these rewrites and taking the product

$$d_{v_S}^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_{v_T}^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] ,$$

we obtain

$$\begin{aligned} v_S(\pi_1 y) \times v_T(\pi_2 y) &- \sum_{I_2} (-1)^{|I_2|+1} \; v_S(\pi_1 y) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) \\ &- \sum_{I_1} (-1)^{|I_1|+1} \; v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 y) \\ &+ \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|} \; v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) . \end{aligned}$$

But at each index I_2,

$$v_S(\pi_1 y) = v_S(\pi_1 \bigcup_{i \in I_2} x_i)$$

as $\pi_1 y \subseteq^- \pi_1 \bigcup_{i \in I_2} x_i$. Similarly, at each index I_1,

$$v_T(\pi_2 y) = v_T(\pi_2 \bigcup_{i \in I_1} x_i)\,.$$

Hence the product becomes

$$\begin{aligned} v_S(\pi_1 y) \times v_T(\pi_2 y) &- \sum_{I_2} (-1)^{|I_2|+1}\ v_S(\pi_1 \bigcup_{i \in I_2} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) \\ &- \sum_{I_1} (-1)^{|I_1|+1}\ v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_1} x_i) \\ &+ \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|}\ v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i)\,. \end{aligned}$$

To simplify this further, we observe that

$$\{x_i \mid i \in I_1\}\uparrow\ \&\ \{x_i \mid i \in I_2\}\uparrow \iff \{x_i \mid i \in I_1 \cup I_2\}\uparrow\ .$$

The "$\Leftarrow$" direction is clear. We show "$\Rightarrow$." Assume $\{x_i \mid i \in I_1\}\uparrow$ and $\{x_i \mid i \in I_2\}\uparrow$. We obtain $\{\pi_1 x_i \mid i \in I_1\}\uparrow$ and $\{\pi_1 x_i \mid i \in I_2\}\uparrow$ as the projection map π_1 preserves consistency. Hence $\bigcup_{i \in I_1} \pi_1 x_i$ and $\bigcup_{i \in I_2} \pi_1 x_i$ are configurations of S. Furthermore, by assumption,

$$\pi_1 y \subseteq^+ \bigcup_{i \in I_1} \pi_1 x_i \quad \text{and} \quad \pi_1 y \subseteq^- \bigcup_{i \in I_2} \pi_1 x_i\,.$$

As S, a strategy over the race-free game $A^\perp \| B$, is automatically race-free—Lemma 5.6—we obtain

$$\bigcup_{i \in I_1 \cup I_2} \pi_1 x_i \in \mathcal{C}(S)$$

by Proposition 5.5. Similarly, because T is race-free, we obtain

$$\bigcup_{i \in I_1 \cup I_2} \pi_2 x_i \in \mathcal{C}(T)\,.$$

Together these entail

$$\bigcup_{i \in I_1 \cup I_2} x_i \in \mathcal{C}(T) \circledast \mathcal{C}(S)\,,$$

i.e. $\{x_i \mid i \in I_1 \cup I_2\}\uparrow$, as required. Notice too that

$$\pi_1 \bigcup_{i \in I_1} x_i \subseteq^- \pi_1 \bigcup_{i \in I_1 \cup I_2} x_i \quad \text{and} \quad \pi_2 \bigcup_{i \in I_2} x_i \subseteq^- \pi_2 \bigcup_{i \in I_1 \cup I_2} x_i\,,$$

which ensure

$$v_S(\pi_1 \bigcup_{i \in I_1} x_i) = v_S(\pi_1 \bigcup_{i \in I_1 \cup I_2} x_i) \quad \text{and} \quad v_T(\pi_2 \bigcup_{i \in I_2} x_i) = v_T(\pi_2 \bigcup_{i \in I_1 \cup I_2} x_i)\,,$$

so that

$$v(\bigcup_{i\in I_1\cup I_2} x_i) = v_S(\pi_1 \bigcup_{i\in I_1} x_i) \times v_T(\pi_2 \bigcup_{i\in I_2} x_i)\,.$$

We can now further simplify the product to

$$\begin{aligned} v(y) &- \sum_{I_2}(-1)^{|I_2|+1}\ v(\bigcup_{i\in I_2} x_i) \\ &- \sum_{I_1}(-1)^{|I_1|+1}\ v(\bigcup_{i\in I_1} x_i) \\ &+ \sum_{I_1,I_2}(-1)^{|I_1|+|I_2|}\ v(\bigcup_{i\in I_1\cup I_2} x_i)\,. \end{aligned}$$

Noting that any subset I for which $\varnothing \neq I \subseteq \{1,\cdots,n\}$ either lies entirely within $\{1,\cdots,m\}$, entirely within $\{m+1,\cdots,n\}$, or properly intersects both, we have finally reduced the product to

$$v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_I x_i)\,,$$

with indices those I which satisfy $\varnothing \neq I \subseteq \{1,\cdots,n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$, *i.e.* the product reduces to $d_v^{(n)}[y; x_1\cdots, x_n]$ as required. $\square$

Corollary 14.24. *The assignment* $(v_T \circledast v_S)(x) =_{\text{def}} v_S(\pi_1 x) \times v_T(\pi_2 x)$ *to* $x \in \mathcal{C}(T)\circledast\mathcal{C}(S)$ *yields a configuration-valuation on the stable family* $\mathcal{C}(T)\circledast\mathcal{C}(S)$.

Proof. From Proposition14.22 we have requirement (1); by Lemma 14.11(i) we need only verify requirement (2), the 'drop condition,' for p-covering intervals, which we can always permute into the form covered by Lemma 18.4—any p-event of $\mathcal{C}(T) \circledast \mathcal{C}(S)$ has a +ve component on one and only one side. $\square$

Example 14.25. The assumption that games are race-free is needed for Corollary 18.5. Consider the composition of strategies $\sigma : \varnothing \longrightarrow\!\!\!\!\!\rightarrow B$ and $\tau : B \longrightarrow\!\!\!\!\!\rightarrow \varnothing$ where B is the game comprising the two moves $\oplus$ and $\ominus$ in conflict with each other—a game with a race. Suppose σ assigns probability 1 to playing $\oplus$ and τ assigns probability 1 to playing $\ominus$, in the dual game. Then the "drop condition" required for the corollary fails.

We can now complete the definition of the composition of probabilistic strategies:

Lemma 14.26. *Let A, B and C be race-free event structure with polarity. Let $\sigma : S \to A^\perp \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^\perp \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$ be probabilistic strategies. Assigning $(v_T \odot v_S)(x) =_{\text{def}} v_S(\Pi_1 x) \times v_T(\Pi_2 x)$ to $x \in \mathcal{C}(T\odot S)$ yields a configuration-valuation on $T\odot S$ which with $\tau\odot\sigma : T\odot S \to A^\perp \| C$ forms a probabilistic strategy from A to C.*

Proof. We need to show that the assignment $w(x) =_{\text{def}} v_S(\Pi_1 x) \times v_T(\Pi_2 x)$ to $x \in \mathcal{C}(T \odot S)$ is a configuration-valuation on $T \odot S$. We use that $v(z) =_{\text{def}} v_S(\pi_1 z) \times v_T(\pi_2 z)$, for $z \in \mathcal{C}(T) \circledast \mathcal{C}(S)$, is a configuration-valuation on $\mathcal{C}(T) \circledast \mathcal{C}(S)$

Recalling, for $x \in \mathcal{C}(T \odot S)$, that $\bigcup x \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ with $\Pi_1 x = \pi_1 \bigcup x$ and $\Pi_2 x = \pi_2 \bigcup x$, we obtain

$$w(x) =_{\text{def}} v_S(\Pi_1 x) \times v_T(\Pi_2 x) = v_S(\pi_1 \bigcup x) \times v_T(\pi_2 \bigcup x) = v(\bigcup x)\,.$$

Consequently,

$$w(\varnothing) = v(\bigcup \varnothing) = v(\varnothing) = 1\,.$$

The function w inherits requirement (1) to be a configuration-valuation from v because

$x \xrightarrow{p}\!\!\subset y$ with p −ve in $T \odot S$ implies $\bigcup x \xrightarrow{top(p)}\!\!\subset \bigcup y$ with $top(p)$ −ve in $\mathcal{C}(T) \circledast \mathcal{C}(S)$.

To see this observe that $top(p)$ either has the form $(s, *)$ or $(*, t)$. Suppose $top(p) = (*, t)$. Suppose $e \rightarrowtriangle_{\bigcup y} (*, t)$. Then, by Lemma 3.27,

either (i) $e = (s', t')$ and $t' \rightarrowtriangle_T t$ or (ii) $e = (*, t')$ and $t' \rightarrowtriangle_T t$.

But (i) would violate the −-innocence of τ. Hence (ii) and being 'visible' the prime $[e]_{\bigcup y} \in x$ ensuring $e \in \bigcup x$. As all $\rightarrowtriangle_{\bigcup y}$-predecessors of $(*, t)$ are in $\bigcup x$ we obtain $\bigcup x \xrightarrow{(*,t)}\!\!\subset \bigcup y$. The proof in the case where $top(p) = (s, *)$ is similar.

Similarly, w inherits requirement (2) from v, as w.r.t. w,

$$\begin{aligned} d_w^{(n)}[y; x_1, \cdots, x_n] &= w(y) - \sum_I (-1)^{|I|+1} w(\bigcup_{i \in I} x_i) \\ &= v(\bigcup y) - \sum_I (-1)^{|I|+1} v(\bigcup \bigcup_{i \in I} x_i) \\ &= v(\bigcup y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} (\bigcup x_i)) \\ &\geq 0\,, \end{aligned}$$

whenever $y \subseteq^+ x_1, \cdots, x_n$ in $\mathcal{C}(T \odot S)$. (Above, the index I ranges over sets satisfying $\varnothing \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$.) □

A copy-cat strategy is easily turned into a probabilistic strategy, as is any deterministic strategy:

Lemma 14.27. *Let S be a deterministic event structure with polarity. Defining $v_S : \mathcal{C}(S) \to [0, 1]$ to satisfy $v_S(x) = 1$ for all $x \in \mathcal{C}(S)$, we obtain a probabilistic event structure with polarity.*

Proof. Clearly

$$x \subseteq^- y \implies v_S(x) = v_S(y) = 1$$

for all $x, y \in \mathcal{C}(S)$. As S is deterministic,

$$y \subseteq^+ x \;\&\; y \subseteq^+ x' \implies x \cup x' \in \mathcal{C}(S)\,,$$

for all $y, x, x' \in \mathcal{C}(S)$. For the remaining requirement, a simple induction shows that for all $n \geq 1$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = 0$$

whenever $y \subseteq^+ x_1, \cdots, x_n$. The basis, when $n = 1$, is clear as

$$d_v^{(1)}[y; x] = v_S(y) - v_S(x) = 1 - 1 = 0$$

when $y \subseteq^+ x$. For the induction step, assuming $y \subseteq^+ x_1, \cdots, x_n$ with $n > 1$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] = 0 - 0 = 0\,,$$

from the induction hypothesis. □

Definition 14.28. We say a probabilistic event structure with polarity is *deterministic* when its configuration valuation assigns 1 to every finite configuration (provided it is race-free it will necessarily also be deterministic as an event structure with polarity—see the proposition immediately below). We say a probabilistic strategy $\sigma : S \to A$ with configuration-valuation v on $\mathcal{C}(S)$ is *deterministic* when the probabilistic event structure S, v is deterministic.

Proposition 14.29. *If a race-free probabilistic event structure with polarity is deterministic, as defined above, then the event structure with polarity itself is deterministic.*

Proof. Assume S, v, a race-free probabilistic event structure with polarity, is deterministic, as defined above. Suppose $y \xrightarrow{+}\!\!\!\subset x_1$ and $y \xrightarrow{+}\!\!\!\subset x_2$. We must have $x_1 \uparrow x_2$ as otherwise the drop condition would be violated. This with race-freeness implies that the event structure with polarity S itself is deterministic by Lemma 5.1. □

Recall that race-freeness of a game A ensures that $\mathbb{C}_A$ is deterministic. Hence as a direct corollary of Lemma 14.27:

Corollary 14.30. *Let A be a race-free game. The copy-cat strategy from A to A comprising $\gamma_A : \mathbb{C}_A \to A^\perp \| A$ with configuration-valuation $v_{\mathbb{C}_A} : \mathcal{C}(\mathbb{C}_A) \to [0,1]$ satisfying $v_{\mathbb{C}_A}(x) = 1$, for all $x \in \mathcal{C}(\mathbb{C}_A)$, forms a probabilistic strategy.*

Example 14.31. Let A be the empty game $\varnothing$, B be the game consisting of two concurrent +ve events b_1 and b_2, and C the game with a single +ve event c. We illustrate the composition of two probabilistic strategies $\sigma : \varnothing \rightarrow\!\!\!\!\!\!+\; B$ and $\tau : B \rightarrow\!\!\!\!\!\!+\; C$.

$$\begin{tikzcd}
S \arrow[d, "\sigma"'] & \oplus \arrow[d, dotted] & \oplus \arrow[d, dotted] & T \arrow[d, "\tau"'] & \ominus \arrow[d, dotted] & \ominus \arrow[r] \arrow[d, dotted] & \oplus \arrow[d, dotted] \\
B & b_1 & b_2 & B^\perp \| C & b_1 & b_2 & c
\end{tikzcd}$$

The strategy σ plays b_1 with probability 2/3 and b_2 with probability 1/3 (and plays both with probability 0). The strategy τ does nothing if just b_1 is played and plays the single +ve event c of C with probabilty 1/2 if b_2 is played. Their composition yields the strategy $\tau \odot \sigma : \varnothing \twoheadrightarrow C$ which plays c with probability $1/6$, so has a $5/6$ chance of doing nothing.

The example illustrates how through probability we can track the presence of terminal configurations within a set of results despite their not being $\subseteq$-maximal. The empty configuration is such a terminal configuration; it could be the final result of the composition as could the configuration $\{c\}$. Such terminal but incomplete results can appear in a composition of strategies through the strategies being partial, in that one or both strategies do not respond in all cases—the example above. Such partial strategies can appear as the composition of two strategies through the occurrence of deadlocks because the two strategies impose incompatible causal dependencies on moves in game at which they interact. $\square$

Remark on schedulers Often in compositional treatments of probabilistic processes one sees a use of "schedulers" to "resolve the nondeterminism" due to openness to the environment [**?**]. Here the use of schedulers is replaced by that of counterstrategy to resolve the nondeterminism. The counterstrategy may be deterministic (so straightforwardly a deterministic probabilistic strategy), in which case it resolves the nondeterminism by selecting at most one play for Opponent.

14.3 2-cells, a bicategory

We have thus extended composition of strategies to composition of probabilistic strategies. This doesn't yet yield a bicategory of probabilistic strategies. The extra structure of configuration-valuations in strategies has to be respected in our choice of 2-cell. The investigation of a suitable notion of 2-cell is the subject of the next section.

We first look for an analogue of the well-known result allowing a probability distribution to be pushed forward across an continuous (or measurable) function. This is not immediate as the configuration-valuations associated with strategies take account of Opponent moves so do not correspond to traditional probability distributions.

Example 14.32. It seems impossible to push forward configuration valuations across arbitrary 2-cells. For example, consider the game A comprising two conflicting Opponent move and one Player move:

$$\begin{array}{c} \oplus \\ \\ \ominus_1 \sim\!\sim\!\sim \ominus_2\,. \end{array}$$

Let one probabilistic strategy comprise

$$\begin{tikzcd} \oplus_1 & \oplus_2 \\ \ominus_1 \arrow[u, rightarrowtail] \arrow[r, phantom, "\sim\!\sim\!\sim"] & \ominus_2 \arrow[u, rightarrowtail] \end{tikzcd}$$

with obvious map σ, where the left Player move occurs with probability p_1 and the Player move on the right with probability p_2 according to a configuration-valuation v, *i.e.* $v(\{\ominus_1, \oplus_1\}) = p_1$ and $v(\{\ominus_2, \oplus_2\}) = p_2$. Take another strategy to be the identity map A to A. It seems compelling to make the push forward of v across σ assign p_1 to the configuration $\{\ominus_1, \oplus\}$ and p_2 to the configuration $\{\ominus_2, \oplus\}$. What value should the push forward of v assign to the configuration $\{\oplus\}$? Because configuration-valuations are invariant under Opponent moves, it has to be simultaneously p_1 and p_2 —impossible if $p_1 \neq p_2$.

We shall now show the following theorem showing how to push forward configuration valuations across maps which are both rigid and receptive; in particular it will allow us to push forward a configuration valuation across a rigid map between strategies.[2]

Theorem14.35. Let $f : S \to S'$ be a receptive and rigid map between event structures with polarity. Let v be a configuration-valuation on S. Then, taking

$$v'(y) =_{\rm def} \sum_{x: fx = y} v(x)$$

for $y \in \mathcal{C}(S')$, defines a configuration-valuation, written fv, on S'. (An empty sum gives 0 as usual.)

The proof of the theorem proceeds in the following steps, needed to cope with the fact sums can be infinite while also involving negative terms.

Lemma 14.33. *Let $f : S \to S'$ be a receptive and rigid map between event structures with polarity. Let v be a configuration-valuation on S. Then, taking*

$$v'(y) =_{\rm def} \sum_{x: fx = y} v(x)$$

we have $v'(y) \in [0, 1]$, for $y \in \mathcal{C}(S')$. Moreover, $v'(\varnothing) = 1$ and $y \subseteq^- y'$ in $\mathcal{C}(S')$ implies $v'(y) = v'(y')$.

Proof. We check that for $y \in \mathcal{C}(S')$ the assignment $v'(y)$ is in $[0, 1]$. Choose a covering chain

$$\varnothing \xrightarrow{t_1}\!\!\!\!\subset y_1 \xrightarrow{t_2}\!\!\!\!\subset \cdots \xrightarrow{t_n}\!\!\!\!\subset y_n = y$$

up to y. As f is rigid for each $x \in \mathcal{C}(S)$ s.t. $fx = y$ there is a corresponding covering chain

$$\varnothing \xrightarrow{s_1}\!\!\!\!\subset x_1 \xrightarrow{s_2}\!\!\!\!\subset \cdots \xrightarrow{s_n}\!\!\!\!\subset x_n = x$$

with $f(s_i) = t_i$ for $0 < i \leq n$. Consider the tree with sub-branches all initial sub-chains of covering chains up to each x s.t. $fx = y$; the tree has the empty covering chain as its root and configurations x, where $fx = y$, as its maximal nodes. Because f is receptive the tree only branches at its +ve coverings,

[2] An alternative, more general proof, for edc strategies, is given later—see Theorem 19.6.

associated with different, possibly infinitely many, s_i which map to a +ve event t_i. The corresponding configurations x_i are pairwise incompatible. Although such configurations x_i may form an infinite set, by the drop condition for v, the values of any finite subset will have sum less than or equal to $v(x_{i-1})$, a property which must therefore also hold for the sum of values of all the x_i. The value remains constant across any −ve event. Hence, working up the tree from the root we obtain that $\sum_{x:fx=y} v(x) \le 1$.

Clearly, $v'(\varnothing) = v(\varnothing) = 1$. Suppose $y \subseteq^- y'$ in $\mathcal{C}(S')$. From the properties of f, x s.t. $fx = y$ determines a unique x' s.t. $x \subseteq^- x'$ and $fx' = y'$, and *vice versa*; in this correspondence $v(x) = v(x')$, as v is a configuration-valuation. Consequently, the sums yielding $v'(y)$ and $v'(y')$ have the same component values and are the same. □

For v' to be a configuration valuation it remains to verify that v' satisfies the +ve drop condition. We first show this for a special case:

Lemma 14.34. *Let $f : S \to S'$ be a receptive and rigid map between event structures with polarity. Assume that S has only finitely many +ve events. Then, v' as defined above in Lemma 14.33 is a configuration valuation.*

Proof. Suppose $y \overset{+}{\longrightarrow\!\!\subset} y_1, \cdots, y_n$. We claim that

$$d_{v'}^{(n)}[y; y_1, \cdots, y_n] = \sum_{x:fx=y} d_v^{(n)}[x; X(x)]$$

so is non-negative, where

$$X(x) =_{\text{def}} \{x' \mid x {-\!\!\subset} x' \ \&\ fx' \in \{y_1, \cdots, y_n\}\}\,.$$

The notation $d_v^{(n)}[x; X(x)]$ is justifiable as the drop function is invariant under permutation and repetition of arguments. Recall

$$d_{v'}^{(n)}[y; y_1, \cdots, y_n] =_{\text{def}} v'(y) - \sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v'(\bigvee_{i\in I} y_i)\,.$$

The claim follows because by the rigidity of f any non-zero contribution

$$(-1)^{|I|+1} v'(\bigcup_{i\in I} y_i)$$

is the sum of contributions

$$(-1)^{|I|+1} v(\bigcup_{i\in I} x_i)\,,$$

a summand of $d_v^{(n)}[x; X(x)]$, over x s.t. there are $x_i \in X(x)$ with $fx_i = y_i$ for all $i \in I$. □

We can now complete the proof of the theorem.

Theorem 14.35. *Let $f : S \to S'$ be a receptive and rigid map between event structures with polarity. Let v be a configuration-valuation on S. Then, taking*

$$v'(y) =_{\text{def}} \sum_{x : fx = y} v(x)$$

for $y \in \mathcal{C}(S')$, defines a configuration-valuation, written fv, on S'.

Proof. We use a slight variation on the $\trianglelefteq$ approximation order between event structures from [4, 2]. We write $S_0 \trianglelefteq S_1$ to mean there is a *receptive* rigid inclusion map between event structures with polarity from S_0 to S_1. Together all $S_0 \trianglelefteq S$ where S_0 has finitely many $+$-events form a directed subset of approximations to S; their $\trianglelefteq$-least upper bound is S got as their union. Such S_0 are associated with receptive rigid maps $f_0 : S_0 \to S'$ got as restrictions of f,

$$\begin{tikzcd} S \arrow[r, "f"] & S' \\ S_0 \arrow[u, hook] \arrow[ur, "f_0"'] & \end{tikzcd}$$

and configuration-valuations v_{S_0} got as restrictions v.

Let $y \overset{+}{\longrightarrow\!\!\subset} y_1, \cdots, y_n$ in $\mathcal{C}(S')$. We claim that

$$d_v[y; y_1, \cdots, y_n] = \lim_{S_0 \trianglelefteq S} d^{S_0}[y; y_1, \cdots, y_n] \qquad (\dagger)$$

i.e., that $d_v[y; y_1, \cdots, y_n]$ is the limit of $d^{S_0}[y; y_1, \cdots, y_n]$, the drop functions got by pushing forward v_{S_0} along f_0 to a configuration-valuation for S'—justified by Lemma 14.34.

Let $\epsilon > 0$. For each $I \subseteq \{1, \cdots, n\}$ there is large enough $S_I \trianglelefteq S$ s.t. for all $\trianglelefteq$-larger S_0,

$$0 \le v(\bigvee_{i \in I} y_i) - v_{S_0}(\bigvee_{i \in I} y_i) \le \epsilon / 2^n .$$

(When $I = \emptyset$ take $\bigvee_{i \in I} y_i = y$.) Taking S_1 to be $\trianglelefteq$-larger than all S_I where $I \subseteq \{1, \cdots, n\}$, we get for all S_2 with $S_1 \trianglelefteq S_2$ that

$$|d_v[y; y_1, \cdots, y_n] - d^{S_2}[y; y_1, \cdots, y_n]| < 2^n \epsilon / 2^n = \epsilon .$$

As ϵ was arbitrary we deduce $(\dagger)$, ensuring $d_v[y; y_1, \cdots, y_n] \ge 0$, as required. □

Consequently, we can push forward a configuration-valuation across a rigid 2-cell between strategies—recall that 2-cells are automatically receptive. Given this it is sensible to adopt the following definition of 2-cell between probabilistic strategies. A 2-cell from a probabilistic strategy $v, \sigma : S \to A^\perp \| B$ to a probabilistic strategy $v', \sigma' : S' \to A^\perp \| B$ is a rigid map $f : S \to S'$ for which both $\sigma = \sigma' f$ and the push-forward $fv \le v'$, *i.e.* for any finite configuration of S' the value $(fv)(x) \le v'(fx)$.

Such 2-cells include receptive rigid embeddings f which preserve the value assigned by configuration-valuations, so $(fv)(x) = v'(fx)$ when $x \in \mathcal{C}(S)$; notice that the push-forward fv will assign value 0 to any configuration not in the image of f, so not impose any additional constraint on the values v' takes outside the image of f. Rigid embeddings, first introduced by Kahn and Plotkin [?] provide a method for defining strategies recursively. One way to characterize those maps $f : S \to S'$ of event structures which are rigid embeddings is as injective functions on events for which the inverse relation f^{op} is a (partial) map of event structures $f^{\mathrm{op}} : S' \to S$.

In turn, 2-cells based on rigid embeddings include as special case that in which the function f is an inclusion. Receptive rigid embeddings which are inclusions give a (slight variant on a) well-known approximation order $\trianglelefteq$ on event structures. The order $\trianglelefteq$ forms a 'large cpo' and is useful when defining event structures recursively [4, 2]. With some care in choosing the precise construction of composition it provides an enrichment of probabilistic strategies and an elementary technique for defining probabilistic strategies recursively. Spelt out, when $v, \sigma : S \to A^\perp \| B$ and $v', \sigma' : S' \to A^\perp \| B$ are probabilistic strategies, we write

$$(v, \sigma) \trianglelefteq (v', \sigma')$$

iff $S \trianglelefteq S'$, the associate inclusion map $i : S \hookrightarrow S'$ makes $\sigma = \sigma' i$ and $v(x) = v'(x)$ for all $x \in \mathcal{C}(S)$. There can be many different, though isomorphic, $\trianglelefteq$-minimal probabilistic strategies, differing only in their choices of initial $-$-events; to be receptive they must start with copies of initial $-$-events of the game. Any chain

$$(v_0, \sigma_0) \trianglelefteq (v_1, \sigma_1) \trianglelefteq \cdots \trianglelefteq (v_n, \sigma_n) \trianglelefteq \cdots$$

has a least upper bound got by taking the union of the event structures.

We now show that 2-cells between probabilistic strategies compose horizontally.

First, recall from Section 4.3.2, the concrete way to define composition of strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ as $\tau \odot \sigma : T \odot S \to A^\perp \| C$ where

$$T \odot S = (S \times T \upharpoonright R) \downarrow V$$

for suitable restricting set R and projecting set V; from Section 4.3.3 that $T \circledast S =_{\mathrm{def}} (S \times T \upharpoonright R)$ can be characterised as a pullback of total maps. We observed in Section 4.5 that composition sends two rigid cells $f : \sigma \Rightarrow \sigma'$ and $g : \tau \Rightarrow \tau'$ to a rigid 2-cell $g \odot f : \tau \odot \sigma \to \tau' \odot \sigma'$.

For probabilistic strategies $v_S, \sigma : S \to A^\perp \| B$ and $v_T, \tau : T \to B^\perp \| C$ we write $v_T \odot v_S$, respectively, $v_T \circledast v_S$ for the configuration-valuations on $T \odot S$ and $T \circledast S$ in the composition $(v_T, \tau) \odot (v_S, \sigma)$ and the composition without hiding $(v_T, \tau) \circledast (v_S, \sigma)$. Recalling how $v_T \circledast v_S$ is defined, we imediately obtain

$$(v_T \circledast v_S((x) = v_T(\Pi_2 x) \times v_S(\Pi_1 x)\,,$$

for $x \in \mathcal{C}(T \circledast S)$, and from how $v_T \odot v_S$ is defined, that

$$(v_T \odot v_S)(y) = (v_T \circledast v_S)([y]_{T \circledast S)}\,,$$

for $y \in \mathcal{C}(T \odot S)$.

To show that 2-cells compose functorially we must first attend to how configuration-valuations are pushed forward by composition on 2-cells.

Lemma 14.36. *Let $f : \sigma \to \sigma'$ be a rigid 2-cell between strategies $\sigma : S \to A^\perp \| B$ and $\sigma' : S' \to A^\perp \| B$. Let $g : \tau \to \tau'$ be a rigid 2-cell between strategies $\tau : T \to B^\perp \| C$ and $\tau' : T' \to B^\perp \| C$. Let v_S be a configuration-valuation for S and v_T a configuration-valuation for T. Then,*

$$(g \odot f)(v_T \odot v_S) = (g v_T) \odot (f v_S)$$

and

$$(g \circledast f)(v_T \circledast v_S) = (g v_T) \circledast (f v_S)\,.$$

Proof. We first consider composition without hiding and lay out the relevant maps:

$$\begin{tikzcd} S \arrow[d, "f"'] & T \circledast S \arrow[l, "\Pi_1"'] \arrow[r, "\Pi_2"] \arrow[d, "g \circledast f"] & T \arrow[d, "g"] \\ S' & T' \circledast S' \arrow[l, "\Pi'_1"] \arrow[r, "\Pi'_2"'] & T' \end{tikzcd}$$

The push-forward configuration-valuation $(g \circledast f)(v_T \circledast v_S)$ at $x' \in \mathcal{C}(T' \circledast S')$ has value

$$((g \circledast f)(v_T \circledast v_S))(x') = \sum_{x : g \circledast f x = x'} (v_T \circledast v_S)(x)\,.$$

Because f and g are rigid, configurations $x \in \mathcal{C}(T \circledast S)$ such that $(g \circledast f)x = x'$ are in 1-1 correspondence with pairs $x_1 \in \mathcal{C}(S)$, $x_2 \in \mathcal{C}(T)$ such that $f x_1 = \Pi'_1 x'$ and $g x_2 = \Pi'_2 x'$; the correspondence takes x to the pair $\Pi_1 x$, $\Pi_2 x$. (Clearly, if $(g \circledast f)x = x'$ then $x_1 = \Pi_1 x$ satisfies $f x_1 = \Pi'_1 x'$ and $x_2 = \Pi_2 x$ satisfies $g x_2 = \Pi'_2 x'$; the converse holds because by rigidity the pairing x' determines between $\Pi'_1 x'$ and $\Pi'_2 x'$ copies to a pairing between x_1 and x_2, yielding a configuration x.) Consequently,

$$\begin{aligned} ((g \circledast f)(v_T \circledast v_S))(x') &= \sum_{x : (g \circledast f)x = x'} (v_T \circledast v_S)(x) \\ &= \sum_{x : (g \circledast f)x = x'} v_S(\Pi_1 x) \times v_T(\Pi_2 x) \\ &= \sum_{x_1 : f x_1 = \Pi'_1 x'} v_S(x_1) \times \sum_{x_2 : g x_2 = \Pi'_2 x'} v_T(x_2) \\ &= (f v_S)(\Pi'_1 x') \times (g v_T)(\Pi'_2 x') \\ &= ((g v_T) \circledast (f v_S))(x')\,, \end{aligned}$$

showing $(g \circledast f)(v_T \circledast v_s) = (g v_T) \circledast (f v_S)$, as required.

The configuration-valuation $v_T \odot v_S$ of $T \odot S$ is given by

$$(v_T \odot v_S)(y) = (v_T \circledast v_S)([y]_{T \circledast S})$$

for all $y \in \mathcal{C}(T \odot S)$. The map $g \odot f$ acts on $y \in \mathcal{C}(T \odot S)$ so

$$(g \odot f)y = (g \circledast f)[y]_{T \circledast S}\,.$$

(For readability, in the following we shall suppress the subscripts specifying the event structure within which the down-closure is taking place.)

On $y' \in \mathcal{C}(T' \odot S')$ the push-forward of $(v_T \odot v_S)$ yields

$$((g \odot f)(v_T \odot v_S))(y') = \sum_{y:(g\odot f)y=y'} (v_T \odot v_S)(y)\,.$$

However, $y \in \mathcal{C}(T \odot S)$ such that $(g \odot f)y = y'$ are in 1-1 correspondence with $x \in \mathcal{C}(T \circledast S)$ such that $(g \circledast f)x = [y']$; the correspondence takes $y \in \mathcal{C}(T \odot S)$ to $[y] \in\in \mathcal{C}(T \circledast S)$. (This is because $g \circledast f$ is rigid and $g \odot f$ is the restriction of $g \circledast f$ to 'visible' events.) Hence

$$\begin{aligned} ((g \odot f)(v_T \odot v_S))(y') &= \sum_{y:(g\odot f)y=y'} (v_T \odot v_S)(y) \\ &= \sum_{x:(g\circledast f)x=[y']} (v_T \circledast v_S)(x) \\ &= ((g \circledast f)(v_T \circledast v_S))([y']) \\ &= ((g v_T) \circledast (f v_S))([y']) \\ &= ((g v_T) \odot (f v_S))(y')\,, \end{aligned}$$

as required to show $(g \odot f)(v_T \odot v_S) = (g v_T) \odot (f v_S)$. □

Lemma 14.37. *Composition of probabilistic strategies is functorial w.r.t. 2-cells, and functorial w.r.t. those 2-cells which are rigid embeddings.*

Proof. In the absence of probability we have functoriality. We need to check that the extra constraints on 2-cells between probabilistic strategies are respected by composition. Let $f : (v_S, \sigma) \Rightarrow (v_{S'}, \sigma')$ and $g : (v_T, \tau) \Rightarrow (v_{T'}, \tau')$ be 2-cells between probabilistic strategies. We adopt the convention that for instance σ has the form $\sigma : S \to A^\perp \| B$ with a configuration-valuation v_S on S. We need to check that

$$((g \odot f)(v_T \odot v_S))(y') \le (v_{T'} \odot v_{S'})(y')\,,$$

for all $y' \in \mathcal{C}(T' \odot S')$.

We first consider composition without hiding where the relevant map is $g \circledast f$, making the following diagram commute:

$$\begin{tikzcd} S \arrow[d, "f"'] & T \circledast S \arrow[l, "\Pi_1"'] \arrow[r, "\Pi_2"] \arrow[d, "g \circledast f"] & T \arrow[d, "g"] \\ S' & T' \circledast S' \arrow[l, "\Pi_1'"] \arrow[r, "\Pi_2'"'] & T' \end{tikzcd}$$

We require that

$$((g \circledast f)(v_T \circledast v_S))(x') \le (v_{T'} \circledast v_{S'})(x')$$

for all configurations x' of $T' \circledast S'$. But, by Lemma 14.36, letting $x' \in \mathcal{C}(T' \circledast S')$, we see

$$
\begin{aligned}
((g \circledast f)(v_T \circledast v_S))(x') &= ((gv_T) \circledast (fv_S))(x') \\
&= (gv_T)(\Pi_2 x') \times (fv_S)(\Pi_1 x') \\
&\leq v_{T'}(\Pi_2 x') \times v_{S'}(\Pi_1 x') \\
&= (v_{T'} \circledast v_{S'})(x')\,.
\end{aligned}
$$

On $y' \in \mathcal{C}(T' \odot S')$ we require

$$((g \odot f)(v_T \odot v_S))(y') \leq (v_{T'} \odot v_{S'})(y')\,.$$

However,

$$
\begin{aligned}
((g \odot f)(v_T \odot v_S))(y') &= ((gv_T) \odot (fv_S))(y') \\
&= ((gv_T) \circledast (fv_S))([y']) \\
&= ((g \circledast f)(v_T \circledast v_S))([y']) \\
&\leq (v_{T'} \circledast v_{S'})([y']) \\
&= (v_{T'} \odot v_{S'})(y')\,.
\end{aligned}
$$

It has been long established that operations of traditional process algebras preserve rigid embeddings. From [4] we obtain that the operation $T \circledast S$ is functorial w.r.t. rigid embeddings. (In fact, in [4] the stronger result is shown that the operations preserve, and are continuous, w.r.t. $\trianglelefteq$, rigid embedding which are inclusions.) Projection is not considered there. However, in general if $f : S \to S'$ is a rigid embedding of event structures and subsets $V \subseteq E$, $V' \subseteq E'$ satisfy

$$e \in V \iff f(e) \in V'\,, \text{ for all } e \in E\,,$$

then $f \restriction V : E \downarrow V \to E' \downarrow V'$ is a rigid embedding. For this reason $T \odot S$ abtained from $T \circledast S$ by projection is also functorial w.r.t. rigid embeddings. $\square$

Combining the results of this section:

Theorem 14.38. *Race-free games with probabilistic strategies with composition and copy-cat defined as in Lemma 14.26 and Corollary 14.30 inherit the structure of a a bicategory from that of games with strategies. 2-cells between probabilistic strategies are now restricted to rigid maps satisfying the conditions explained above. The bicategory restricts to one in which the cells are rigid embeddings.*

Important remark There is a more general definition of 2-cell for probabilistic strategies pointed out by Hugo Paquet, a definition which has the advantage of being strictly more general in that it does not require the underlying 2-cell on strategies be rigid. According to this definition, a 2-cell $f : \sigma, v \Rightarrow \sigma', v'$ between probabilistic strategies $\sigma : S \to A$ with configuration valuation v and $\sigma : S \to A$ with configuration valuation v' is a two cell $f : \sigma \Rightarrow \sigma'$ of strategies for which

$$v(x) \leq v'(fx)$$

for all $x \in \mathcal{C}(S)$. This definition is strictly more general than the rigid 2-cell used for most of this section; a rigid 2-cells is one of this more general kind by the following argument. Suppose $f : \sigma, v \Rightarrow \sigma', v'$ is a rigid 2-cell between probabilistic strategies, *i.e.* such that the push forward fv is less than or equal to v', pointwise, *i.e.*

$$(fv)(y) =_{\mathrm{def}} \sum_{x' : fx' = y} \leq v'(y)$$

on $y \leq \mathcal{C}(S)'$. Then certainly, for $x \in \mathcal{C}(S)$,

$$v(x) \leq \sum_{x' : fx' = fx} = (fv)(fx) \leq v'(fx)\,,$$

as required of a 2-cell according to the more general definition.

14.3.1 A category of probabilistic rigid-image strategies

We extend the results of Section 4.6 on rigid-image strategies to probabilistic rigid-image strategies. We show here that the order-enriched category $\mathbf{Strat}_0$ of rigid-image strategies supports probability to give us an order-enriched category of probabilistic rigid-image strategies. A probabilistic rigid-image strategy over a game A comprises a rigid-image strategy $\sigma : S \to A$ together with a configuration-evaluation v for S. Given probabilistic rigid image strategies $v_S, \sigma : S \to A^\perp \| B$ and $v_T, \tau : T \to B^\perp \| C$ their composition comprises $(\tau \odot \sigma)_0 : (T \odot S)_0 \to A^\perp \| C$, the rigid image of $\tau \odot \sigma$, with configuration-valuation $(v_T \odot v_S)_0$ the push-forward along the map $T \odot S \to (T \odot S)_0$ to the rigid image of the configuration valuation $v_T \odot v_S$.

Taking rigid images yields a functor from the bicategory of probabilistic strategies to the order-enriched category of probabilistic rigid-image strategies. A strategy $\sigma : S \to A$ has a *rigid image* comprising

$$\begin{tikzcd} S \arrow[r, two heads, "f_0"] \arrow[dr, "\sigma"'] & S_0 \arrow[d, "\sigma_0"] \\ & A \end{tikzcd}$$

where f_0 is rigid epi and σ_0 is a strategy with universal property:

$$\begin{tikzcd} S \arrow[r, two heads, "f"] \arrow[rr, bend left, "f_0"] \arrow[dr, "\sigma"'] & S' \arrow[r, dashed] \arrow[d, "\sigma'"] & S_0 \arrow[dl, "\sigma_0"] \\ & A & \end{tikzcd}$$

A probabilistic strategy $\sigma : S \to A$ with configuration-valuation v of S has rigid image the probabilistic strategy $\sigma_0 : S_0 \to A$ with configuration-valuation the push-forward $v_0 =_{\rm def} f_0 v$. As could be hoped, the determination of the probabilistic rigid-image strategy v_0, σ_0 from a probabilistic strategy v, σ is functorial.

From Section 4.6, we know that the operation of forming the rigid-image of a strategy is functorial w.r.t. rigid 2-cells. The key extra fact needed for this to be functorial for the extension to probabilistic strategies is that the configuration-valuation assigned to the rigid-image of $\tau \odot \sigma$ equals that assigned in the composition of rigid-image strategies $(\tau_0 \odot \sigma_0)_0$, which we might write as:

$$v_{(\tau\odot\sigma)_0} = v_{(\tau_0\odot\sigma_0)_0}\,.$$

We also have

$$v_{(\tau\circledast\sigma)_0} = v_{(\tau_0\circledast\sigma_0)_0}\,.$$

We show the former in detail. The argument for the latter is analogous.

Suppose $v_S, \sigma : S \to A\perp\|B$ be a probabilistic strategy. Let $f : \sigma \Rightarrow \sigma_0$ be the rigid 2-cell connecting the strategy σ with its rigid image. Let $(v_S)_0 =_{\rm def} f v_S$ be its push forward across f, giving us the configuration-valuation associated with the rigid-image strategy. Suppose $v_T, \tau : T \to B\perp\|C$. Let $g : \tau \Rightarrow \tau_0$ be the rigid 2-cell connecting it with its rigid image; again write $(v_T)_0$ for the push-forward to a configuration-valuation of its rigid image. Write $h : \tau_0\odot\sigma_0 \Rightarrow (\tau_0\odot\sigma_0)_0$ for the 2-cell from : $\tau_0\odot\sigma_0$ to its rigid image. The push-forward of the configuration-valuation of the composition $\tau\odot\sigma$ to its rigid image is

$$\begin{aligned}(v_T\odot v_S)_0 &= (h(g\odot f))(v_T\odot v_S)\\ &= h((g\odot f)(v_T\odot v_S))\\ &= h(g v_T \odot f v_S)\\ &= h((v_T)_0\odot(v_S)_0)\\ &= ((v_T)_0\odot(v_S)_0)_0\,,\end{aligned}$$

the composition of the push-forwards in the category of probabilistic rigid-image strategies. We conclude that the action taking a probabilistic strategy to its probabilistic rigid-image strategy is functorial.

Is anything lost in moving to probabilistic rigid-image strategies? A negative answer is provided by the next result if we are considering probabilistic strategies as characterised by the probabilistic experiments we can perform on them. By virtue of the following proposition, a probabilistic strategy and its probabilistic rigid-image will always induce the same probability distribution on the game whenever they are composed with a probabilistic counterstrategy.

Proposition 14.39. *Let $f : (\sigma, v) \Rightarrow (\sigma', v')$ be a 2-cell between probabilistic strategies $v, \sigma : S \to A$ and $v', \sigma' : S' \to A$ for which the push-forward $fv = v'$. Let $v_T, \tau : T \to A^\perp$ be a probabilistic counterstrategy. Then*

$$\begin{tikzcd} T\circledast S \arrow[r, "\tau\circledast f"] \arrow[dr, "\tau\circledast\sigma"'] & T\circledast S' \arrow[d, "\tau\circledast\sigma'"] \\ & A \end{tikzcd}$$

commutes and the push-forward $(\tau \circledast f)(v_T \circledast v) = v_T \circledast v'$. *Moreover,* $T \circledast S$ *with* $v_T \circledast v$ *and* $T \circledast S'$ *with* $v_T \circledast v'$ *are probabilistic event structures determining continuous valuations* w *and* w' *respectively. The push-forwards of* w *and* w' *across the maps* $\tau \circledast \sigma$ *and* $\tau \circledast \sigma'$ *respectively to continuous valuations on the open sets of* $\mathcal{C}^\infty(A)$ *are the same.*

Proof. The commuting diagram simply expresses that $\tau \circledast f : \tau \circledast \sigma \Rightarrow \tau \circledast \sigma'$ is a 2-cell of partial strategies. We have

$$(\tau \circledast f)(v_T \circledast v) = v_T \circledast (fv) = v_T \circledast v' .$$

None of the events of $T \circledast S$ and $T \circledast S'$ are those of Opponent (all events are neutral) ensuring they form probabilistic event structures with configuration-valuations $v_T \circledast v$ and $v_T \circledast v'$, respectively. As such they determine continuous valuations w and w' on open sets of configurations $\mathcal{C}^\infty(T \circledast S)$ and $\mathcal{C}^\infty(T \circledast S')$, respectively. In this situation the push-forward across the rigid 2-cell $\tau \circledast f$ agrees with standard push-forward of probability theory: for U an open set of $\mathcal{C}^\infty(T \circledast S')$,

$$w'(U) = w((\tau \circledast f)^{-1}U) .$$

The continuous valuations w and w' push-forward (in the sense of probability theory) across the obviously-continuous maps of event structures $\tau \circledast \sigma$ and $\tau \circledast \sigma'$. For instance, the push-forward of w is the continuous valuation assigning

$$w((\tau \circledast \sigma)^{-1}V)$$

to an open set $V \subseteq \mathcal{C}^\infty(A)$. The commuting diagram ensures that both push-forwards to open sets of $\mathcal{C}^\infty(A)$ are the same. □

14.4 Probabilistic processes—an early version

As an indication of the expressivity of probabilistic strategies we sketch how they straightforwardly include a simple language of probabilistic processes, reminiscent of a higher-order CCS. For this section only, write $\sigma : A$ to mean σ is a probabilistic strategy in game A. Probabilistic strategies are closed under the following operations.

Composition $\sigma \odot \tau : A \| C$, if $\sigma : A \| B$ and $\tau : B^\perp \| C$. Hiding is automatic in a synchronized composition directly based on the composition of strategies.

Simple parallel composition $\sigma \| \tau : A \| B$, if $\sigma : A$ and $\tau : B$. Note that simple parallel composition can be regarded as a special case of synchronized composition: via the identification of $\sigma \| \tau$ with $\tau \odot \sigma$, taking $\sigma : A^\perp \longrightarrow\!\!\!\!\!\rightarrow \varnothing$ and $\tau : \varnothing \longrightarrow\!\!\!\!\!\rightarrow B$, the operation $\sigma \| \tau$ yields a probabilistic strategy. Supposing $\sigma : S \to A$ and $\tau : T \to B$ and S and T have configuration valuations v_S and v_T, respectively, then the configuration valuation v for $S \| T$ satisfies $v(x) = v_S(x_1) \times v_T(x_2)$, for

$x \in \mathcal{C}(S\|T)$.

Pullback if $\sigma_1 : A$ and $\sigma_2 : A$ we can form their pullback:

$$\begin{tikzcd} & S_1 \wedge S_2 \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, dotted, "\sigma_1 \wedge \sigma_2"] & \\ S_1 \arrow[dr, "\sigma_1"'] & & S_2 \arrow[dl, "\sigma_2"] \\ & \Gamma^\perp \| \Delta\,. & \end{tikzcd}$$

If σ_1 and σ_2 are associated with configuration-valuations v_1 and v_2 respectively then we tentatively take the configuration-valuation of the pullback to be $v(x) = v_1(\Pi_1 x) \times v_2(\Pi_2 x)$ for $x \in \mathcal{C}(S_1 \wedge S_2)$.

To check that v is indeed a configuration-valuation we embed configurations of $S_1 \wedge S_2$ in those of $S_1\|S_2$ as described in the next lemma, so inheriting the conditions required of v from those of the configuration-valuation of $\sigma_1\|\sigma_2$.

Lemma 14.40. *Define*

$$\psi : \mathcal{C}(S_1 \wedge S_2) \to \mathcal{C}(S_1\|S_2)$$

by $\psi(x) = \Pi_1 x\|\Pi_2 x$ *for* $x \in \mathcal{C}(S_1 \wedge S_2)$. *Then,*

(i) ψ *is injective,*

(ii) ψ *preserves unions, and*

(iii) ψ *reflects compatibility, and in particular +-compatibility: if* $x \subseteq^+ y$ *and* $x \subseteq^+ z$ *in* $\mathcal{C}(S_1 \wedge S_2)$ *and* $\psi(y) \cup \psi(z) \in \mathcal{C}(S_1\|S_2)$, *then* $y \cup z \in \mathcal{C}(S_1 \wedge S_2)$.

Proof. Consider the pullback $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$, π_1, π_2 in stable families of σ_1 and σ_2, regarded as maps between families of configurations. Configurations $\mathcal{C}(S_1 \wedge S_2)$ are order isomorphic, under inclusion, to configurations $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$. See the end of Section 3.3.4 for the detailed construction of pullbacks of stable families. It is thus sufficient to show that $\phi : \mathcal{C}(S_1) \wedge \mathcal{C}(S_2) \to \mathcal{C}(S_1\|S_2)$, where $\phi(x) = \pi_1 x\|\pi_2 x$ for $x \in \mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$, satisfies conditions (i), (ii) and (iii) in place of ψ. (i) Injectivity follows because configurations in the pullback of stable families are determined by their projections; the nature of events of the pullback fixes their synchronisations. (ii) is obvious. (iii) To show ϕ reflects compatibility, assume $x \subseteq y$ and $x \subseteq z$ in $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$ and $\phi(y) \cup \phi(z) \in \mathcal{C}(S_1\|S_2)$. Inspecting the construction of the pullback $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$ it is now easy to check that $y \cup z$ satisfies the conditions needed to be in $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$, as required. □

Corollary 14.41. *Taking* $v(x) = v_1(\Pi_1 x) \times v_2(\Pi_2 x)$ *for* $x \in \mathcal{C}(S_1 \wedge S_2)$ *defines a configuration-valuation of* $S_1 \wedge S_2$.

Proof. The assignment $x \mapsto v_1(x_1) \times v_2(x_2)$, for $x \in \mathcal{C}(S_1\|S_2)$ determines a configuration-valuation of $S_1\|S_2$. The one non-obvious condition required of v to be a configuration-valuation is the $+$-drop condition. This follows directly from the $+$-drop condition holding in $\mathcal{C}(S_1\|S_2)$ because ψ reflects $+$-compatibility. $\square$

Input prefixing $\sum_{i\in I} \ominus.\sigma_i : \sum_{i\in I} \ominus.A_i$, if $\sigma_i : A_i$, for $i \in I$, where I is countable.

Output prefixing $\sum_{i\in I} p_i \oplus.\sigma_i : \sum_{i\in I} \oplus.A_i$, if $\sigma_i : A_i$, for $i \in I$, where I is countable, and $p_i \in [0,1]$ for $i \in I$ with $\sum_{i\in I} p_i \le 1$. If $\sum_{i\in I} p_i < 1$, there is non-zero probability of terminating without any action. By design $(\sum_{i\in I} \oplus.A_i)^\perp = \sum_{i\in I} \ominus.A_i^\perp$.

General probabilistic sum More generally we can define $\bigoplus_{i\in I} p_i\sigma_i : A$, for $\sigma_i : A$ and I countable with sub-probability distribution $p_i, i \in I$. The operation makes the $+$-events of different components conflict and re-weights the configuration-valuation on the components according to the sub-probability distribution. In order for the sum to remain receptive, the initial $-$ve events of the components over a common event in the game A must be identified.

Relabelling, the composition $f_*\sigma : B$, if $\sigma : A$ and $f : A \to B$, possibly partial on $+$ve events but always defined on $-$ve events, is receptive and innocent in the sense of Definition 4.6. Then the composition of maps $f\sigma : S \to B$ is receptive and innocent. Its defined part, taken to be $f_*\sigma : B$, is given by the factorization

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "\sigma"'] & S\downarrow D \arrow[d, "f_*\sigma"] \\ & A\,, \end{tikzcd}$$

where D is the subset of S at which $f\sigma$ is defined, is a strategy over B. If the configuration-valuation on S is v then that on $S \downarrow D$ is given by $x \mapsto v([x])$, for $x \in \mathcal{C}(S\downarrow D)$, where $[x]$ is the down-closure of x in S. The map $f_*\sigma : B$ is a strategy because, directly from the definition of innocence of partial maps, the projection $S \to S \downarrow D$ reflects immediate causal dependencies *from* $+$ve events and *to* $-$ve events. The function $x \mapsto v([x])$, for $x \in \mathcal{C}(S\downarrow D)$, is a configuration valuation: First, clearly $v[\varnothing]) = v(\varnothing) = 0$. Second, if $x \subseteq^- y$ in $\mathcal{C}(S\downarrow D)$, then $[x] \subseteq^- [y]$ in $\mathcal{C}(S)$ directly from the $-$-innocence of f, ensuring $v([x]) = v([y])$. Third, the drop condition is inherited from v. Assuming $y \stackrel{+}{\longrightarrow}\!\!\!\subset x_1,\cdots,x_n$ in $\mathcal{C}(S\downarrow D)$ we obtain $[y] \subseteq^+ [x_1],\cdots,[x_n]$ in $\mathcal{C}(S)$ because f is only undefined on $+$ve events. Hence, by the drop condition for v,

$$v([y]) - \sum_I (-1)^{|I|+1} v(\bigcup_{i\in I}[x_i]) \ge 0\,,$$

where I ranges over subsets $\varnothing \ne I \subseteq \{1,\cdots,n\}$ s.t. $\{[x_i] \mid i \in I\}\!\uparrow_S$. But,

$$\{[x_i] \mid i \in I\}\!\uparrow_S \iff \{x_i \mid i \in I\}\!\uparrow_{S\downarrow V}\,,$$

and down-closure commutes with unions. So

$$v([y]) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} [x_i]) = v([y]) - \sum_I (-1)^{|I|+1} v([\bigcup_{i \in I} x_i]) ,$$

where in the latter expression I ranges over subsets $\emptyset \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\} \uparrow_{S \downarrow V}$.

In particular, the composition $f\sigma : B$, if $\sigma : A$ and $f : A \to B$ is itself a strategy, *i.e.* total, receptive and innocent.

Pullback $f^*\sigma : A$, if $\sigma : B$ and $f : A \to B$ is a map of event structures, possibly partial, which reflects +-consistency in the sense that

$$y \overset{+}{\longrightarrow\!\!\subset} x_1, \cdots, x_n \ \& \ \{f x_i \mid 1 \leq i \leq n\} \uparrow \implies \{x_i \mid 1 \leq i \leq n\} \uparrow \ .$$

The strategy $f^*\sigma$ is got by the pullback

$$\begin{tikzcd} S' \arrow[r, "f'"] \arrow[d, "f^*\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "\sigma"] \\ A \arrow[r, "f"'] & B\,. \end{tikzcd}$$

Then, the map f' also reflects +-consistency. This fact ensures we define a configuration-valuation $v_{S'}$ on S' by taking $v_{S'}(x) = v_S(f'x)$, for $x \in \mathcal{C}(S')$. If $\sigma : S \to B$ is a strategy then so is $f^*\sigma : S' \to A$. Pullback along $f : A \to B$ may introduce events and causal links, present in A but not in B. The pullback operation subsumes the operations of prefixing $\ominus.\sigma$ and $\oplus.\sigma$ and we can recover the previous prefix sums if we also have have sum types—see below.

Sum types If A_i, $i \in I$, is a countable family of games, we can form their sum, the game $\sum_{i \in I} A_i$ as the sum of event structures. If $\sigma : A_j$, for $j \in I$, we can create the probabilistic strategy $j\,\sigma : \sum_{i \in I} A_i$ in which we extend σ with those initial −ve events needed to maintain receptivity. A probabilistic strategy of sum type $\sigma : \sum_{i \in I} A_i$ projects to a probabilistic strategy $(\sigma)_j : A_j$ where $j \in I$.

Abstraction $\lambda x : A.\sigma : A \multimap B$. Because probabilistic strategies form a monoidal-closed bicategory, with tensor $A \| B$ and function space $A \multimap B =_{\mathrm{def}} A^\perp \| B$, they support an (linear) λ-calculus, which in this context permits process-passing as in [**?**].

Recursive types and probabilistic processes can be dealt with along standard lines [4].

The types as they stand are somewhat inflexible. For example, that maps of event structures are locally injective would mean that simple labelling of events as in say CCS could not be directly captured through typing. However, this can be remedied by introducing monads, but doing this in sufficient generality would involve the introduction of symmetry.

In the pullback operations we have relied on certain maps being stable under pullback. The following two propositions make good our debt, and use techniques from open maps [**?**].

Proposition 14.42. *If $\sigma : S \to B$ is a strategy then so is $f^*\sigma : S' \to A$.*

Proof. Define an *étale* map (w.r.t. to a path category $\mathcal{P}$) to be like an open map, but where the lifting is unique. It is straightforward to show that the pullback of an étale map is étale. In fact, strategies can be regarded as étale maps, from which the proposition follows. Within the category of event structures with polarity and partial maps, take the path subcategory $\mathcal{P}$ to comprise all finite elementary event structures with polarity and take a typical map $f : p \to q$ in $\mathcal{P}$ to be a map such that:

(i) if $e \rightarrowtriangle_p e'$ with e −ve and e' +ve and both $f(e)$ and $f(e')$ defined, then $f(e) \rightarrowtriangle_q f(e')$; and

(ii) all events in q not in the image fp are −ve.

It can be checked that w.r.t. this choice of $\mathcal{P}$ the étale maps are precisely those maps which are strategies. □

Proposition 14.43. *If $f : A \to B$ reflects +-consistency, then so does $f' : S' \to S$.*

Proof. As +-consistency-reflecting maps are special kinds of open maps, known to be stable under pullback. An appropriate path category comprises: all finite event structures with polarity for which there is a subset M of $\leq$-maximal +-events s.t. a subset X is consistent iff $X \cap M$ contains at most one event of M—all finite elementary event structures with polarity are included as M, the chosen subset of $\leq$-maximal +-events, may be empty; maps in the path category are rigid maps of event structures with polarity whose underlying functions are bijective on events. □

14.5 The metalanguage on probabilistic strategies

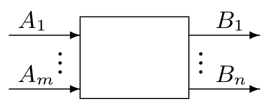
The metalanguage of games and strategies is largely stable under the addition of probability. Though for instance we shall need to restrict to race-free games in order to have identities w.r.t. the composition of probabilistic strategies.

In the language for probabilistic strategies, race-free games $A, B, C, \cdots$ will play the role of types. There are operations on games of forming the dual $A^\perp$, simple parallel composition $A \| B$, sum $\Sigma_{i\in I} A_i$ as well as recursively-defined games —the latter rest on well-established techniques [4] and will be ignored here. The operation of sum of games is similar to that of simple parallel composition but where now moves in different components are made inconsistent; we restrict its use to those cases in which it results in a game which is race-free.

Terms have typing judgements:

$$x_1 : A_1, \cdots, x_m : A_m \vdash \ t \ \dashv y_1 : B_1, \cdots, y_n : B_n \ ,$$

where all the variables are distinct, interpreted as a probabilistic strategy from the game $\vec{A} = A_1 \| \cdots \| A_m$ to the game $\vec{B} = B_1 \| \cdots \| B_n$. We can think of the term t as a box with input and output wires for the variables:



The idea is that t denotes a probabilistic strategy $S \to \vec{A}^\perp \| \vec{B}$ with configuration valuation v. The term t describes witnesses, finite configurations of S, to a relation between finite configurations $\vec{x}$ of $\vec{A}$ and $\vec{y}$ of $\vec{B}$, together with their probability conditional on the Opponent moves involved.

Duality The duality, that a probabilistic strategy from A to B can equally well be seen as a probabilistic strategy from $B^\perp$ to $A^\perp$, is caught by the rules:

$$\frac{\Gamma, x : A \vdash t \dashv \Delta}{\Gamma \vdash t \dashv x : A^\perp, \Delta} \qquad \frac{\Gamma \vdash t \dashv x : A, \Delta}{\Gamma, x : A^\perp \vdash t \dashv \Delta}$$

Composition The composition of probabilistic strategies is described in the rule

$$\frac{\Gamma \vdash t \dashv \Delta \qquad \Delta \vdash u \dashv \mathrm{H}}{\Gamma \vdash \exists \Delta. [\, t \parallel u \,] \dashv \mathrm{H}}$$

which, in the picture of strategies as boxes, joins the output wires of one strategy to input wires of the other.

Probabilistic sum For I countable and a sub-probability distribution $p_i, i \in I$, we can form the probabilistic sum of strategies of the same type:

$$\frac{\Gamma \vdash t_i \dashv \Delta \quad i \in I}{\Gamma \vdash \Sigma_{i \in I} p_i t_i \dashv \Delta \, .}$$

In the probabilistic sum of strategies, of the same type, the strategies are glued together on their initial Opponent moves (to maintain receptivity) and only commit to a component with the occurrence of a Player move, from which component being determined by the distribution $p_i, i \in I$. We use $\perp$ for the empty probabilistic sum, when the rule above specialises to

$$\Gamma \vdash \perp \dashv \Delta \, ,$$

which denotes the minimum strategy in the game $\Gamma^\perp \| \Delta$—it comprises the initial segment of the game $\Gamma^\perp \| \Delta$ consisting of its initial Opponent events.

Conjoining two strategies The pullback of a strategy across a map of event structures is itself a strategy [**?**]. We can use the pullback of one strategy against another to conjoin two probabilistic strategies of the same type:

$$\frac{\Gamma \vdash t_1 \dashv \Delta \quad \Gamma \vdash t_2 \dashv \Delta}{\Gamma \vdash t_1 \wedge t_2 \dashv \Delta}$$

Such a strategy acts as the two component strategies agree to act jointly. In the case where t_1 and t_2 denote the probabilistic strategies $\sigma_1 : S_1 \to \Gamma^\perp \| \Delta$ with configuration valuation v_1 and $\sigma_2 : S_2 \to \Gamma^\perp \| \Delta$ with v_2 the strategy $t_1 \wedge t_2$ denotes the pullback

$$\begin{tikzcd}
 & S_1 \wedge S_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, dotted, "\sigma_1 \wedge \sigma_2"] & \\
S_1 \arrow[dr, "\sigma_1"'] & & S_2 \arrow[dl, "\sigma_2"] \\
 & \Gamma^\perp \| \Delta &
\end{tikzcd}$$

with configuration valuation $x \mapsto v_1(\pi_1 x) \times v_2(\pi_2 x)$ for $x \in \mathcal{C}(S_1 \wedge S_2)$.

Copy-cat terms Copy-cat terms are a powerful way to lift maps or relations expressed in terms of maps to strategies. Along with duplication they introduce new "causal wiring." Copy-cat terms have the form

$$x : A \vdash \ gy \sqsubseteq_C fx \ \dashv y : B\,,$$

where $f : A \to C$ and $g : B \to C$ are maps of event structures preserving polarity. (In fact, f and g may even be "affine" maps, which don't necessarily preserve empty configurations, provided $g\varnothing \sqsubseteq_C f\varnothing$—see [**?**].) This denotes a deterministic strategy—so a probabilistic strategy with configuration valuation constantly one—provided f reflects $-$-compatibility and g reflects $+$-compatibility. The map g reflects $+$-compatibility if whenever $x \subseteq^+ x_1$ and $x \subseteq^+ x_2$ in the configurations of B and $fx_1 \cup fx_2$ is consistent, so a configuration, then so is $x_1 \cup x_2$. The meaning of f reflecting $-$-compatibility is defined analogously.

A term for copy-cat arises as a special case,

$$x : A \vdash y \sqsubseteq_A x \dashv y : A\,,$$

as do terms for the jth injection into and jth projection out of a sum $\Sigma_{i \in I} A_i$ w.r.t. its component A_j,

$$x : A_j \vdash y \sqsubseteq_{\Sigma_{i \in I} A_i} jx \dashv y : \Sigma_{i \in I} A_i$$

and

$$x : \Sigma_{i \in I} A_i \vdash jy \sqsubseteq_{\Sigma_{i \in I} A_i} x \dashv y : A_j\,,$$

as well as terms which split or join 'wires' to or from a game $A \| B$.

In particular, a map $f : A \to B$ of games which reflects $-$-compatibility lifts to a deterministic strategy $f_! : A \twoheadrightarrow B$:

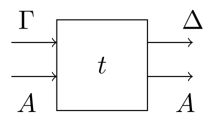
$$x : A \vdash \ y \sqsubseteq_B fx \ \dashv y : B\,.$$

A map $f : A \to B$ which reflects +-compatibility lifts to a deterministic strategy $f^* : B \mathrel{\circ\!\!\!\rightarrow} A$:

$$y : B \vdash \; fx \sqsubseteq_B y \;\; \dashv \;\; x : A \,.$$

The construction $f^* \odot t$ denotes the pullback of a strategy t in B across the map $f : A \to B$. It can introduce extra events and dependencies in the strategy. It subsumes the operations of prefixing by an initial Player or Opponent move on games and strategies.

Trace A probabilistic *trace*, or feedback, operation is another consequence of such "wiring." Given a probabilistic strategy $\Gamma, x : A \vdash t \dashv y : A, \Delta$ represented by the diagram



we obtain

$$\Gamma, \Delta^\perp \vdash t \dashv x : A^\perp, y : A$$

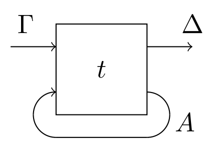
which post-composed with the term

$$x : A^\perp, y : A \vdash x \sqsubseteq_A y \dashv ,$$

denoting the copy-cat strategy $\gamma_{A^\perp}$, yields

$$\Gamma \vdash \exists x : A^\perp, y : A. \, [\, t \parallel x \sqsubseteq_A y \,] \dashv \Delta \,,$$

representing its trace:



The composition introduces causal links from the Player moves of $y : A$ to the Opponent moves of $x : A$, and from the Player moves of $x : A$ to the Opponent moves of $y : A$—these are the usual links of copy-cat $\gamma_{A^\perp}$ as seen from the left of the turnstyle. If we ignore probabilities, this trace coincides with the feedback operation which has been used in the semantics of nondeterministic dataflow (where only games comprising solely Player moves are needed) [3].

Duplication Duplications of arguments is essential if we are to support the recursive definition of strategies. We duplicate arguments through a probabilistic strategy $\delta_A : A \mathrel{\circ\!\!\!\rightarrow} A \| A$. Intuitively it behaves like the copy-cat strategy but where a Player move in the left component may choose to copy from either of the two components on the right. In general the technical definition is involved, even without probability—see [**?**]. The introduction of probability begins to reveal a limitation within probabilistic strategies as we have defined them, a point we will follow up on in the next section. We can see the issue in the second of two simple examples. The first is that of δ_A in the case where the game A consists of a single Player move $\oplus$. Then, δ_A is the deterministic strategy

$$\begin{tikzcd}[row sep=small] & \oplus \\ \ominus \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & \oplus \end{tikzcd}$$

in which the configuration valuation assigns one to all finite configurations —we have omitted the obvious map to the game $A^\perp \| A \| A$. In the second example, assume A consists of a single Opponent move $\ominus$. Now δ_A is no longer deterministic and takes the form

$$\begin{tikzcd}[row sep=small] \oplus \arrow[d, squiggly, no head] & \ominus \arrow[l, Rightarrow] \\ \oplus & \ominus \arrow[l, Rightarrow] \end{tikzcd}$$

and the strategy is forced to choose probabilistically between reacting to the upper or lower move of Opponent in order to satisfy the drop condition of its configuration valuation. Given the symmetry of the situation, in this case any configuration containing a Player move is assigned value a half by the configuration valuation associated with δ_A. (In the definition of the probabilistic duplication for general A the configuration valuation is distributed uniformly over the different ways Player can copy Opponent moves.) But this is odd: in the second example, if the Opponent makes only one move there is a 50% chance that Player will not react to it! There are mathematical consequences too. In the absence of probability δ_A forms a comonoid with counit $\perp : A \twoheadrightarrow \varnothing$. However, as a probabilistic strategy δ_A is no longer a comonoid—it fails associativity. It is hard to see an alternative definition of a probabilistic duplication strategy within the limitations of the event structures we have been using. We shall return to duplication, and a simpler treatment through a broadening of event structures in the next section.

Recursion Once we have duplication strategies we can treat recursion. Recall that 2-cells, the maps between probabilistic strategies, include the approximation order $\trianglelefteq$ between strategies. The order forms a 'large complete partial order' with a bottom element the minimum strategy $\perp$. Given $x : A, \Gamma \vdash t \dashv y : A$, the term $\Gamma \vdash \mu x : A. t \dashv y : A$ denotes the $\trianglelefteq$-least fixed point amongst probabilistic strategies $X : \Gamma \twoheadrightarrow A$ of the $\trianglelefteq$-continuous operation $F(X) = t \odot (\mathrm{id}_\Gamma \| X) \odot \delta_\Gamma$. (With one exception, F is built out of operations which it's been shown can be be defined concretely in such a way that they are $\trianglelefteq$-continuous; the one exception which requires separate treatment is the 'new' operation of projection, used to hide synchronisations.) With probability, as δ_Γ is no longer a comonoid not all the "usual" laws of recursion will hold, though the unfolding law will hold by definition.

There are important special cases though, when we can avoid the problems with duplication, for example, when we restrict all types and type constructions to games comprising purely Player moves—then duplication strategies are deterministic; we obtain a language for *probabilistic dataflow*, like nondeterministic dataflow but with probabilistic choice.

14.5.1 Payoff

Given a probabilistic strategy $v_S, \sigma : S \to A$ and counter-strategy $v_T, \tau : T \to A^\perp$ we obtain

$$\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
S \arrow[dr, "\sigma"'] & & T \arrow[dl, "\tau"] \\
 & A &
\end{tikzcd}$$

with valuation $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$, for $x \in \mathcal{C}(P)$, on the pullback P—a probabilistic event structure, with probability measure $\mu_{\sigma,\tau}$. Define $f =_{\text{def}} \sigma\pi_1 = \tau\pi_2$. Adding *payoff* as a Borel measurable function $X : \mathcal{C}^\infty(A) \to \mathbb{R}$ the *expected payoff* is obtained as the Lebesgue integral

$$\begin{aligned}
\mathbf{E}_{\sigma,\tau}(X) &=_{\text{def}} \int_{x \in \mathcal{C}^\infty(P)} X(f(x))\ d\mu_{\sigma,\tau}(x) \\
&= \int_{y \in \mathcal{C}^\infty(A)} X(y)\ d\mu_{\sigma,\tau} f^{-1}(y)\,,
\end{aligned}$$

where we can choose either to integrate over $\mathcal{C}^\infty(P)$ with measure $\mu_{\sigma,\tau}$, or over $\mathcal{C}^\infty(A)$ with measure $\mu_{\sigma,\tau} f^{-1}$.

14.5.2 A simple value-theorem

Let A be a game with payoff X. Its dual is the game $A^\perp$ with payoff $-X$. If A, X and B, Y are two games with payoff, their parallel composition $(A, X) \parr (B, Y)$ is the game with payoff $(A \| B, X + Y)$.

Let A be a game with payoff X. Define

$$\begin{aligned}
\mathrm{val(A, X)} &=_{\text{def}} \sup_\sigma \inf_\tau \mathrm{E}_{\sigma,\tau}(\mathrm{X}) \\
\mathrm{val(A^\perp, -X)} &=_{\text{def}} \sup_\tau \inf_\sigma \mathrm{E}_{\tau,\sigma}(\mathrm{-X}) = -\inf_\tau \sup_\sigma \mathrm{E}_{\sigma,\tau}(\mathrm{X})\,.
\end{aligned}$$

The game A, X is said to have a value if

$$\mathrm{val(A, X)} = -\mathrm{val(A^\perp, -X)} = \mathrm{E}_{\sigma_0,\tau_0}(\mathrm{X})\,,$$

its value then being $\mathrm{val(A, X)}$.

The following theorem says that a Nash equilibrium—expressed in properties (1) and (2)—determines a value for a game with payoff.

Theorem 14.44. *Let A be a game with payoff X. Suppose there are a strategy σ_0 and a counterstrategy τ_0 s.t.*

$$\begin{aligned}
&(1) \forall \tau,\ \textit{a counterstrategy. } E_{\sigma_0,\tau}(X) \geq E_{\sigma_0,\tau_0}(X) \ \textit{ and} \\
&(2) \forall \sigma,\ \textit{a strategy. } E_{\sigma,\tau_0}(X) \leq E_{\sigma_0,\tau_0}(X)\,.
\end{aligned}$$

Then, the game A, X has a value and $E_{\sigma_0,\tau_0}(X)$ is the value of the game.

Proof. Letting σ stand for strategies and τ for counterstrategies, we have

$$\mathrm{val}(\mathrm{A}) =_{\mathrm{def}} \sup_{\sigma} \inf_{\tau} \mathrm{E}_{\sigma,\tau}(\mathrm{X})$$
$$\mathrm{val}(\mathrm{A}^{\perp}) =_{\mathrm{def}} \sup_{\tau} \inf_{\sigma} \mathrm{E}_{\tau,\sigma}(-\mathrm{X}) = -\inf_{\tau} \sup_{\sigma} \mathrm{E}_{\sigma,\tau}(\mathrm{X})\,.$$

We require

$$\mathrm{val}(\mathrm{A}) = -\mathrm{val}(\mathrm{A}^{\perp}) = \mathrm{E}_{\sigma_0,\tau_0}(\mathrm{X})\,.$$

For all strategies σ,

$$\inf_{\tau} E_{\sigma,\tau}(X) \le E_{\sigma,\tau_0}(X) \le E_{\sigma_0,\tau_0}(X)$$

by (2). Therefore

$$\sup_{\sigma} \inf_{\tau} E_{\sigma,\tau}(X) \le E_{\sigma_0,\tau_0}(X)\,.$$

Also

$$\sup_{\sigma} \inf_{\tau} E_{\sigma,\tau}(X) \ge \inf_{\tau} E_{\sigma_0,\tau}(X) \ge E_{\sigma_0,\tau_0}(X)$$

by (1). Hence

$$\sup_{\sigma} \inf_{\tau} E_{\sigma,\tau}(X) = E_{\sigma_0,\tau_0}(X)\,. \tag{3}$$

Dually,

$$\sup_{\sigma} E_{\sigma,\tau}(X) \ge E_{\sigma_0,\tau}(X) \ge E_{\sigma_0,\tau_0}(X)$$

by (1). Therefore

$$\inf_{\tau} \sup_{\sigma} E_{\sigma,\tau}(X) \ge E_{\sigma_0,\tau_0}(X)\,.$$

Also,

$$\inf_{\tau} \sup_{\sigma} E_{\sigma,\tau}(X) \le \sup_{\sigma} E_{\sigma,\tau_0}(X) \le E_{\sigma_0,\tau_0}(X)$$

by (2). Hence

$$\inf_{\tau} \sup_{\sigma} E_{\sigma,\tau}(X) = E_{\sigma_0,\tau_0}(X)\,. \tag{4}$$

From (3) and (4) it follows that

$$\mathrm{val}(\mathrm{A}) = -\mathrm{val}(\mathrm{A}^{\perp}) = \mathrm{E}_{\sigma_0,\tau_0}(\mathrm{X})\,,$$

the value of the game, as required. □

Chapter 15

Quantum games

We first explore a definition of quantum event structure in which events are associated with projection or unitary operators. It is shown how this structure induces configuration-valuations, and hence probability measures, on compatible parts of the domain of configurations of the event structure. This elementary situation is not preserved by the projection operation on event structures, so we move to a more general definition. We conclude with a brief exploration of quantum games and strategies. A quantum game is taken to be a quantum event structure in which events carry polarities and a strategy in a quantum game as a probabilistic strategy in its event structure.

15.1 Simple quantum event structures

Throughout let $\mathcal{H}$ be a Hilbert space over the complex numbers, with countable basis. For operators A, B on $\mathcal{H}$ we write $[A, B] =_{\mathrm{def}} AB - BA$.

Definition 15.1. A *(simple) quantum event structure* (over $\mathcal{H}$) comprises an event structure $(E, \leq, \mathrm{Con})$ together with an assignment Q_e of projection or unitary operators on $\mathcal{H}$ to events $e \in E$ such that for all $x \in \mathcal{C}(E), e_1, e_2 \in E$ for which $x \xrightarrow{e_1}\!\!\subset x_1$ and $x \xrightarrow{e_2}\!\!\subset x_2$,

$$x_1 \uparrow x_2 \implies [Q_{e_1}, Q_{e_2}] = 0\,,$$

i.e. the two events occur concurrently at x implies their associated operators commute. Say the quantum event structure is *strong* when

$$x_1 \uparrow x_2 \iff [Q_{e_1}, Q_{e_2}] = 0\,,$$

i.e. the two events occur concurrently at x *iff* their associated operators commute.

Definition 15.2. Given a finite configuration, $x \in \mathcal{C}(E)$, define the operator A_x to be the composition $Q_{e_n} Q_{e_{n-1}} \cdots Q_{e_2} Q_{e_1}$ for some covering chain

$$\varnothing \xrightarrow{e_1}\!\!\subset x_1 \xrightarrow{e_2}\!\!\subset x_2 \cdots \xrightarrow{e_n}\!\!\subset x_n = x$$

in $\mathcal{C}(E)$. This is well-defined as for any two covering chains up to x the sequences of events are Mazurkiewicz trace equivalent, *i.e.* obtainable, one from the other, by successively interchanging concurrent events. In particular $A_\varnothing$ is the identity operator on $\mathcal{H}$.

Proposition 15.3. *In a strong quantum event structure* $(E, \leq, \mathrm{Con})$ *with assignment of operators* Q *the consistency predicate* Con *is determined in a pairwise fashion, i.e. for any finite subset of events* X*,*

$$X \in \mathrm{Con} \iff \forall e_1, e_2 \in X.\ \{e_1, e_2\} \in \mathrm{Con}\,.$$

Writing $e_1 \# e_2 \iff_{\mathrm{def}} \{e_1, e_2\} \notin \mathrm{Con}$*,*

$$e_1 \# e_2 \iff \exists e_1' \leq e_1, e_2' \leq e_2.\ [e_1] \cup [e_2) \in \mathrm{Con}\ \&\ [e_1) \cup [e_2] \in \mathrm{Con}\ \&\ [P_{e_1}, P_{e_2}] \neq 0\,.$$

Proof. Observe that if $\{e_1, e_2\} \in \mathrm{Con}$ with both $x \overset{e_1}{\longrightarrow\!\subset} x_1$ and $x \overset{e_2}{\longrightarrow\!\subset} x_2$, then $x_1 \uparrow x_2$. To see this argue from $\{e_1, e_2\} \in \mathrm{Con}$, $x \overset{e_1}{\longrightarrow\!\subset} x_1$ and $x \overset{e_2}{\longrightarrow\!\subset} x_2$ that $[e_1) \cup [e_2) \overset{e_1}{\longrightarrow\!\subset} [e_1]$ and $[e_1) \cup [e_2) \overset{e_2}{\longrightarrow\!\subset} [e_2]$ where $[e_1] \uparrow [e_2]$ follows directly from the consistency of $\{e_1, e_2\}$. It follows that $[Q_{e_1}, Q_{e_2}] = 0$, whence $x_1 \uparrow x_2$, as E, Q is a strong quantum event structure. A simple induction on the size of a finite pairwise-consistent down-closed subset of events X shows it to be a configuration. As a finite set is consistent iff its down-closure is consistent, the result follows. $\square$

Example 15.4. In the quantum event structure E with assignment of projection operators P_e to events e, assume the event structure E comprises solely concurrent events. In other words, no event causally depends on any other and any two events are concurrent. This is an example of a strong quantum event structure. Each projection operator P_e commutes with every other $P_{e'}$. Therefore the eigenvectors of all the projection operators P_e extend to an orthonormal basis of $\mathcal{H}$. Each projection operator corresponds to that subset of basis vectors it fixes. Under this correspondence, a composition of projection operators is associated with the intersection of the sets of fixed basis vectors. In other words, for any finite configuration x, the operator A_x is the projection operator which fixes precisely those basis vectors which are fixed by all the P_e, for $e \in x$.

Example 15.5. Consider an event structure consisting of two events e_1, e_2 incomparable under $\leq$ with $\{e_, e_2\} \notin \mathrm{Con}$. Only assignments of operators to e_1, e_2 for which $[Q_{e_1}, Q_{e_2}] \neq 0$ will yield a *strong* quantum event structure.

Example 15.6. Consider an event structure consisting of two events for which $e_1 \leq e_2$. Any assignment of projection operators to e_1, e_2 will yield a strong quantum event structure.

Example 15.7. Let (M, L, I) be a Mazurkiewicz trace language consisting of an alphabet L with independence relation I and subset of strings $M \subseteq L^*$, so M is closed under prefixes and I-closed in the sense that if $sabt \in M$ and aIb

then $sbat \in M$. Assume an assignment of projection and unitary operators Q_a to symbols $a \in \Sigma$ such that

$$a\,I\,b \implies [Q_a, Q_b] = 0\,.$$

Then, M determines a quantum event structure: as shown in [2], M determines an event structure with events e associated with the minimal ways a symbol, say a, appears in a string in M—then the operator assigned to e is Q_a. If we assume that

$$sa \in M \;\&\; sb \in M \;\&\; a\,I\,b \implies sab \in M\,.$$

and an assignment of operators Q_a to symbols $a \in \Sigma$ such that

$$a\,I\,b \iff a \neq b \;\&\; [Q_a, Q_b] = 0\,,$$

then M determines a *strong* quantum event structure.

The unitary and projection operators of $\mathcal{H}$ form a Mazurkiewicz trace language, and in turn a strong quantum event structure.

Definition 15.8. Take as Mazurkiewicz trace language that with alphabet comprising (names for) all the unitary and projection operators on $\mathcal{H}$ with all strings of such and with independence relation

$$A\ I\ B \iff A \neq B \;\&\; [A, B] = 0\,,$$

between operators A and B. The Mazurkiewicz trace language determines a strong quantum event structure, associated with the Hilbert space $\mathcal{H}$.

15.2 From quantum to probabilistic

Consider a quantum event structure with an initial state given by a density operator ρ on $\mathcal{H}$. While it does not make sense to attribute a probability distribution globally, over the whole space of configurations $\mathcal{C}^\infty(E)$, there is a sensible probability distribution on compatible configurations of the event structure. Below, by an unnormalized density operator we mean a positive, self-adjoint operators with trace less than or equal to one.

Theorem 15.9. *Let E, Q be a simple quantum event structure with initial state a density operator ρ. Each configuration $x \in \mathcal{C}(E)$ is associated with an unnormalized density operator $\rho_x =_{\mathrm{def}} A_x \rho A_x^\dagger$ and a value in $[0,1]$ given by $v(x) =_{\mathrm{def}} \mathrm{Tr}(\rho_x) = \mathrm{Tr}(A_x^\dagger A_x \rho)$. For any $w \in \mathcal{C}^\infty(E)$, the function v restricts to a configuration-valuation v_w on finite configurations in the family of configurations $\mathcal{F}_w =_{\mathrm{def}} \{x \in \mathcal{C}^\infty(E) \mid x \subseteq w\}$; hence v_w extends to a unique probability measure q_w on $\mathcal{F}_w$.*

Proof. We show v restricts to a configuration-valuation on $\mathcal{F}_w$. As $A_\varnothing = \mathrm{id}_{\mathcal{H}}$, $v(\varnothing) = \mathrm{Tr}(\rho) = 1$. By Lemma 14.11, we need only to show $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ when $y \overset{e_1}{\longrightarrow\!\!\!\subset} x_1, \cdots, y \overset{e_n}{\longrightarrow\!\!\!\subset} x_n$ in $\mathcal{F}_w$.

First, observe that if for some event e_i the operator Q_{e_i} is unitary, then $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$. W.l.o.g. suppose e_n is assigned the unitary operator U. Then, $A_{x_n} = U A_y$ so

$$v(x_n) = \mathrm{Tr}(A_{x_n}^\dagger A_{x_n} \rho) = \mathrm{Tr}(A_y^\dagger U^\dagger U A_y \rho) = \mathrm{Tr}(A_y^\dagger A_y \rho) = v(y)\,.$$

Let $\varnothing \neq I \subseteq \{1, \cdots, n\}$. Then, either $\bigcup_{i \in I} x_i = \bigcup_{i \in I} x_i \cup x_n$ or $\bigcup_{i \in I} x_i \xrightarrow{e_n}\!\!\subset \bigcup_{i \in I} x_i \cup x_n$. In the either case—in the latter case by an argument similar to that above,

$$v(\bigcup_{i \in I} x_i) = v(\bigcup_{i \in I} x_i \cup x_n)\,.$$

Consequently,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] =& d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] \\ =& v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) - v(x_n) + \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i \cup x_n) \\ =& \ 0 \end{aligned}$$

—above index I is understood to range over sets for which $\varnothing \neq I \subseteq \{1, \cdots, n\}$.

It remains to consider the case where all events e_i are assigned projection operators P_{e_i}. As $x_1, \cdots, x_n \subseteq w$ we must have that all the projection operators $P_{e_1}, \cdots, P_{e_n}$ commute. (Locally the situation resembles that of Example 15.4.)

As $[P_{e_i}, P_{e_j}] = 0$, for $1 \leq i, j \leq n$, we can assume an orthonormal basis which extends the sub-basis of eigenvectors of all the projection operators P_{e_i}, for $1 \leq i \leq n$. Let $y \subseteq x \subseteq \bigcup_{1 \leq i \leq n} x_i$. Define P_x to be the projection operator got as the composition of all the projection operators P_e for $e \in x \setminus y$—this is a projection operator, well-defined irrespective of the order of composition as the relevant projection operators commute. Define B_x to be the set of those basis vectors fixed by the projection operator P_x. In particular, P_y is the identity operator and B_y the set of all basis vectors. When $x, x' \in \mathcal{C}(E)$ with $y \subseteq x \subseteq \bigcup_{1 \leq i \leq n} x_i$ and $y \subseteq x' \subseteq \bigcup_{1 \leq i \leq n} x_i$,

$$B_{x \cup x'} = B_x \cap B_{x'}\,.$$

Also,

$$P_x|\psi\rangle = \sum_{i \in B_x} \langle i|\psi\rangle\, |i\rangle\,,$$

so

$$\langle\psi|P_x|\psi\rangle = \sum_{i \in B_x} \langle i|\psi\rangle\langle\psi|i\rangle = \sum_{i \in B_x} |\langle i|\psi\rangle|^2\,,$$

for all $|\psi\rangle \in \mathcal{H}$.

Assume $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$, where the ψ_k are normalised and all the p_k are positive with sum $\sum_k p_k = 1$. For x with $y \subseteq x \subseteq \bigcup_{1\le i\le n} x_i$,

$$\begin{aligned} v(x) &= \mathrm{Tr}(A_x^\dagger A_x \rho) \\ &= \mathrm{Tr}(A_y^\dagger P_x^\dagger P_x A_y \rho) \\ &= \mathrm{Tr}(A_y^\dagger P_x A_y \sum_k p_k |\psi_k\rangle\langle\psi_k|) \\ &= \sum_k p_k \,\mathrm{Tr}(A_y^\dagger P_x A_y |\psi_k\rangle\langle\psi_k|) \\ &= \sum_k p_k \langle A_y\psi_k | P_x | A_y \psi_k\rangle \\ &= \sum_{i\in B_x} \sum_k p_k |\langle i|A_y\psi_k\rangle|^2 \\ &= \sum_{i\in B_x} r_i\,, \end{aligned}$$

where we define $r_i =_{\mathrm{def}} \sum_k p_k |\langle i|A_y\psi_k\rangle|^2$, necessarily a non-negative real for $i \in B_x$.

We now establish that

$$d_v^{(n)}[y; x_1, \cdots, x_n] = \sum_{i\in B_y \smallsetminus B_{x_1}\cup\cdots\cup B_{x_n}} r_i\,,$$

for all $n \in \omega$, by mathematical induction—it then follows directly that its value is non-negative.

The base case of the induction, when $n = 0$, follows as

$$d_v^{(0)}[y;\,] = v(y) = \sum_{i\in B_y} r_i\,,$$

a special case of the result we have just established.

For the induction step, assume $n > 0$. Observe that

$$B_y \smallsetminus B_{x_1} \cup \cdots \cup B_{x_{n-1}} = (B_y \smallsetminus B_{x_1} \cup \cdots \cup B_{x_n}) \,\dot\cup\, (B_{x_n} \smallsetminus B_{x_1\cup x_n} \cup \cdots \cup B_{x_{n-1}\cup x_n})\,,$$

where as signified the outer union is disjoint. Hence,

$$\sum_{i\in B_y \smallsetminus B_{x_1}\cup\cdots\cup B_{x_{n-1}}} r_i = \sum_{i\in B_y \smallsetminus B_{x_1}\cup\cdots\cup B_{x_n}} r_i + \sum_{i\in B_{x_n} \smallsetminus B_{x_1\cup x_n}\cup\cdots\cup B_{x_{n-1}\cup x_n}} r_i\,,$$

By definition,

$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\mathrm{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n]$$

—making use of the fact that we are only forming unions of compatible configurations. From the induction hypothesis,

$$\begin{aligned} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] &= \sum_{i\in B_y \smallsetminus B_{x_1}\cup\cdots\cup B_{x_{n-1}}} r_i \\ \text{and } d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] &= \sum_{i\in B_{x_n} \smallsetminus B_{x_1\cup x_n}\cup\cdots\cup B_{x_{n-1}\cup x_n}} r_i\,. \end{aligned}$$

Hence
$$d_v^{(n)}[y; x_1, \cdots, x_n] = \sum_{i \in B_y \setminus B_{x_1} \cup \cdots \cup B_{x_n}} r_i \,,$$
ensuring $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$, as required.

By Theorem 14.14, the configuration-valuation v_w extends to a unique probability measure on $\mathcal{F}_w$. □

Interpretation. We can regard $w \in \mathcal{C}^\infty(E)$ as a quantum experiment. The experiment specifies unitary and projection operators to apply and in which order. The order being partial permits commuting operators to be applied concurrently, independently of each other. The experiment can end in an element of $\mathcal{F}_w$ with chance given by the probability measure got from the configuration-valuation v_w. To say an experiment ends or results in $w' \in \mathcal{F}_w$ means it succeeds in the confirmation, observation or test associated with w', but goes no further.

In particular, we may take w to be a maximal configuration, obtaining a maximal part of the space configurations over which it is sensible to attribute a probability distribution. Compatible parts of the domain of configurations of a quantum event structure with an initial state carry an intrinsic probability distribution. With the reading of configurations as histories the theorem is reminiscent of the consistent/decoherent histories view of quantum computation. Note however that the consistency/decoherence conditions traditional in that approach have been replaced here, in the case of simple quantum event structures, by compatibility w.r.t. the inclusion order on configurations, and that compatibility respects traditional quantum notions of commuting observables.

Example 15.10. Let E comprise the quantum event structure with two concurrent events e_0 and e_1 associated with projectors P_0 and P_1, where necessarily $[P_0, P_1] = 0$. Assume an initial state $|\psi\rangle\langle\psi|$. The configuration $\{e_0, e_1\}$ is associated with the following probability distribution. The probability that e_0 succeeds is $||P_0|\psi\rangle||^2$, that e_1 succeeds $||P_1|\psi\rangle||^2$, and that both succeed is $||P_1 P_0|\psi\rangle||^2$.

In the case where P_0 and P_1 commute because $P_0 P_1 = P_1 P_0 = 0$, the events e_0 and e_1 are mutually exclusive. There is probability zero of both events e_0 and e_1 succeeding, probability $||P_0|\psi\rangle||^2$ of e_0 succeeding, $||P_1|\psi\rangle||^2$ of e_1 succeeding, and probability $1 - ||P_0|\psi\rangle||^2 - ||P_1|\psi\rangle||^2$ of getting stuck at the empty configuration where neither event succeeds.

A special case of this is the measurement of a qubit in state ψ, the measurement of 0 where $P_0 = |0\rangle\langle 0|$, and the measurement of 1 where $P_1 = |1\rangle\langle 1|$, though here $||P_0|\psi\rangle||^2 + ||P_1|\psi\rangle||^2 = 1$, as a measurement of the qubit will determine a result of either 0 or 1.

Example 15.11. The measurement of two qubits with entanglement. *****

Example 15.12. Let E comprise the event structure with three events e_1, e_2, e_3 with trivial causal dependency and consistency relation generated by taking

$\{e_1, e_2\} \in \mathrm{Con}$ and $\{e_2, e_3\} \in \mathrm{Con}$—so $\{e_1, e_3\} \notin \mathrm{Con}$. To be a quantum event structure we must have $[Q_{e_1}, Q_{e_2}] = 0$, $[Q_{e_2}, Q_{e_3}] = 0$ and, to be strong, that $[Q_{e_1}, Q_{e_3}] \neq 0$. The maximal configurations are $\{e_1, e_2\}$ and $\{e_2, e_3\}$. Assume an initial state $|\psi\rangle\langle\psi|$. The first maximal configuration is associated with a probability distribution where e_1 occurs with probability $\|Q_{e_1}|\psi\rangle\|^2$ and e_2 occurs with probability $\|Q_{e_2}|\psi\rangle\|^2$. The second maximal configuration is associated with a probability distribution where e_2 occurs with probability $\|Q_{e_2}|\psi\rangle\|^2$ and e_3 occurs with probability $\|Q_{e_3}|\psi\rangle\|^2$.

15.3 An extension

Recall that by an unnormalized density operator we mean a positive, self-adjoint operators with trace less than or equal to one.

Theorem 15.9shows how a quantum event structure with initial state induces a probabilistic event structure on the sub event structure comprising the events of a configuration. We can generalise this to sub event structures with inconsistent events provided immediately conflicting events are associated with operators whose composition is 0. (Accordingly in the sub event structure if an event is associated with a unitary operator then it can only be in immediate conflict with an event associated with the 0 operator.)

First let's be precise on what we mean by a sub event structure. Let $E_0 = (E_0, \leq_0, \mathrm{Con}_0)$ and $E = (E, \leq, \mathrm{Con})$ be event structures. Write $E_0 \trianglelefteq E$ iff E_0 is a down-closed subset of E with

$$e' \leq_0 e \text{ iff } e', e \in E_0 | \& \ e' \leq e, \text{ and}$$

$$X \in \mathrm{Con}_0 \text{ iff } X \subseteq_{\mathrm{fin}} E_0 \ \& \ X \in \mathrm{Con};$$

in other words, E_0 is a substructure of E. In this case,

$$x \in \mathcal{C}^\infty(E_0) \text{ iff } x \subseteq E_0 \text{ and } x \in \mathcal{C}^\infty(E).$$

Theorem 15.13. *Let E, Q be a simple quantum event structure with initial state a density operator ρ. Each configuration $x \in \mathcal{C}(E)$ is associated with an unnormalized density operator $\rho_x =_{\mathrm{def}} A_x \rho A_x^\dagger$ and a value in $[0,1]$ given by $v(x) =_{\mathrm{def}} \mathrm{Tr}(\rho_x) = \mathrm{Tr}(A_x^\dagger A_x \rho)$.*

Let $E_0 \trianglelefteq$ be a sub event structure of E for which

$$\text{whenever } x \xrightarrow{e_1}\!\subset x_1 \text{ and } x \xrightarrow{e_2}\!\subset x_2 \text{ with } x_1 \not\uparrow x_2 \text{ in } \mathcal{C}^\infty(E_0) \text{ then } Q_{e_1} Q_{e_2} = 0.$$

Then the restriction v_0 of v to the finite configurations of E_0 is a configuration-valuation; hence v_0 extends to a unique probability measure on $\mathcal{C}^\infty(E_0)$.

Proof. As $A_\varnothing = \mathrm{id}_{\mathcal{H}}$, $v(\varnothing) = \mathrm{Tr}(\rho) = 1$. By Lemma 14.11, we need only to show $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ when $y \xrightarrow{e_1}\!\subset x_1, \cdots, y \xrightarrow{e_n}\!\subset x_n$ in $\mathcal{C}(E_0)$.

To this end construct a finite quantum event structure E_1, Q_1 as the event structure with events

$$y \cup \bigcup_{1 \leq i \leq n} x_i$$

and causal dependency and assigment of operators inherited from E (and E_0) but with all finite subsets of events consistent. Note that any immediate conflicts between events e_i and e_j at y amongst the events $e_1, \cdots, e_n$ are replaced by instances of the concurrency relation e_i *co* e_j. For such 'new' instances of concurrency we shall have $[Q_{e_i} Q_{e_j}] = 0$ as both compositions $Q_{e_i} Q_{e_j}$ and $Q_{e_j} Q_{e_i}$ are 0. Thus E_1, Q_1 is a quantum event structure. The event structure E_1 may have configurations which are not configurations of E_0. However such additional configurations z will be associated with the operator $A_z = 0$ by the assumption on E_0. Consequently, the value of the drop $d_v^{(n)}[y; x_1, \cdots, x_n]$ in E_0 equals that of $d_v^{(n)}[y; x_1, \cdots, x_n]$ in E_1. But by Theorem 15.9 the drop in E_1 is always non-negative, yielding the required drop condition for E_0. $\Box$

15.3.1 A notion of distributed quantum tests

We can refine our description of quantum experiments. We base the idea on *confusion-free* event structures in which conflict (inconsistency) is localised at cells.

Let $E = (E, \leq, \mathrm{Con})$ be an event structure. Say two events $e_1, e_2 \in E$ are in *immediate conflict* at a configuration $x \in \mathcal{C}^\infty(E)$ iff both $x \cup \{e_1\}, x \cup \{e_2\} \in \mathcal{C}^\infty(E)$ and yet their union $x \cup \{e_1, e_2\}$ is not a configuration. Say E *has binary conflict* iff

$$X \in' Con \iff X \subseteq_{\mathrm{fin}} E \ \&\ \forall e_1, e_2 \in X.\ \{e_1, e_2\} \in \mathrm{Con}\,.$$

Then, defining the *conflict* relation by

$$e_1 \# e_2 \iff \{e_1, e_2\} \notin \mathrm{Con}\,,$$

as set is consistent iff it is conflct-free, *i.e.* no pairs of events within it are in conflict. We can further define $e_1 \#_\mu e_2$, the *immediate-conflict* relation, iff e_1 and e_2 are in immediate conflict at some configuration.

Say an event structure E is *confusion-free* iff it has binary conflict, the relation $\#_\mu \cup \mathrm{id}_E$ is an equivalence relation and moreover

$$e_1 \#_m u e_2 \implies [e_1) = [e_2)\,.$$

In this case we call the equivalence classes of $\#_\mu \cup \mathrm{id}_E$ *cells.*

It follows that iff an event e in a cell c is enabled at a configuration x, all the events of c are enabled as well. In this sense conflict is localised at cells. A finite subset is inconsistent iff it has two events which share distinct events from a common cell in their causal history. Consequently, a configuration is a down-closed subset of events in which no two distinct events belong to a common cell. Confusion-free event structures correspond to deterministic concrete data structures [**?**, **?**] and are those event structures derived from confusion-free occurrence nets [**?**].

A form of distributed *quantum test* is represented by a quantum event structure E, Q where E is a confusion-free event structure, $Q_e \neq 0$ for all events e, and for any two distinct events e_1, e_2 of a common cell $Q_{e_1} Q_{e_2} = 0$. This formalises the idea of a making local measurements in a distributed fashion where the outcomes of measurements determine those future measurements to make. It follows that any event e associated with a unitary operation Q_e is the sole member of its cell. Note the measurements need not be complete in that the sum of the operators associated with a cell need not be the identity.

Proposition 15.14. *In a quantum test E, Q if Q_e is unitary, for an event $e \in E$, then the cell of e is a singleton.*

By Theorem 15.13, once provided with an initial state ρ, such a quantum test forms a probabilistic event structure with configuration-valuation $v(x) =_{\text{def}} \text{Tr}(A_x \rho A_x^\dagger)$ on its finite configurations x.

Example 15.15. A single measurement by the following quantum test***

Example 15.16. Quantum teleportation can be represented by the following quantum test***

15.3.2 Measurement with values

To support measurements yielding values we associate values with configurations of a quantum event structure E, Q, in the form of a measurable function, $V : \mathcal{C}^\infty(E) \to \mathbb{R}$. If the experiment results in $x \in \mathcal{C}^\infty(E)$ we obtain $V(x)$ as the measurement value resulting from the experiment. By Theorem 15.9, assuming an initial state given by a density operator ρ, we obtain a probability measure q_w on the sub-configurations of $w \in \mathcal{C}^\infty(E)$. This is interpreted as giving a probability distribution on the final results of an experiment w. Accordingly, w.r.t. an experiment $w \in \mathcal{C}^\infty(E)$, the expected value is

$$\mathbf{E}_w(V) =_{\text{def}} \int_{x \in \mathcal{F}_w} V(x) \; dq_w(x)$$

—*cf.* Section 14.5.1.

Traditionally quantum measurement is associated with an Hermitian operator A on $\mathcal{H}$ where the possible values of a measurement are eigenvalues of A. How is this realized by a quantum event structure? Suppose the Hermitian operator has spectral decomposition

$$A = \sum_{i \in I} \lambda_i P_i$$

where orthogonal projection operators P_i are associated with eigenvalue λ_i. The projection operators satisfy $\sum_{i \in I} P_i = \text{id}_{\mathcal{H}}$ and $P_i P_j = 0$ if $i \neq j$.

Form the quantum event structure with concurrent events e_i, for $i \in I$, and $Q(e_i) = P_i$. Because the projection operators are orthogonal, $[P_i, P_j] = 0$ when $i \neq j$, so we do indeed obtain a (strong) quantum event structure. Let $V(\{e_i\}) = \lambda_i$, and take arbitrary values on all other configurations. The event structure has a single, maximum configuration $w =_{\text{def}} \{e_i \mid i \in I\}$. It is the experiment w which will correspond to traditional measurement via A. Assume an initial state $|\psi\rangle\langle\psi|$. As above, the expected value of the experiment w is

$$\mathbf{E}_w(V) = \int_{x \in \mathcal{F}_w} V(x) \; dq_w(x) \,.$$

It can be checked that the probability ascribed to each of the singleton configurations $\{e_i\}$ is $\langle\psi|P_i|\psi\rangle$, and is zero elsewhere. Consequently,

$$\mathbf{E}_w(V) = \sum_{i\in I} \lambda_i \langle\psi|P_i|\psi\rangle = \langle\psi|A|\psi\rangle$$

—the well-known expression for the expected value of the measurement A on pure state $|\psi\rangle$.

Example 15.17. The spin state of a spin-1/2 particle is an element of two-dimensional Hilbert space, $\mathcal{H}_2$. Traditionally the Hermitian operator for measuring spin in a particular fixed direction is

$$|{\uparrow}\rangle\langle{\uparrow}| \;-\; |{\downarrow}\rangle\langle{\downarrow}|\,.$$

It has eigenvectors $|{\uparrow}\rangle$ (spin up) with eigenvalue +1 and $|{\downarrow}\rangle$ (spin down) with eigenvalue −1. Accordingly, its quantum event structure comprises the two concurrent events u associated with projector $|{\uparrow}\rangle\langle{\uparrow}|$ and d with projector $|{\downarrow}\rangle\langle{\downarrow}|$. Its configurations are:

$$\begin{tikzcd} & \{u,d\} & \\ \{u\} \arrow[ur, hook] & & \{d\} \arrow[ul, hook'] \\ & \varnothing \arrow[ul, hook] \arrow[ur, hook'] & \end{tikzcd}$$

The value associated with the configuration $\{u\}$ is +1, and that with $\{d\}$ is −1. Given an initial pure state $\psi = a|{\uparrow}\rangle + b|{\downarrow}\rangle$, the probability of the experiment $\{u,d\}$ yielding value +1 is $|a|^2$ and that of yielding −1 is $|b|^2$. The probability that the experiment ends in configurations $\varnothing$ or $\{u,d\}$ is zero. Its expected value is $|a|^2 - |b|^2$. This would be the average value resulting from measuring the spin of a large number of particles initially in pure state ψ. □

15.4 Probabilistic quantum experiments

It can be useful, or even necessary, to allow the choice of which quantum measurements to perform to be made probabilistically. For example, experiments to invalidate the Bell inequalities, to demonstrate the non-locality of quantum physics, make use of probabilistic quantum experiments.

Formally, a probability distribution over quantum experiments can be realized by a total map of event structures $f : P \to E$ where P, v is a probabilistic event structure and E, Q is a quantum event structure; the configurations of E correspond to quantum experiments assigned probabilities through P. Through the map f we can integrate the probabilistic and quantum features. Via the map f, the event structure E inherits a configuration valuation, making it itself a probabilistic event structure; we can see this indirectly by noting that if v_o is a continuous valuation on the open sets of P then $v_o f^{-1}$ is a continuous valuation on the open sets of E. On the other hand, via f the event structure P becomes a quantum event structure; an event $p \in P$ is interpreted as operation $Q(f(p))$. Of course, f can be the identity map, as is so in the example below.

Suppose E, Q is a quantum event structure with initial state ρ and a measurable value function $V : \mathcal{C}^\infty(E) \to \mathbb{R}$. Recall, from Section 15.3.2, that the expected value of a quantum experiment $w \in \mathcal{C}^\infty(E)$ is

$$\mathbf{E}_w(V) =_{\text{def}} \int_{w' \in \mathcal{F}_w} V(x)\ dq_w(w'),$$

where q_w is the probability measure induced on $\mathcal{F}_w$ by Q and ρ. The expected value of a probabilistic quantum experiment $f : P \to E$, where P, v is a probabilistic event structure is

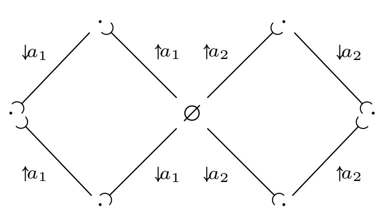
$$\int_{w \in \mathcal{C}^\infty(E)} \mathbf{E}_w(V)\ d\mu f^{-1}(w),$$

where μ is the probability measure induced on $\mathcal{C}^\infty(P)$ by the configuration-valuation v.

Example 15.18. Imagine an observer who randomly chooses between measuring spin in a first fixed direction $\mathbf{a_1}$ or in a second fixed direction $\mathbf{a_2}$. Assume that the probability of measuring in the $\mathbf{a_1}$-direction is p_1 and in the $\mathbf{a_2}$-direction is p_2, where $p_1 + p_2 = 1$. The two directions $\mathbf{a_1}$ and $\mathbf{a_2}$ correspond to choices of bases $|{\uparrow} a_1\rangle$, $|{\downarrow} a_1\rangle$ and $|{\uparrow} a_2\rangle$, $|{\downarrow} a_2\rangle$ in $\mathcal{H}_2$. We describe this scenario as a probabilistic quantum experiment. The quantum event structure has four events, ${\uparrow} a_1, {\downarrow} a_1, {\uparrow} a_2, {\downarrow} a_2$, in which ${\uparrow} a_1, {\downarrow} a_1$ are concurrent, as are ${\uparrow} a_2, {\downarrow} a_2$; all other pairs of events are in conflict. The event ${\uparrow} a_1$ is associated with measuring spin up in direction $\mathbf{a_1}$ and the event ${\downarrow} a_1$ with measuring spin down in direction $\mathbf{a_1}$. Similarly, events ${\uparrow} a_2$ and ${\downarrow} a_2$ correspond to measuring spin up and down, respectively, in direction $\mathbf{a_2}$. Correspondingly, we associate events with the following projection operators:

$$Q({\uparrow} a_1) = |{\uparrow} a_1\rangle\langle{\uparrow} a_1|\,, \qquad Q({\downarrow} a_1) = |{\downarrow} a_1\rangle\langle{\downarrow} a_1|\,,$$
$$Q(u_2) = |{\uparrow} a_2\rangle\langle{\uparrow} a_2|\,, \qquad Q(d_2) = |{\downarrow} a_2\rangle\langle{\downarrow} a_2|\,.$$

The configurations of the event structure take the form



where we have taken the liberty of inscribing the events just on the covering intervals. Measurement in the $\mathbf{a_1}$-direction corresponds to the configuration

$\{\uparrow a_1, \downarrow a_1\}$—the configuration to the far left in the diagram—and in the $\mathbf{a_2}$-direction to the configuration $\{\uparrow a_2, \downarrow a_2\}$—that to the far right. To describe that the probability of the measurement in the $\mathbf{a_1}$-direction is p_1 and that in the $\mathbf{a_2}$-direction is p_2, we assign a configuration valuation v for which

$$\begin{aligned} &v(\{\uparrow a_1, \downarrow a_1\}) = v(\{\uparrow a_1\}) = v(\{\downarrow a_1\}) = p_1\,, \\ &v(\{\uparrow a_2, \downarrow a_2\}) = v(\{\uparrow a_2\}) = v(\{\downarrow a_2\}) = p_2 \ \text{ and } \ v(\varnothing) = 1\,. \end{aligned}$$

Such an probabilistic quantum experiment is not very interesting on its own. But imagine that there are two similar observers A and B measuring the spins in directions $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{b_1}$, $\mathbf{b_2}$, respectively, of two particles created so that together they have zero angular momentum, ensuring they have a total spin of zero in any direction. Then quantum mechanics predicts some remarkable correlations between the observations of A and B, even at distances where their individual choices of what directions to perform their measurements could not possibly be communicated from one observer to another. For example, were both observers to choose the same direction to measure spin, then if one measured spin up then other would have to measure spin down even though the observers were light years apart.

We can describe such scenarios by a probabilistic quantum experiment which is essentially a simple parallel composition of two versions of the (single-observer) experiment above. In more detail, make two copies of the single-observer event structure: that for A, the event structure E_A, has events $\uparrow a_1, \downarrow a_1, \uparrow a_2, \downarrow a_2$, while that for B, the event structure E_B, has events $\uparrow b_1, \downarrow b_1, \uparrow b_2, \downarrow b_2$. Assume they possess configuration valuations v_A and v_B, respectively, determined by the probabilistic choices of directions made by A and B. Write Q_A and Q_B for the respective assignments of projection operators to events of E_A and E_B. The probabilistic event structure for the two observers together is got as $E_A \| E_B$ with configuration valuation $v(x) = v_A(x_A) \times v_B(x_B)$, for $x \in \mathcal{C}(E_A \| E_B)$, where x_A and x_B are projections of x to configurations of A and B. In this compound system an event such as *e.g.* $\uparrow a_1$ is interpreted as the projection operator $Q_A(\uparrow a_1) \otimes \mathrm{id}_{\mathcal{H}_2}$ on the Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_2$, where the combined state of the two particles belongs. We can capture the correlation or anti-correlation of the observers' measurements for spin through a value function on configurations given by

$$V(\{\uparrow a_i, \uparrow b_j\}) = V(\{\downarrow a_i, \downarrow b_j\}) = 1\,, \quad V(\{\uparrow a_i, \downarrow b_j\}) = V(\{\downarrow a_i, \uparrow b_j\}) = -1\,, \ \text{and}$$
$$V(x) = 0 \text{ otherwise.}$$

For example, assuming an initial state, the correlation between A observing in direction $\mathbf{a_i}$ and B in direction $\mathbf{b_j}$ is $\mathbf{E}_w(V)$ where w is the experiment

$$\{\uparrow a_i, \downarrow a_i, \uparrow b_j, \downarrow b_j\}\,.$$

□

15.5 More general quantum event structures

Definition 15.19. A *(general) quantum event structure* comprises an event structure $(E, \leq, \mathrm{Con})$ together with a functor Q from the partial-order $(\mathcal{C}(E), \subseteq)$ (regarded as a category) to the monoid of 1-bounded operators on $\mathcal{H}$ (regarded as a one-object category) which satisfy

$$\mathrm{id}_{\mathcal{H}} - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} Q(y, \bigcup_{i\in I} x_i)^{\dagger} Q(y, \bigcup_{i\in I} x_i)$$

is a positive operator, for all $y \subseteq x_1, \cdots, x_n$ with $\{x_1, \cdots, x_n\}\uparrow$.

Proposition 15.20. *Assume an assignment $Q(x,y)$ of 1-bounded operators on $\mathcal{H}$ to all covering intervals $x \mathrel{-\!\!\!\subset} y$ in $\mathcal{C}(E)$, such that*

$$Q(x_1, y)\, Q(x, x_1) = Q(x_2, y)\, Q(x, x_2)$$

whenever

$$\begin{tikzcd}
& y & \\
x_1 \arrow[ur, dash] & & x_2 \arrow[ul, dash] \\
& x \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

and

$$\mathrm{id}_{\mathcal{H}} - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} Q(y, \bigcup_{i\in I} x_i)^{\dagger} Q(y, \bigcup_{i\in I} x_i)$$

is a positive operator, whenever $y \mathrel{-\!\!\!\subset} x_1, \cdots, x_n$ with $\{x_1, \cdots, x_n\}\uparrow$. Then, extending Q to all intervals $x \subseteq y$ by defining

$$Q(x,y) =_{\mathrm{def}} Q(x_{n-1}, y)\, Q(x_{n-2}, x_{n-1}) \cdots Q(x, x_1)$$

for any covering chain

$$x \mathrel{-\!\!\!\subset} x_1 \mathrel{-\!\!\!\subset} \cdots \mathrel{-\!\!\!\subset} x_{n-2} \mathrel{-\!\!\!\subset} x_{n-1} \mathrel{-\!\!\!\subset} y$$

yields a general quantum event structure E, Q.

Corollary 15.21. *A simple quantum event structure E with assignment $e \mapsto Q_e$ of unitary or projection operators to events e, determines a general quantum event structure E, Q for which $Q(x,y) = Q_e$ when $x \overset{e}{\mathrel{-\!\!\!\subset}} y$.*

Theorem 15.22. *Let E, Q be a general quantum event structure with initial state a density operator ρ. Each configuration $x \in \mathcal{C}(E)$ is associated with an unnormalized density operator*

$$\rho_x =_{\mathrm{def}} Q(\emptyset, x)\, \rho\, Q(\emptyset, x)^{\dagger}$$

and a value in $[0,1]$ *given by*

$$v(x) =_{\rm def} {\rm Tr}(\rho_x) = {\rm Tr}(Q(\varnothing, x)^\dagger Q(\varnothing, x)\, \rho)\,.$$

For any $w \in \mathcal{C}^\infty(E)$*, the function* v *restricts to a configuration-valuation* v_w *on finite configurations in the family of configurations*

$$\mathcal{F}_w =_{\rm def} \{x \in \mathcal{C}^\infty(E) \mid x \subseteq w\}\,;$$

hence v_w *extends to a unique probability measure* q_w *on the Borel sets of* $\mathcal{F}_w$*.*

We would like a result showing how to realize a general quantum event structure from a simple quantum event structure by projection, possibly with tracing-out.

15.6 Quantum strategies

We define a *quantum game* to comprise $A, pol, \mathcal{H}_A, Q$ where A, pol is a race-free event structure with polarity and A, Q is a quantum event structure, with Hilbert space $\mathcal{H}_A$. A quantum game *with initial state* is a quantum game with ρ a density operator.

A *strategy* in a quantum game A, pol, Q comprises a probabilistic strategy in A, so a strategy $\sigma : S \to A$ together with configuration-valuation v on $\mathcal{C}(S)$.

Given a strategy $v_S, \sigma : S \to A$ and counter-strategy $v_T, \tau : T \to A^\perp$ in a quantum game A, Q we obtain a probabilistic event structure P via pull-back, *viz.*

$$\begin{tikzcd} & P \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, "\vee" near start] & \\ S \arrow[dr, "\sigma"'] & & T \arrow[dl, "\tau"] \\ & A & \end{tikzcd}$$

with a configuration-valuation $v(x) =_{\rm def} v_S \Pi_1(x) \times v_T \Pi_2(x)$ on finite configurations $x \in \mathcal{C}(P)$. This induces a probabilistic measure μ on the event structure P. Write $f =_{\rm def} \sigma\Pi_1 = \tau\Pi_2$. We can interpret $f : P \to A$ as the probabilistic quantum experiment which results from the interaction of the strategy σ and the counter-strategy τ.

Suppose now the quantum game has an initial state ρ. We now investigate the probability the interaction of σ with τ produces a result in a Borel subset U of of $\mathcal{C}^\infty(A)$, that the probabilistic experiment the interaction induces succeeds in U.

First note that P becomes a quantum event structure via the map f to the quantum event structure A: the assignment of operators is given by the composition of Q with f. By Theorems 15.9 and 15.22, w.r.t. any $x \in \mathcal{C}^\infty(P)$, we obtain a probability measure q_x on $\mathcal{F}_x =_{\text{def}} \{x' \in \mathcal{C}^\infty(P) \mid x' \subseteq x\}$. Write f_x for the restriction of f to $\mathcal{F}_x$. The expression

$$q_x(f_x^{-1}U)$$

gives the probability of obtaining a result in U conditional on $x \in \mathcal{C}^\infty(P)$. I believe (***but haven't yet proved***) that the function

$$x \mapsto q_x(f_x^{-1}U)$$

from $\mathcal{C}^\infty(P)$ to $[0,1]$ is measurable, making the function a random variable. If so, the probability of a result in $U \subseteq \mathcal{C}^\infty(A)$ is given by the Lebesgue integral

$$\int q_x(f_x^{-1}U)\, d\mu(x)\,.$$

We examine some special cases.

Consider the case where σ and τ are deterministic, with configuration valuations assigning one to each finite configuration. Then, P will also be deterministic in the sense that all its finite subsets will be consistent. It will thus have a single maximal configuration $w \in \mathcal{C}^\infty(P)$. The configuration-valuation v will assign one to each finite configuration of P. In this case the probability measure on Borel subsets V of $\mathcal{C}^\infty(P)$ is simple to describe:

$$\mu(V) = \begin{cases} 1 & \text{if } w \in V\,, \\ 0 & \text{otherwise,} \end{cases}$$

leading to

$$\int q_x(f_x^{-1}U)\, d\mu(x) = q_w(f^{-1}U)\,.$$

Consider now the case where Opponent initially offers $n \in \{1, \cdots, N\}$ mutually-inconsistent alternatives to Player and resumes with a deterministic strategy. Suppose too that initially Player chooses amongst the alternatives probabilistically, choosing option n with probability p_n, and then resumes deterministically. This will result in an event structure P taking the form of a prefixed sum $\sum_{1 \le n \le N} e_n.P_n$ in which all the events of P_n causally depend on event e_n. In this situation,

$$\int q_x(f_x^{-1}U)\, d\mu(x) = \sum_{1 \le n \le N} p_n \cdot q_{w_n}(f_n^{-1}U)\,,$$

where w_n is the maximal configuration of $e_n.P_n$ and $f_n : e_n.P_n \to A$ is the restriction of f, for $1 \le n \le N$.

Example 15.23. Quantum-coin tossing demonstrates the extra power quantum moves can have over classical moves. Initially Player and Opponent are presented with a quantum coin in the form of a qubit, the two bits being associated with heads H or tails T. ***

15.7 A bicategory of quantum games

Quantum games inherit the structure of a bicategory from probabilistic games. A strategy *from* a quantum game A *to* a quantum game B is a strategy in the quantum game $A^\perp \| B$. For this to make sense we have to extend the definitions of simple parallel composition and dual to quantum games. Assume A and B are quantum games. In defining their simple parallel composition $A\|B$ and dual $A^\perp$ we take:

$$\mathcal{H}_{A\|B} = \mathcal{H}_A \otimes \mathcal{H}_B, \quad Q_{A\|B}(1,a) = Q_A \otimes \mathrm{id}_{\mathcal{H}_B} \ \text{ and } \ Q_{A\|B}(2,b) = \mathrm{id}_{\mathcal{H}_A} \otimes Q_B\,;$$
$$\mathcal{H}_{A^\perp} = \mathcal{H}_A \ \text{ and } \ Q_{A^\perp} = Q_A\,.$$

Although we do obtain a bicategory of quantum games in this way, it is not likely to be the final story. One possible awkwardness is that we need to supply initial states, before we can determine the probabilities of quantum experiments. Perhaps the simple parallel composition of games, $A\|B$, is not the most appropriate for quantum games in that it would appear to exclude moves introducing entanglement between the two games. A more apt parallel composition might obtain by basing games directly on Hilbert spaces with parallel composition as tensor; then quantum games can result, *e.g.* by Definition 15.8. There is also the issue of adjoining value functions (*cf.* Section 15.3.2) to quantum games in a way that respects their bicategorical structure. Providing a structured account and analysis of quantum experiments, as in the simple experiment discussed in Example 15.18, should provide guidelines.

Acknowledgments I originally tried unsuccessfully to build a definition of quantum event structures around the decoherence/consistency conditions used in the decoherent/consistent histories approach to quantum theory; the conditions appear to be too sensitive to what one considers to be the initial and final events of a finite configuration. Both Prakash Panangaden and Samson Abramsky suggested the alternative of basing compatiblity more directly, and more traditionally, on the commutation of operators, which led to the definitions above.

Chapter 16

Event structures with disjunctive causes

*****introduction[1]

16.1 Motivation

within distributed strategies

hiding and parallel causes

how to attribute differing probabilities to differing parallel causes

More generally, through a careful analysis of the "ways" in which events occur, also

solves the problems of how to mix probability with nondeterminism, and higher-order

provides a compositional way to build up probability spaces **** For "convenience" probabilists generally separate the probability space from the space of values*** would be interesting to learn if this is sometimes used to build up probability space in a compositional fashion from simpler spaces.

16.2 Disjunctive causes and general event structures

Probabilistic strategies, as presented previously, do not cope with stochastic behaviour such as races as in the game

$$\ominus \rightsquigarrow \oplus \, .$$

[1]This and the following chapter are based on joint work with Marc de Visme for his M1 report for ENS Paris written while he was on an internship at Cambridge, Spring 2015.

To do such we would expect to have to equip events in the strategy with stochastic rates (which isn't hard to do if synchronisation events are not hidden). So this is to be expected. But at present probabilistic strategies do not cope with benign Player-Player races either! Consider the game

$$\begin{array}{ccc} & \oplus & \\ \ominus & & \ominus \end{array}$$

where Player would like a strategy in which they play a move iff Opponent plays one of theirs. We might stipulate that Player wins if a play of any $\ominus$ is accompanied by the play of $\oplus$ and *vice versa*. Intuitively a winning strategy would be got by assigning watchers (in the team Player) for each $\ominus$ who on seeing their $\ominus$ race to play $\oplus$. This strategy should win with certainty against any counter-strategy: no matter how Opponent plays one or both of their moves at least one of the watchers will report this with the Player move. But we cannot express this with event structures. The best we can do is a probabilistic strategy

$$\begin{tikzcd} \oplus \arrow[r, squiggly, dash] & \oplus \\ \ominus \arrow[u] & \ominus \arrow[u] \end{tikzcd}$$

with configuration valuation assigning 1/2 to configurations containing either Player move and 1 otherwise. Against a counter-strategy with Opponent playing one of their two moves with probability 1/2 this strategy only wins half the time. In fact, the strategy together with the counter-strategy form a Nash equilibrium when a winning configuration for Player is assigned payoff +1 and a loss −1 — see Section **??**. This strategy really is the best we can do presently in that it is optimal amongst those expressible using the simple (prime) event structures.

If we are to be able to express the intuitively strategy which wins with certainty we need to develop distributed probabilistic strategies to allow 'disjunctive' causal dependence as in 'general event structures' $(E, \vdash, \mathrm{Con})$ which allow *e.g.* two distinct compatible causes $X \vdash e$ and $Y \vdash e$. In this specific strategy both Opponent moves would enable the Player move, with all events being consistent.

But, as we'll see, for general event structures there is problem with the operation of hiding.

16.3 General event structures and families

A *general event structure*[**?**, **?**] is a structure $(E, \mathrm{Con}, \vdash)$ where E is a set of event occurrences, the consistency relation Con is a non-empty collection of finite subsets of E satisfying

$$X \subseteq Y \in \mathrm{Con} \implies X \in \mathrm{Con}$$

and the *enabling relation* $\vdash \subseteq \mathrm{Con} \times E$ satisfies

$$Y \in \mathrm{Con} \;\&\; Y \supseteq X \;\&\; X \vdash e \implies Y \vdash e\,.$$

A *configuration* is a subset of E which is

consistent: $X \subseteq_{\text{fin}} x \implies X \in \text{Con}$ and

secured: $\forall e \in x \exists e_1, \cdots, e_n \in x.\ e_n = e \ \&\ \forall i \leq n.\ \{e_1, \cdots, e_{i-1}\} \vdash e_i$.

Write $\mathcal{C}^\infty(E)$ for the configurations of E and $\mathcal{C}(E)$ for its finite configurations.

The notion of secured has been expressed through the existence of a securing chain to express an enabling of an event within a set which is a complete enabling in the sense that everything in the securing chain is itself enabled by earlier members of the chain. One can imagine more refined ways in which to express complete enablings which are rather like proofs, perhaps as trees or partial orders in which events are enabled by those events earlier in the order. Later the idea that complete enablings are consistent partial orders of events in which all events are enabled by earlier events in the order will play an important role in generalising general event structures to structures suitable for supporting strategies with parallel causes and their attendant constructions.

A *map* of general event structures $f : (E, \text{Con}, \vdash) \to (E', \text{Con}', \vdash')$ is a partial function $f : E \rightharpoonup E'$ such that

$X \in \text{Con} \implies fX \in \text{Con}' \ \&\ \forall e_1, e_2 \in X.\ f(e_1) = f(e_2) \implies e_1 = e_2$ and

$X \vdash e \ \&\ f(e)$ is defined $\implies fX \vdash' f(e)$.

It follows that the image fx of a configuration x of E is itself a configuration and moreover that

$$\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2 .$$

Maps compose as partial functions with identity maps being identity functions. Write $\mathcal{GES}$ for the category of general event structures.

A *family of configurations* comprises a family $\mathcal{F}$ of sets such that

if $X \subseteq \mathcal{F}$ is finitely compatible in $\mathcal{F}$ then $\bigcup X \in F$; and

if $e \in x \in \mathcal{F}$ then there exists a securing chain $e_1, \cdots, e_n = e$ in x s.t. $\{e_1, \cdots, e_i\} \in \mathcal{F}$ for all $i \leq n$.

The latter condition is equivalent to saying (i) that whenever $e \in x \in \mathcal{F}$ there is a finite $x_0 \in \mathcal{F}$ s.t. $e \in x_0 \in \mathcal{F}$ and (ii) that if $e, e' \in x$ and $e \neq e'$ then there is $y \in \mathcal{F}$ with $y \subseteq x$ s.t. $e \in y \iff e' \neq y$. The elements of the underlying set $\bigcup \mathcal{F}$ stand for *events*.

Such a family is *stable* when for any compatible non-empty subset X of $\mathcal{F}$ its intersection $\bigcap X$ is a member of $\mathcal{F}$.

A configuration $x \in \mathcal{F}$ is *irreducible, with top element* e iff $e \in x$ and $\forall y \in \mathcal{F}.\ e \in y \subseteq x$ implies $y = x$. Notice that because the top element of an irreducible has a securing chain the irreducible is a finite set with a unique top element, e. Irreducibles coincide with complete join irreducibles w.r.t. the order of inclusion. There is a maximum configuration $\widehat{x}$ strictly included in any irreducible x with top e; so $\widehat{x} \overset{e}{\longrightarrow\!\!\subset} x$. It is tempting to think of irreducibles as representing minimal complete enablings (as I did for a while). But, as sets, irreducibles both lack sufficient structure: in the formulation we are led to, several minimal complete enabling can correspond to the same irreducible; and are not general enough: in our formulation of minimal complete enabling there are minimal complete enablings whose underlying set is not an irreducible.

A map between families of configurations from $\mathcal{F}$ to $\mathcal{G}$ is a partial function $f : \bigcup \mathcal{F} \rightharpoonup \bigcup \mathcal{G}$ between their events such that for any $x \in \mathcal{F}$ its image $fx \in \mathcal{G}$ and

$$\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2 \,.$$

Maps compose as partial functions with identity maps being identity functions. We obtain a category $\mathcal{SFam}$ of families of configurations.

The forgetful functor from $\mathcal{GES}$ to $\mathcal{SFam}$ taking a general event structure to its family of configurations has a left adjoint, which constructs a canonical general event structure from a family: Let $\mathcal{A}$ be a family of configurations with underlying events A. Construct a general event structure

$$ges(\mathcal{A}) =_{\mathrm{def}} (A, \mathrm{Con}, \vdash)$$

with

- $X \in \mathrm{Con}$ iff $X \subseteq_{\mathrm{fin}} y$, for some $y \in \mathcal{A}$, and
- $X \vdash a$ iff $a \in A$, $X \in \mathrm{Con}$ and $a \in y \subseteq X \cup \{a\}$, for some $y \in \mathcal{A}$.

The unit of the adjunction has typical component $\mathrm{id}_A : \mathcal{A} \to \mathcal{C}^\infty(ges(\mathcal{A}))$ given as the identity function on events.

Theorem 16.1. *Let $\mathcal{A} \in \mathcal{SFam}$ with underlying set A. Then, $\mathcal{A} = \mathcal{C}^\infty(ges(\mathcal{A}))$.*

Suppose $B = (B, \mathrm{Con}_B, \vdash_B) \in \mathcal{GES}$ and that $g : \mathcal{A} \to \mathcal{C}^\infty(B)$ is a map in $\mathcal{Fam}_\equiv$. Then, $g : ges(\mathcal{A}) \to B$ in $\mathcal{GES}$.

The functor from $\mathcal{GES}$ to $\mathcal{SFam}$ taking a map of general event structures to the corresponding map of families of configurations has a left adjoint acting as ges on objects. The unit of the adjunction has typical component $\mathrm{id}_A : \mathcal{A} \to \mathcal{C}^\infty(ges(\mathcal{A}))$ given as the identity function on events A of a family of configurations $\mathcal{A}$.

The above yields a coreflection of families of configurations in general event structures. It cuts down to an equivalence between families of configurations and *replete* event structures. Say a general event structure $(E, \mathrm{Con}, \vdash)$ is *replete* when ϵ_E is an isomorphism. A general event structure E is replete iff

$$\begin{aligned}
&\forall e \in E \exists X \in \mathrm{Con}.\ X \vdash e \,, \\
&\forall X \in \mathrm{Con} \exists x \in \mathcal{C}(E).\ X \subseteq x \text{ and} \\
&X \vdash e \implies \exists x \in \mathcal{C}(E).\ e \in x \ \&\ x \subseteq X \cup \{e\} \,.
\end{aligned}$$

The last condition is equivalent to stipulating that each minimal enabling $X \vdash e$—where X is a minimal consistent set enabling e—corresponds to an irreducible configuration $X \cup \{e\}$.

Sometimes when it's important to disambiguate general event structures from those we have studied previously we shall use 'prime event structures' for event structures of the form $(E, \leq, \mathrm{Con})$. We can regard such a prime event structure as a (replete) general event structure $(E, \mathrm{Con}, \vdash)$ where $X \vdash e$ iff $X \in \mathrm{Con}$, $e \in E$ and $[e) \subseteq X$.

Clearly the partial functions which are maps of prime event structures can be understood as maps of the associated general event structures. We obtain a full embedding of prime event structures $\mathcal{SE}$ in $\mathcal{GES}$, and indeed in $\mathcal{F}$ as the general event structures in the image are replete. Neither of these is a left adjoint (despite what is claimed in [**?**]). However, later, in Section 16.12, we shall recover an adjunction from prime to (replete) general event structures at the slight cost of adding an equivalence relation to prime event structures and their maps.

Remark Although general event structures do not support hiding, so do not support strategies fully, their relative simplicity recommends them as a model for strategies with parallel causes provided they carry unhidden neutral events (so called *partial strategies* [**?**]), which have advantages when it comes to operational semantics and more discriminating equivalences. This line of research is being followed up in the PhD work of Tamas Kispeter.

16.4 The problem

With one exception, all the operations we have used in building strategies and, in particular, the bicategory of strategies extend easily to general event structures. The one exception, that of hiding, has been crucial in building a bicategory.

We present an argument to show general event structures are not closed under hiding. The following describes a general event structure.

> Events: a, b, c, d and e.
>
> Enabling: (1) $b, c \vdash e$ and (2) $d \vdash e$, with all events other than e being enabled by the empty set.
>
> Consistency: all subsets are consistent unless they contain the events a and b; in other words, the events a and b are in conflict.

Any configuration will satisfy the assertion

$$(a \wedge e) \implies d$$

because if e has occurred it has to have been enabled by (1) or (2) and if a has occurred its conflict with b has prevented the enabling (1), so e can only have occurred via enabling (2).

Now imagine the event b is hidden, so allowed to occur invisibly in the background. The "configurations after hiding" are those obtained by hiding (*i.e.* removing) the invisible event b from the configurations of the original event structure. The assertion above will still hold of the configurations after hiding.

There isn't a general event structure with events a, c, d and e, and configurations those which result when we hide (or remove) b from the configurations of the original event structure. One way to see this is to observe that amongst the configurations after hiding we have

$$\{c\} \mathrel{-\!\!\!\subset} \{c, e\} \text{ and } \{c\} \mathrel{-\!\!\!\subset} \{a, c\}$$

where both $\{c, e\}$ and $\{a, c\}$ have upper bound $\{a, c, d, e\}$, and yet $\{a, c, e\}$ is not a configuration after hiding as it fails to satisfy the assertion. (In the configurations of any general event structure if $x \mathrel{-\!\!\!\subset} y$ and $x \mathrel{-\!\!\!\subset} z$ and y and z are bounded above, then $y \cup z$ is a configuration.)

The first general event structure can be built out of the composition *without hiding* of strategies described by general event structures, one from a game A to a game B and the other from B to C; the second structure, not a general event structure, arises when hiding the events over the intermediate game B.

To obtain a bicategory of strategies with disjunctive causes we need to support hiding. We need to look for structures more general than general event structures. The example above gives a clue: the inconsistency is one of inconsistency between (minimal complete) enablings rather than events.

16.5 Adding disjunctive causes to prime event structures

To cope with disjunctive causes and hiding we must go beyond general event structures. We introduce structures in which we *objectify* cause; a minimal complete causal enabling is no longer an instance of a relation but a structure that realises that instance (*cf.* a proof in contrast to an entailment, or judgement of theorem-hood). This is in order to express inconsistency between minimal complete enablings, inexpressible as inconsistencies on events, that can arise when hiding.

Fortunately we can do this while staying close to prime event structures. The twist is to regard "disjunctive events" as comprising subsets of events of a prime event structure, the events of which are thought of as representing "prime causes," *i.e.* a particular formalisation of minimal complete enablings. Technically, we do this by extending prime event structures with an equivalence relation on its events.

In detail, an event structure with equivalence (an ese) is a structure

$$(P, \leq, \mathrm{Con}_P, \equiv)$$

where $(P, \leq, \mathrm{Con}_P)$ satisfies the axioms of a (prime) event structure and $\equiv$ is an equivalence relation on P.

The intention is that the events of P represent *prime causes* while the $\equiv$-equivalence classes of P represent *disjunctive events*: p in P is a prime cause of the event $\{p\}_{\equiv}$. Notice there may be several prime causes of the same event and that these may be parallel causes in the sense that they are consistent with each other and causally independent.

A *configuration* of the ese is a configuration of $(P, \leq, \mathrm{Con}_P)$ and we shall use the notation of earlier on event structures $\mathcal{C}^\infty(P)$ and $\mathcal{C}(P)$ for its configurations, respectively finite configurations. Say a configuration is *unambiguous* when it has no two distinct elements which are $\equiv$-equivalent. We modify the relation of concurrency and say $p_1, p_2 \in P$ are *concurrent* and write $p_1 co\, p_2$ iff $[p_1] \cup [p_2]$ is an unambiguous configuration of P and neither $p_1 \leq p_2$ nor $p_2 \leq p_1$.

An ese dissociates the two roles of enabling and atomic action conflated in the events of a prime event structures. The elements of P are to be thought of as minimal complete enablings and the equivalence classes as actions representing the occurrence of at least one prime cause.

When the equivalence relation $\equiv$ of an ese is the identity we essentially have a prime event structure. This view is reinforced in our choice of maps. A map from $(P, \leq_P, \mathrm{Con}_P, \equiv_P)$ to $(Q, \leq_Q, \mathrm{Con}_Q, \equiv_Q)$ is a partial function $f : P \rightharpoonup Q$ which *preserves* $\equiv$, *i.e.*

> if $p_1 \equiv_P p_2$ then either both $f(p_1)$ and $f(p_2)$ are undefined or both defined with $f(p_1) \equiv_Q f(p_2)$

s.t. for all $x \in \mathcal{C}(P)$

(i) the direct image $fx \in \mathcal{C}(Q)$, and

(ii) $\forall p_1, p_2 \in x.\ f(p_1) \equiv_Q f(p_2) \implies p_1 \equiv_P p_2$.

Maps compose as partial functions with the usual identity.

We sometimes use an alternative description of maps:

Proposition 16.2. *A map of* ese's from P to Q is a partial function $f : P \rightharpoonup Q$ which preserves $\equiv$ s.t.

(i) for all $X \in \mathrm{Con}_P$ the direct image $fX \in \mathrm{Con}_Q$ and $\forall p_1, p_2 \in X.\ f(p_1) \equiv_Q f(p_2) \implies p_1 \equiv_P p_2$, and

(ii) whenever $q \leq_Q f(p)$ there is $p' \leq_P p$ s.t. $f(p') = q$.

Such maps preserve the concurrency relation.

We regard two maps $f_1, f_2 : P \to Q$ as equivalent, and write $f_1 \equiv f_2$, iff they are equi-defined and yield equivalent results, *i.e.*

if $f_1(p)$ is defined then so is $f_2(p)$ and $f_1(p) \equiv_Q f_2(p)$, and
if $f_2(p)$ is defined then so is $f_1(p)$ and $f_1(p) \equiv_Q f_2(p)$.

Composition respects $\equiv$: if $f_1, f_2 : P \to Q$ with $f_1 \equiv f_2$ and $g_1, g_2 : Q \to R$ with $g_1 \equiv g_2$, then $g_1 f_1 \equiv g_2 f_2$. Write $\mathcal{ES}_\equiv$ for the category of ese's; it is enriched in the category of sets with equivalence relations (sometimes called setoids).

Ese's support a hiding operation. Let $(P, \leq, \mathrm{Con}_P, \equiv)$ be an ese. Let $V \subseteq P$ be a $\equiv$-closed subset of 'visible' events. Define the *projection* of P on V, to be $P{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V, \equiv_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$ and $v \equiv_V v'$ iff $v \equiv v'$ & $v, v' \in V$.

Hiding is associated with a factorisation of partial maps. Let

$$f : (P, \leq_P, \mathrm{Con}_P, \equiv_P) \to (Q, \leq_Q, \mathrm{Con}_Q, \equiv_Q)$$

be a partial map between two ese's. Let

$$V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\}\,.$$

Then f factors into the composition

$$P \xrightarrow{f_0} P{\downarrow}V \xrightarrow{f_1} Q$$

of f_0, a partial map of ese's taking $p \in P$ to itself if $p \in V$ and undefined otherwise, and f_1, a total map of ese's acting like f on V. We call f_1 the *defined part* of the partial map f. Because $\equiv$-equivalent maps share the same domain of definition, $\equiv$-equivalent maps will determine the same projection and $\equiv$-equivalent defined parts. We say a map $f : E \to E'$ is a *projection* if its defined part is an isomorphism. The factorisation is characterised to within isomorphism by the following universal characterisation: for any factorisation $P \xrightarrow{g_0} P_1 \xrightarrow{g_1} Q$ where g_0 is partial and g_1 is total there is a (necessarily total) unique map $h : P{\downarrow}V \to P_1$ such that

$$\begin{tikzcd} P \arrow[r, "f_0"] \arrow[dr, "g_0"'] & P{\downarrow}V \arrow[r, "f_1"] \arrow[d, dashed, "h"] & Q \\ & P_1 \arrow[ur, "g_1"'] & \end{tikzcd}$$

commutes.

16.6 Equivalence families

We shall relate ese's to general event structures by an adjunction (strictly, a form of pseudo adjunction or biadjunction as it shall rely on the enrichment by equivalence). This will provide a way to embed families of configurations and so replete general event structures in ese's. The adjunction will factor through a more basic adjunction to families of configurations which also bear an equivalence relation on their underlying sets (we'll call them equivalence-families). This latter adjunction provides a full embedding of ese's in ef's and is itself important as it provides a way to do key constructions such as bipullback within ese's; just as it can be hard to constructions such as pullback within event structures, so that we often rely on first carrying out the constructions in stable families.

A family with equivalence or an *equivalence-family* (ef) is a family of configurations $\mathcal{A}$ with an equivalence relation $\equiv_A$ on its underlying set $A =_{\mathrm{def}} \bigcup \mathcal{A}$.

We can identify a family of configurations $\mathcal{A}$ with the equivalence family $(\mathcal{A}, =)$, taking the equivalence to be simply equality on the underlying set.

Let $(\mathcal{A}, \equiv_A)$ and $(\mathcal{B}, \equiv_B)$ be ef's, with respective underlying sets A and B. A map $f : (\mathcal{A}, \equiv_A) \to (\mathcal{B}, \equiv_B)$ is a partial function $f : A \rightharpoonup B$ which preserves $\equiv$ s.t. $x \in \mathcal{A} \implies fx \in \mathcal{B}\ \&\ \forall a_1, a_2 \in x.\ f(a_1) \equiv_B f(a_2) \implies a_1 \equiv_A a_2$. Composition is composition of partial functions. We regard two maps

$$f_1, f_2 : (\mathcal{A}, \equiv_A) \to (\mathcal{B}, \equiv_B)$$

as equivalent, and write $f_1 \equiv f_2$, iff they are equidefined and yield equivalent results. Composition respects $\equiv$. This yields a category of equivalence families $\mathcal{F}am_\equiv$; it is enriched in the category of sets with equivalence relations.

Later *stable* ef's will come to play an important role. In an equivalence family $(\mathcal{A}, \equiv_A)$ say a configuration $x \in \mathcal{A}$ is *unambiguous* iff

$$\forall a_1, a_2 \in x.\ a_1 \equiv_A a_2 \implies a_1 = a_2\,.$$

An equivalence family $(\mathcal{A}, \equiv_A)$, with underlying set of events A, is *stable* iff it satisfies

$$\forall x, y, z \in \mathcal{A}.\ x, y \subseteq z\ \&\ z \text{ is unambiguous} \implies x \cap y \in \mathcal{A} \text{ and}$$
$$\forall a \in A, x \in \mathcal{A}.\ a \in x \implies \exists z \in \mathcal{A}.\ z \text{ is unambiguous }\ \&\ a \in z \subseteq x\,.$$

In effect a stable equivalence family contains a stable subfamily of unambiguous configurations out of which all other configurations are obtainable as unions. Local to any unambiguous configuration there is a partial order on its events.

Clearly we can regard an ese $(P, \leq, \mathrm{Con}, \equiv_P)$ as an ef $(\mathcal{C}^\infty(P), \equiv_P)$ and a function which is a map of ese's as a map between the associated ef's and this operation is functorial. However, the converse, how to construct an ese from a family, is much less clear. To do so we follow up on the idea introduced in Section 16.3 of basing minimal complete enablings on partial orders. A minimal complete enabling will correspond to an *extremal (causal) realisations* with top. realisations and how to obtain extremal realisations, among these the primes with a top element, will be our topic over the next few sections.

16.7 Realisations

Let $\mathcal{A}$ be a family of configurations with underlying set A.

Definition 16.3. A *(causal) realisation* comprises a partial order

$$(E, \leq)\,,$$

its *carrier*, such that the set $\{e' \in E \mid e' \leq e\}$ is finite for all events $e \in E$, together with a function

$$\rho : E \to A$$

s.t. its image $\rho E \in \mathcal{A}$ and

$$\forall e \in E.\ \rho\{e' \in E \mid e' \leq e\} \in \mathcal{A}\,.$$

(Equivalently, instead of the latter condition, we can say ρ sends down-closed subsets of its carrier E to configurations of $\mathcal{A}$.)

We say the realisation ρ is *injective* when ρ is injective as a function.

We define maps between realisations $(E, \leq), \rho$ and $(E', \leq'), \rho'$ as partial surjective functions $f : E \rightharpoonup E'$ s.t.

$$\forall e \in E.\ f(e) \text{ is defined } \implies \rho(e) = \rho'(f(e))\ \&\ f\{e_0 \in E \mid e_0 \leq e\} \supseteq \{e' \in E' \mid e' \leq' f(e)\}\,.$$

Equivalently we could define such a map as a partial surjective function $f : E \rightharpoonup E'$ which preserves down-closed subsets and satisfies $\rho(e) = \rho'(f(e))$ when $f(e)$ is defined. It is convenient to write such a map as

$$f : \rho \geq \rho' \quad \text{or} \quad \rho \geq^f \rho'\,.$$

Occasionally we shall write $\rho \geq \rho'$, or the converse $\rho' \leq \rho$, to mean there is a map of realisations from ρ to ρ'.

Such a map factors into a "projection" followed by a total map, as

$$\rho \geq_1^{f_1} \rho_0 \geq_2^{f_2} \rho'$$

where ρ_0 stands for the realisation $(E_0, \leq_0), \rho_0$ where

$$E_0 = \{r \in R \mid f(r) \text{ is defined}\}\,,$$

the domain of definition of f, with $\leq_0$ the restriction of $\leq$, and f_1 is the inverse relation to the inclusion $E_0 \subseteq E$, and f_2 is the total function $f_2 : E_0 \to E'$. We are using $\geq_1$ and $\geq_2$ to signify the two kinds of maps. Notice that $\geq_1$-maps are reverse inclusions. Notice too that $\geq_2$-maps are exactly the total maps of realisations. Total maps $\rho \geq_2^f \rho'$ are precisely those surjective functions f from the carrier of ρ to the carrier of ρ' which preserve down-closed subsets and satisfy $\rho = \rho' f$.

We shall say a realisation ρ is *extremal* when

$$\rho \geq_2^f \rho' \implies f \text{ is an isomorphism}$$

for any realisation ρ'.

16.8 Extremal realisations

Let $\mathcal{A}$ be a configuration family with underlying set A. Any realisation in $\mathcal{A}$ can be coarsened to an extremal realisation.

Lemma 16.4. *For any realisation ρ there is an extremal realisation ρ' with $\rho \geq_2^f \rho'$.*

Proof. The category of realisations with total maps has colimits of total-order diagrams. A diagram d from a total order $(I, \leq)$ to realisations, comprises a collection of total maps of realisations $d_{i,j} : d(i) \to d(j)$ when $i \leq j$ s.t. $d_{i,i}$ is always the identity map and if $i \leq j$ and $j \leq k$ then $d_{i,k} = d_{j,k} \circ d_{i,j}$. We suppose each realisation $d(i)$ has carrier $(E_i, \leq_i)$ with $d(i) : E_i \to A$. We construct the colimit realisation of the diagram as follows.

The elements of the colimit realisation consist of equivalence classes of elements of the disjoint union

$$E =_{\text{def}} \biguplus_{i \in I} E_i$$

under the equivalence

$$(i, e_i) \sim (j, e_j) \iff \exists k \in I.\ i \leq k \ \&\ j \leq k \ \&\ d_{i,k}(e_i) = d_{j,k}(e_j)\,.$$

Consequently we may define a function $\rho_E : E \to A$ by taking $\rho_E(\{e_i\}_\sim) = \rho_i(e_i)$. Because every $d_{i,j}$ is a surjective function, every equivalence class in E has a representative in E_i for every $i \in I$. Moreover, for any $e \in E$ there is $k \in I$ s.t.

$$\{e' \in E \mid e' \leq_E e\} = \{\{e'_k\}_\sim \mid e'_k \leq_k e_k\}\,,$$

where $e = \{e_k\}_\sim$, so is finite. It follows that ρ_E is a realisation. The maps $f_i : \rho_i \geq_2 \rho_E$, where $i \in I$, given by $f_i(e_i) = \{e_i\}_\sim$ form a colimiting cone.

Suppose ρ is a realisation. Consider all total-order diagrams d from a total order $(I, \leq)$ to realisations starting from ρ with $d_{i,j}$ not an isomorphism if $i < j$. Amongst them there is a maximal diagram by Zorn's lemma. From the maximality of the diagram its colimit is necessarily extremal. In more detail, construct a colimiting cone $f_i : d(i) \geq_2 \rho_E$, $i \in I$, with the same notation as above. By maximality of the diagram some f_k must be an isomorphism; otherwise we could extend the diagram by adding a top element to the total order and sending it to ρ_E. If j should satisfy $k < j$ then $f_j \circ d_{k,j} = f_k$ so $f_k^{-1} f_j \circ d_{k,j} = \mathrm{id}_{E_k}$. It would follow that $d_{k,j}$ is injective, as well as surjective, it being a total map of realisations, and consequently that $d_{k,j}$ is an isomorphism—a contradiction. Hence k is the maximum element in $(I, \leq)$. If the colimit were not extremal we could again adjoin a new top element above k thus extending the diagram—a contradiction. □

Corollary 16.5. *Every countable configuration of a family of configurations has an injective extremal realisation.*

Proof. Let x be a countable configuration of a family of configurations $\mathcal{A}$. By serialising the countable configuration,

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$$

where $\{e_1, \cdots, e_n\} \in \mathcal{A}$ for all n, we obtain an injective realisation ρ. By Lemma 16.4 we can coarsen ρ to an extremal realisation ρ' with $\rho \geq_2^f \rho'$. As $\rho = \rho' f$ the surjective function f is also injective, so a bijection, ensuring that the extremal realisation ρ' is also injective. □

The following lemma and corollary are central.

Lemma 16.6. *Assume* $(R, \leq), \rho$, $(R_0, \leq_0), \rho_0$ *and* $(R_1, \leq_1), \rho_1$ *are realisations.*
(i) Suppose $f : \rho \succeq_1^{f_1} \rho_0 \succeq_2^{f_2} \rho_1$. *Then there are maps so that* $f : \rho \succeq_2^{g_2} \rho' \succeq_1^{g_1} \rho_1$, *as shown below:*

$$\begin{tikzcd} \rho \arrow[r, dotted, "g_2"] \arrow[d, "f_1"'] & \rho' \arrow[d, dotted, "g_1"] \\ \rho_0 \arrow[r, "f_2"] & \rho_1 \end{tikzcd}$$

(ii) Suppose $\rho \succeq_1^{f_1} \rho_0$ *where* R_0 *is not a down-closed subset of* R. *Then there are maps so* $f_1 = \rho \succeq_2^{g_2} \rho' \succeq_1^{g_1} \rho_0$ *with* g_2 *not an isomorphism:*

$$\begin{tikzcd} \rho \arrow[r, dotted, "g_2"] \arrow[d, "f_1"'] & \rho' \arrow[dl, dotted, "g_1"] \\ \rho_0 & \end{tikzcd}$$

Proof. (i) Construct the realisation $(R', \leq'), \rho'$ as follows. Define

$$R' = (R \setminus R_0) \cup R_1$$

where w.l.o.g. we assume the sets $R \setminus R_0$ and R_1 are disjoint. Define the function $g_2 : R \to R'$ to act as the identity on elements of $R \setminus R_0$ and as f_2 on elements of R_0. Because f_2 reflects the order so does g_2, and g_2 preserves down-closed subsets.

When $b \in R \setminus R_0$, define

$$a \leq' b \ \text{ iff } \ \exists a_0 \in R.\ a_0 \leq b \ \&\ g_2(a_0) = a\,.$$

When $b \in R_1$, define

$$a \leq' b \ \text{ iff } \ a \in R_1 \ \&\ a \leq_1 b\,.$$

Define ρ' to act as ρ on elements of $R \setminus R_0$ and as ρ_1 on elements of R_1. Then $\rho = \rho' g_2$ directly. To see $\leq'$ is a partial order observe that reflexivity and antisymmetry follow directly from the corresponding properties of $\leq$ and $\leq_1$. Transitivity requires an argument by cases. For example, in the most involved case, where

$$c \leq' a \text{ with } a \in R_1 \text{ and } a \leq' b \text{ with } b \in R \setminus R_0$$

we obtain

$$c \leq_1 a \text{ and } a_0 \leq b$$

for some $a_0 \in R_0$ with $f_2(a_0) = a$. As f_2 is surjective and reflects the order,

$$c_0 \leq_0 a_0 \text{ and } a_0 \leq b$$

for some $c_0, \in R_0$ with $f_2(c_0) = c$. Consequently, $c_0 \leq b$ with $g_2(c_0) = c$, making $c \leq' b$, as required for transitivity.

We should check that ρ' is a realisation. Let $b \in R'$. If $b \in R_1$ then $\rho'[b]' = \rho_1[b]_1 \in \mathcal{C}(A)$. If $b \in R \smallsetminus R_0$ then $\rho'[b]' = \rho g_2[b]$ the image under ρ of the down-closed subset $g_2[b]$, so in $\mathcal{C}(A)$.

We have already remarked that g_2 reflects the order and $\rho = \rho' g_2$ making it a map of realisations. This concludes the proof of (i).
(ii) This follows from the construction of $(R' \leq'), \rho'$ used in (i) but in the special case where f_2 is the identity map. Then $R' = R$ but $\leq' \neq \leq$ as there is $e \in R_0$ with $[e]_0 \subsetneq [e]$ ensuring that $[e]' = [e]_0 \neq [e]$. □

Corollary 16.7. *If ρ is extremal and $\rho \succeq^f \rho'$, then ρ' is extremal and there is ρ_0 s.t. $f : \rho \succeq_1 \rho_0 \cong \rho'$. Moreover, the carrier R_0 of ρ_0 is a down-closed subset of the carrier R of ρ, with order the restriction of that on R.*

Proof. Directly from Lemma 16.6. Assume ρ is extremal and $\rho \succeq^f \rho'$. We can factor f into $\rho \succeq_1^{f_1} \rho_0 \succeq_2^{f_2} \rho'$. From (i), if ρ_0 were not extremal nor would ρ be—a contradiction; hence f_2 is an isomorphism. From (ii), the carrier R_0 of ρ_0 has to be a down-closed subset of the carrier R of ρ, as otherwise we would contradict the extremality of ρ. □

It follows that if ρ is extremal and $\rho \succeq^f \rho'$ then ρ' is extremal and the inverse relation $g =_{\text{def}} f^{-1}$ is an injective function preserving and reflecting down-closed subsets, *i.e.* $g[r'] = [g(r')]$ for all $r' \in R'$. In other words:

Corollary 16.8. *If ρ is extremal and $\rho \succeq^f \rho'$, then ρ' is extremal and the inverse $g =_{\text{def}} f^{-1}$ is a rigid embedding from the carrier of ρ' to the carrier of ρ s.t. $\rho' = \rho f$.*

Lemma 16.9. *Let $(R, \leq), \rho$ be an extremal realisation. Then*

(i) if $r' \leq r$ and $\rho(r) = \rho(r')$ then $r = r'$;

(ii) if $[r) = [r')$ and $\rho(r) = \rho(r')$ then $r = r'$.

Proof. (i) Suppose $r' \leq r$ and $\rho(r) = \rho(r')$. By Corollary 16.8, we may project to $[r]$ to obtain an extremal realisation $\rho_0 : [r] \to A$. Suppose r and r' were unequal. We can define a realisation as the restriction of ρ_0 to $[r)$. The function from $[r]$ to $[r)$ taking r to r' and otherwise acting as the identity function is a map of realisations from the realisation ρ_0 and clearly not an isomorphism, showing ρ_0 to be non-extremal—a contradiction. Hence $r = r'$, as required.
(ii) Suppose $[r) = [r')$ and $\rho(r) = \rho(r')$. Projecting to $[r, r']$ we obtain an extremal realisation. If r and r' were unequal there would be a non-isomorphism map to the realisation obtained by projecting to $[r]$, *viz.* the map from $[r, r']$ to $[r]$ sending r' to r and fixing all other elements. □

By modifying condition (i) in the lemma above a little we obtain a characterisation of extremal realisations:

Lemma 16.10. *Let $(R, \leq), \rho$ be a realisation. Then ρ is extremal iff* *(i) if* $X \subseteq [r)$*, with* X *down-closed and* $r \in R$*, and* $\rho(X \cup \{r\}) \in \mathcal{A}$ *then* $X = [r)$*; and*

(ii) if $[r) = [r')$ *and* $\rho(r) = \rho(r')$ *then* $r = r'$*.*

Proof. "Only if": Assume ρ is extremal. We have already established (ii) in Lemma 16.9. To show (i), suppose X is down-closed and $X \subseteq [r)$ in R with $\rho(X \cup \{r\}) \in \mathcal{A}$. By Corollary 16.8, we may project to $[r]$ to obtain an extremal realisation $\rho_0 : [r] \to A$. Modify the restricted order $[r]$ to one in which $r' \leq r$ iff $r' \in X$, and is otherwise unchanged. The same underlying function ρ_0 remains a realisation, call it ρ_0', on the modified order. The identity function gives us a map $f : \rho_0 \geq_2 \rho_0'$ which is an isomorphism between realisations iff $X = [r)$.
"If": Assume (i) and (ii). Suppose $f : \rho \geq_2 \rho'$, where R', ρ' is a realisation. We show f is injective and order-preserving. As f is presumed to be surjective and to preserve down-closed subsets we can then conclude it is an isomorphism.

To see f is injective suppose $f(r_1) = f(r_2)$. W.l.o.g. we may suppose r_1 and r_2 are minimal in the sense that

$$r_1' \leq r_1 \ \&\ r_2' \leq r_2 \ \&\ f(r_1') = f(r_2') \implies r_1' = r_1 \ \&\ r_2' = r_2\,.$$

Define $r' =_{\text{def}} f(r_1) = f(r_2)$. Then

$$[r'] \subseteq f[r_1] \ \&\ [r'] \subseteq f[r_2]\,.$$

Furthermore, by the minimality of r_1, r_2,

$$[r') \subseteq f[r_1) \ \&\ [r'] \subseteq f[r_2)\,.$$

It follows that

$$[r') \subseteq f[r_1) \cap f[r_2) = f([r_1) \cap [r_2))$$

where the equality is again a consequence of the minimality of r_1, r_2. Taking $X =_{\text{def}} [r_1) \cap [r_2)$ we have $(fX) \cup \{r'\}$ is down-closed in R'. Therefore

$$\rho(X \cup \{r_1\}) = \rho' f(X \cup \{r_1\}) = \rho'(fX \cup \{r'\}) \in \mathcal{A}\,.$$

By condition (ii), $X = [r_1)$. Similarly, $X = [r_2)$, so $[r_1) = [r_2)$. Obviously $\rho(r_1) = \rho' f(r_1) = \rho' f(r_1) = \rho(r_2)$, so we obtain $r_1 = r_2$ by (i).

We now check that f preserves the order. Let $r \in R$. Define

$$X =_{\text{def}} [\{r_1 \leq r \mid f(r_1) < f(r)\}]\,,$$

where the square brackets signify down-closure in R. Then X is down-closed in R by definition and $X \subseteq [r)$. We have $[f(r)] \subseteq f[r]$ whence

$$fX = f[r] \cap [f(r)) = [f(r))\,.$$

Therefore $fX \cup \{f(r)\}$ is down-closed in R', so

$$\rho(X \cup \{r\}) = \rho' f(X \cup \{r\}) = \rho'(fX \cup \{f(r)\}) \in \mathcal{A}\,.$$

Hence $X = [r)$, by (ii). It follows that

$$\begin{aligned} r_1 \rightarrowtriangle r &\implies r_1 \in X \\ &\implies f(r_1) < f(r) \text{ in } R'. \end{aligned}$$

As the order on R is the transitive closure of immediate dependency, this in turn that f preserves the order. □

Lemma 16.11. *There is at most one map between extremal realisations.*

Proof. Let $(R, \leq), \rho$ and $(R', \leq'), \rho'$ be extremal realisations. Let $f, f' : \rho \to \rho'$ be maps with converse relations g and g' respectively. We show the two functions g and g' are equal, and hence so are their converses f and f'. Suppose otherwise that $g \neq g'$. Then there is an $\leq$-minimal $r' \in R'$ for which $g(r') \neq g'(r')$ and $g[r) = g'[r')$. Hence $[g(r')) = [g'(r'))$ and $\rho(g(r')) = \rho'(r') = \rho(g'(r'))$. As ρ is extremal, by Lemma 16.9(ii) we obtain $g(r') = g'(r')$—a contradiction. □

Hence extremal realisations of A under $\preceq$ form a preorder. The *order of extremal realisations* has as elements isomorphism classes of extremal realisations ordered according to the existence of a map between representatives of isomorphism classes. Alternatively, we could take a choice of representative from each isomorphism class and order these according to whether there is a map from one to the other. We say a realisation has a top element when its carrier contains an element which dominates all other elements in the carrier. In fact, the following is a direct corollary of Proposition 16.17 in the next section.

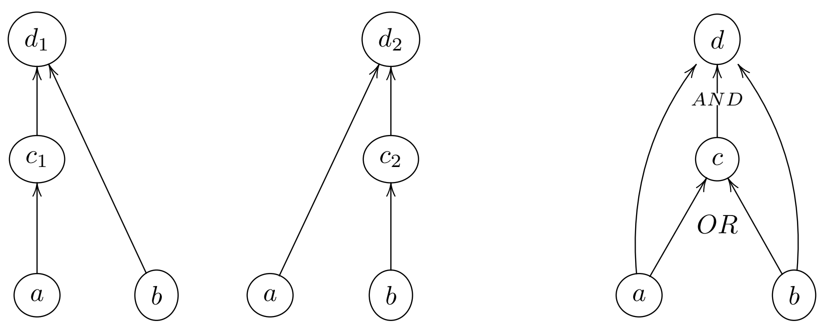
Proposition 16.12. *The order of extremal realisations of a family of configurations $\mathcal{A}$ forms a prime-algebraic domain [1] with complete primes represented by those extremal realisations which have a top element.*

The proofs of the following observations are straightforward. They emphasise that extremal realisations with top are for our purposes (among them to develop probabilistic strategies with parallel causes) an appropriate generalisation of (complete) primes when we move from prime event structures to general event structures.

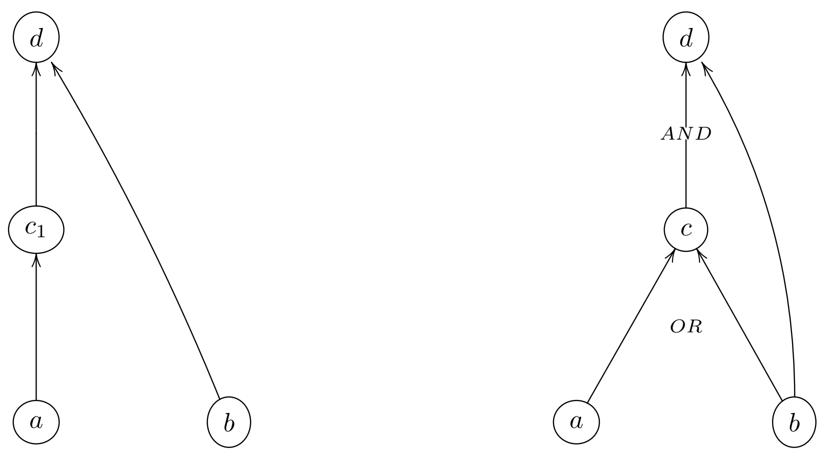
Proposition 16.13. *Let $(A, \leq_A, \mathrm{Con}_A)$ be a prime event structure. For an extremal realisation $(R, \leq_R), \rho$ of $\mathcal{C}^\infty(A)$, the function $\rho : R \to \rho R$ is an order isomorphism between $(R, \leq_R)$ and the configuration $\rho R \in \mathcal{C}^\infty(A)$ ordered by the restriction of $\leq_A$. The function taking an extremal realisation $(R, \leq_R), \rho$ to the configuration ρR is an order isomorphism from the order of extremal realisations of $\mathcal{C}^\infty(A)$ to the configurations of A; extremal realisations with a top correspond complete primes of $\mathcal{C}^\infty(A)$.*

We conclude with examples illustrating the nature of extremal realisations. It is convenient to describe families of configurations by general event structures, taking advantage of the economic representation they provide.

Example 16.14. This and the following example shows that prime extremal realisations do not correspond to irreducible configurations. Below, on the right we show a general event structure with irreducible configuration $\{a, b, c, d\}$. On the left we show two prime extremals with tops d_1 and d_2 which both have the same irreducible configuration $\{a, b, c, d\}$ as their image. The lettering indicates the functions associated with the realisations, *e.g.* events d_1 and d_2 in the partial orders map to d in the general event structure.

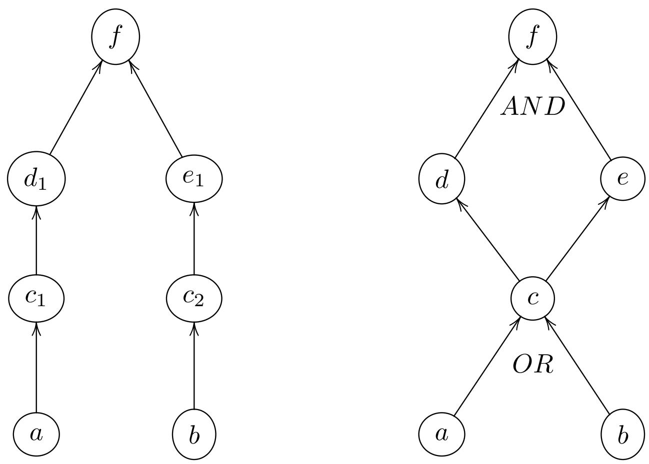


Example 16.15. On the other hand there are prime extremal realisations of which the image is not an irreducible configuration. Below the prime extremal on the left describes a situation where d is enabled by b and c being enabled by a. It has image the configuration $\{a, b, c, d\}$ which is not irreducible, being the union of the two configurations $\{a\}$ and $\{b, c, d\}$.



Example 16.16. It is also possible to have prime extremal realisations in which an event depends on another event having been enabled in two distinct ways,

as in the following extremal realisation on the left.



The extremal describes the event f being enabled by d and e where they are in turn enabled by different ways of enabling c. Although an extremal (with top element) it is clearly not an injective realisation.

16.9 An adjunction from $\mathcal{ES}_{\equiv}$ to $\mathcal{F}am_{\equiv}$

We exhibit an adjunction (precisely, a very simple case of biadjunction) from $\mathcal{ES}_{\equiv}$, the category of ese's, to $\mathcal{F}am_{\equiv}$, the category of equivalence families.

The left adjoint $I : \mathcal{ES}_{\equiv} \to \mathcal{F}am_{\equiv}$ is the full and faithful functor which takes an ese to its family of configurations with the original equivalence.

The right adjoint $er : \mathcal{F}am_{\equiv} \to \mathcal{ES}_{\equiv}$ is defined on objects as follows. Let $\mathcal{A}$ be an equivalence family with underlying set A. Define $er(\mathcal{A}) = (P, \mathrm{Con}_P, \leq_P, \equiv_P)$ where

- P consists of a choice from within each isomorphism class of those extremals p of $\mathcal{A}$ with a top element—we write $top(p)$ for the image of the top element in A;
- Causal dependency $\leq_P$ is $\leq$ on P;
- $X \in \mathrm{Con}_P$ iff $X \subseteq_{\mathrm{fin}} P$ and $top\,[X]_P \in \mathcal{A}$ —the set $[X]_P$ is the $\leq_P$-downwards closure of X, so equal to $\{p' \in P \mid \exists p \in X.\ p' \leq p\}$;
- $p_1 \equiv_P p_2$ iff $p_1, p_2 \in P$ and $top(p_1) \equiv_A top(p_2)$.

Proposition 16.17. *The configurations of P, ordered by inclusion, are order-isomorphic to the order of extremal realisations: an extremal realisation ρ corresponds, up to isomorphism, to the configuration $\{p \in P \mid p \leq \rho\}$ of P; conversely, a configuration x of P corresponds to an extremal realisation $top : x \to A$ with carrier $(x, \leq)$, the restriction of the order of P to x.*

Proof. It will be helpful to recall, from Corollary 16.8 , that if $\rho \geq^f \rho'$ between extremal realisations, then the inverse relation f^{-1} is a rigid embedding of (the carrier of) ρ' in (the carrier of) ρ; so $\rho' \leq \rho$ stands for a rigid embedding. Suppose $x \in \mathcal{C}^\infty(P)$. Then x determines an extremal realisation

$$\theta(x) =_{\rm def} top : (x, \leq) \to A\,.$$

The function $\theta(x)$ is a realisation because each p in x is, and extremal because, if not, one of the p in x would fail to be extremal, a contradiction. Clearly $\rho' \leq \rho$ implies $\theta(\rho') \subseteq \theta(\rho)$. Conversely, it is easily checked that any extremal realisation $\rho : (R, \leq) \to A$ defines a configuration $\{p \in P \mid p \leq \rho\}$. If $x \subseteq y$ in $\mathcal{C}^\infty(P)$ then $\phi(x) \leq \phi(y)$. It can be checked that θ and ϕ are mutual inverses, *i.e.* $\phi\theta(x) = x$ and $\theta\phi(\rho) \cong \rho$ for all configurations x of P and extremal realisations ρ. □

From the above proposition we see that the events of $er(\mathcal{A})$ correspond to completely-prime extremal realisations [1]. This justifies our future use of the term 'prime extremal' instead of the clumsier 'extremal with top element.'

The component of the counit of the adjunction $\epsilon_A : I(er(\mathcal{A})) \to \mathcal{A}$ is given by the function

$$\epsilon_A(p) = top(p)\,.$$

It is a routine check to see that ϵ_A preserves $\equiv$ and that any configuration x of P images under top to a configuration in $\mathcal{A}$, moreover in a way that reflects $\equiv$.

Let $Q = (Q, \mathrm{Con}_Q, \leq_Q, \equiv_Q)$ be an ese and $f : I(Q) \to \mathcal{A}$ a map in $\mathcal{F}am_\equiv$. We shall define a map $h : Q \to er(\mathcal{A})$ s.t. $f = \epsilon_A h$.

We define the map $h : Q \to er(\mathcal{A})$ by induction on the depth of Q. The depth of an event in an event structure is the length of a longest $\leq$-chain up to it—so an initial event has depth 1. We take the depth of an event structure to be the maximum depth of its events. (Because of our reliance on Lemma 16.4, we use the axiom of choice implicitly.)

Assume inductively that $h^{(n)}$ defines a map from $Q^{(n)}$ to $er(\mathcal{A})$ where $Q^{(n)}$ is the restriction of Q to depth below or equal to n such that $f^{(n)}$ the restriction of f to $Q^{(n)}$ satisfies $f^{(n)} = \epsilon_A h^{(n)}$. (In particular, $Q^{(0)}$ is the empty ese and $h^{(0)}$ the empty function.) Then, by Proposition 16.17, any configuration x of $Q^{(n)}$ determines an extremal realisation $\rho_x : h^{(n)}x \to A$ with carrier $(h^{(n)}x, \leq)$.

Suppose $q \in Q$ has depth $n+1$. If $f(q)$ is undefined take $h^{(n+1)}(q)$ to be undefined. Otherwise, note there is an extremal realisation $\rho_{[q)}$ with carrier $(h[q), \leq)$. Extend $\rho_{[q)}$ to a realisation $\rho^\top_{[q)}$ with carrier that of $\rho_{[q)}$ with a new top element $\top$ adjoined, and make $\rho^\top_{[q)}$ extend the function $\rho_{[q)}$ by taking $\top$ to $f(q)$. By Lemma 16.4, there is an an extremal realisation ρ such that $\rho^\top_{[q)} \geq_2 \rho$. Because $\rho_{[q)}$ is extremal $\rho_{[q)} \leq_1 \rho$, so ρ only extends the order of $\rho_{[q)}$ with extra dependencies of $\top$. (For notational simplicity we identify the carrier of ρ with the set $h[q) \cup \{\top\}$.) Project ρ to the extremal with top $\top$. Define this to be the value of $h^{(n+1)}(q)$. In this way, we extend $h^{(n)}$ to a partial function $h^{(n+1)} : Q^{(n+1)} \to er(\mathcal{A})$ such that $f^{(n+1)} = \epsilon_A h^{(n+1)}$. To see that $h^{(n+1)}$ is a map we can use Proposition 16.2. By construction $h^{(n+1)}$ satisfies property (ii) of Proposition 16.2 and the other properties are inherited fairly directly from f via the definition of $er(\mathcal{A})$.

Defining $h = \bigcup_{n\in\omega} h^{(n)}$ we obtain a map $h : Q \to er(\mathcal{A})$ such that $f = \epsilon_A h$.

Suppose $h' : Q \to er(\mathcal{A})$ is a map s.t. $f \equiv \epsilon_A \circ h'$. Then, for any $q \in Q$,

$$top(h'(q)) = \epsilon_A \circ h'(q) \equiv_A f(q) = \epsilon_A \circ h(q) = top(h(q))\,,$$

so $h'(q) \equiv_P h(q)$ in $er(\mathcal{A})$. Thus $h' \equiv h$.

In summary, we have proved the following:

Theorem 16.18. *Let $\mathcal{A} \in \mathcal{F}am_\equiv$. For all $f : I(Q) \to \mathcal{A}$ in $\mathcal{F}am_\equiv$, there is a map $h : Q \to er(\mathcal{A})$ in $\mathcal{ES}_\equiv$ such that $f = \epsilon_A \circ I(h)$ i.e. so the diagram*

$$\begin{tikzcd} A & I(er(\mathcal{A})) \arrow[l, "\epsilon_A"'] \\ & I(Q) \arrow[ul, "f"] \arrow[u, dotted, "I(h)"'] \end{tikzcd}$$

commutes. Moreover, if $h' : Q \to er(\mathcal{A})$ is a map in $\mathcal{ES}_\equiv$ s.t. $f \equiv \epsilon_A \circ I(h')$, i.e. the diagram above commutes up to $\equiv$, then $h' \equiv h$.

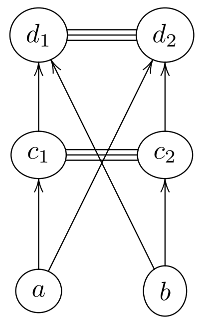
The theorem does not quite exhibit an adjunction, because the usual cofreeness condition specifying an adjunction is weakened to only having uniqueness up to $\equiv$. However the condition it describes does specify an exceedingly simple case of a biadjunction (or pseudo adjunction) between 2-categories—a set together with an equivalence relation (a *setoid*) is a very simple example of a category. As a consequence, whereas with the usual cofreeness condition allows us to extend the right adjoint to arrows, so obtaining a functor, in this case following that same line will only yield a pseudo functor er as right adjoint: thus extended, er will only preserve composition and identities up to $\equiv$.

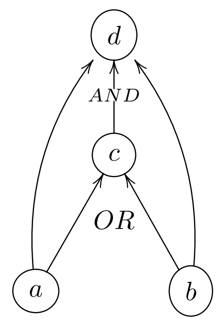
The map

$$(P, \equiv) \to er(\mathcal{C}^\infty(P), \equiv)$$

which takes $p \in P$ to the realisation with carrier $([p], \leq)$, the restriction of the causal dependency of P, with the inclusion function $[p] \hookrightarrow P$ is an isomorphism; recall from Proposition 16.13 that the configurations of a prime event structure correspond to its extremal realisations. Such maps furnish the components of the unit of the adjunction.

Example 16.19. On the right we show a general event structure and on its left the ese which its family of configurations (with equivalence the identity relation) gives rise to under the construction er:





16.10 An adjunction from $\mathcal{F}am_\equiv$ to $\mathcal{GES}$

The right adjoint $fam : \mathcal{GES} \to \mathcal{F}am_\equiv$ is most simply described. Given $(E, \mathrm{Con}, \vdash)$ in $\mathcal{GES}$ it returns the equivalence family $(\mathcal{C}^\infty(E), =)$ in $\mathcal{F}am_\equiv$ comprising the configurations together with the identity equivalence between events that appear within some configuration; the partial functions between events that are maps in $\mathcal{GES}$ are automatically maps in $\mathcal{F}am_\equiv$—the action of fam on maps.

For the effect of the left adjoint $col : \mathcal{F}am_\equiv \to \mathcal{GES}$ on objects, define the *collapse*

$$col(\mathcal{A}) =_{\mathrm{def}} (E, \mathrm{Con}, \vdash)$$

where

- $E = A_\equiv$, the equivalence classes of events in $A =_{\mathrm{def}} \bigcup \mathcal{A}$
- $X \in \mathrm{Con}$ iff $X \subseteq_{\mathrm{fin}} y_\equiv$, for some $y \in \mathcal{A}$
- $X \vdash e$ iff $e \in E$, $X \in \mathrm{Con}$ and $e \in y_\equiv \subseteq X \cup \{e\}$, for some $y \in \mathcal{A}$.

Let $(\mathcal{A}, \equiv) \in \mathcal{F}am_\equiv$. Assume that $\mathcal{A}$ has underlying set A. The unit of the adjunction is defined to have typical component $\eta_A : (\mathcal{A}, \equiv) \to fam(col(\mathcal{A}, \equiv))$ given by

$$\eta_A(a) = \{a\}_\equiv .$$

It is easy to check that η_A is a map in $\mathcal{F}am_\equiv$.

Theorem 16.20. *Suppose that $B = (B, \mathrm{Con}_B, \vdash_B) \in \mathcal{GES}$ and that $g : (\mathcal{A}, \equiv) \to (\mathcal{C}^\infty(B), =)$ is a map in $\mathcal{F}am_\equiv$. Then, there is a unique map $k : col(\mathcal{A}, \equiv) \to B$ in $\mathcal{GES}$ s.t. the diagram*

$$\begin{tikzcd}
(\mathcal{A}, \equiv) \arrow[r, "\eta_A"] \arrow[dr, "g"'] & fam(col(\mathcal{A}, \equiv)) \arrow[d, dotted, "fam(k)"] \\
& (\mathcal{C}^\infty(B), =)
\end{tikzcd}$$

commutes.

Proof. The map $k : col(\mathcal{A}, \equiv) \to B$ is given as the function

$$k(e) = g(a) \text{ where } e = \{a\}_\equiv .$$

It is easily checked to be a map in $\mathcal{GES}$ and moreover to be the unique map from $col(\mathcal{A}, \equiv)$ to B making the above diagram commute. □

Theorem 16.20 determines an adjunction from $\mathcal{F}am_\equiv$ to $\mathcal{GES}$. The construction *col* automatically extends from objects to maps; maps in $\mathcal{F}am_\equiv$ preserve equivalence so collapse to functions preserving equivalence classes.

The counit of the adjunction has components $\epsilon_E : col((\mathcal{C}^\infty(E), =)) \to E$ which send singleton equivalence classes $\{e\}$ to e. The conunit is an isomorphism at precisely those general event structures E which are replete.

16.11 An adjunction from $\mathcal{ES}_\equiv$ to $\mathcal{GES}$

Composing the adjunctions

$$\begin{tikzcd} \mathcal{ES}_\equiv \arrow[r, bend right, "I"'] \arrow[r, phantom, "\top"] & \mathcal{F}am_\equiv \arrow[l, bend right, "er"'] \arrow[r, bend right, "col"'] \arrow[r, phantom, "\top"] & \mathcal{GES} \arrow[l, bend right, "fam"'] \end{tikzcd}$$

we obtain a adjunction

$$\begin{tikzcd} \mathcal{ES}_\equiv \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathcal{GES} \arrow[l, bend right] \end{tikzcd} .$$

Strictly speaking this is only a pseudo adjunction because the first adjunction from $\mathcal{ES}_\equiv$ to $\mathcal{F}am_\equiv$ is only a pseudo adjunction.

The composite adjunction from $\mathcal{ES}_\equiv$ to $\mathcal{GES}$ cuts down to a reflection, in which the counit is a natural isomorphism, when we restrict to the subcategory of $\mathcal{GES}$ where all general event structures are replete. The right adjoint provides a full and faithful embedding of replete general event structures (and so families of configurations) in ese's. Recall the right adjoint constructs an ese out of the prime extremal realisations of a general event structure.

We can ask on what subcategory of $\mathcal{ES}_\equiv$ the adjunction further cuts down to an equivalence of categories. We now provide those extra axioms an ese's should satisfy in order that the subcategory of such is equivalent to that of replete general event structures. This amounts to characterising those ese's which are obtained to within isomorphism as images of replete general event structures under the right adjoint, or equivalently as images of families of configurations. The characterising axioms on an ese $(P, \leq, \mathrm{Con}, \equiv)$ are:

(A) For X a finite down-closed subset of P,
$X \equiv y \ \&\ y \in \mathcal{C}(P) \implies X \in \mathcal{C}(P)$;

(B) For $p, q \in P$, $[p) = [q) \ \&\ p \equiv q \implies p = q$;

(C) For X a down-closed subset of P and $p \equiv q$,
$X \subseteq [p) \ \&\ [q)_\equiv \subseteq X_\equiv \implies X = [p)$;

(D) For $x \in \mathcal{C}(P)$ and $t \in P$,
$x \cup [t] \in \mathcal{C}(P)$ & $(x \cup [t])_\equiv = x_\equiv \cup \{\{t\}_\equiv\} \implies \exists p \in P.\ p \equiv t$ & $x \cup \{p\} \in \mathcal{C}(P)$.

In writing the axioms we have used expressions such as $X \equiv Y$, for subsets X and Y of P, to mean for any $p \in X$ there is $q \in Y$ with $p \equiv q$ and *vice versa*; and $X_\equiv$ to stand for the set of $\equiv$-equivalence classes $\{\{p\}_\equiv \mid p \in X\}$; so $X \equiv Y$ iff $X_\equiv = Y_\equiv$.

Axiom (D) may be replaced by

(D′) For $x, y \in \mathcal{C}(P)$ and $t \in P$,
$x \stackrel{t}{\longrightarrow\!\!\!\!\subset}$ & $x \equiv y \implies \exists p \in P.\ p \equiv t$ & $x \cup \{p\} \in \mathcal{C}(P)$.

Assume (D) and, for $x, y \in \mathcal{C}(P)$, that $x \stackrel{t}{\longrightarrow\!\!\!\!\subset}$ and $x \equiv y$. Then, by (A), $y \cup [t] \in \mathcal{C}(P)$ as $y \cup [t] \equiv x \cup \{t\}$, clearly consistent; whence $y \cup \{p\} \in \mathcal{C}(P)$ for some p by (D). Conversely, assuming (D′) and $x \cup [t] \in \mathcal{C}(P)$ and $(x \cup [t])_\equiv = x_\equiv \cup \{\{t\}_\equiv\}$, in the case where $t \notin x$ we obtain $x \cup [t) \stackrel{t}{\longrightarrow\!\!\!\!\subset}$ and $x \cup [t) \equiv x$; whence $x \cup \{p\} \in \mathcal{C}(P)$ for some p by (D′). This shows (D) follows from (D′) in the case when $t \notin x$; in the case when $t \in x$, axiom (D) is obvious.

Theorem 16.21. *Let $P \in \mathcal{ES}_\equiv$. Then, $P \cong er(\mathcal{A})$ for some equivalence family $\mathcal{A}$ iff P satisfies axioms (A), (B), (C) and (D).*

Proof. We show axioms (A), (B), (C), (D) hold of any ese $P = er(\mathcal{A})$, constructed from a family of configurations $\mathcal{A}$. We obtain P satisfies axiom (A) from the way the consistency of $er(\mathcal{A})$ is defined: if $X \equiv y$, with y a configuration, X inherits consistency from y ensuring that X, assumed down-closed, is a configuration. If $[p) = [q)$ and $p \equiv q$, then p and q correspond to the same extremal realisation with top, so are equal—ensuring (B) holds of P. We obtain (C) via Lemma 16.10(i), as $[p]$ corresponds to an extremal with top p. Given the correspondence between configurations of P and extremal realisations, axiom (D) expresses an obvious extension property of extremal realisations.

Conversely, we now show that if an ese $P = (P, \mathrm{Con}, \leq, \equiv)$ satisfies (A), (B), (C), (D) then there is an isomorphism

$$\eta_P : P \cong er(\mathcal{A})$$

if we take the family of configurations so

$$\mathcal{A} = \mathcal{C}^\infty(\,col(\mathcal{C}^\infty(P), \equiv))\,.$$

Recall, from Proposition 16.17, that the configurations of $er(\mathcal{A})$ correspond to extremal realisations of $col(\mathcal{C}^\infty(P), \equiv)$.

Before we define the map η_P we remark that a configuration x of P determines an extremal realisation of $col(\mathcal{C}^\infty(P), \equiv)$: the realisation has carrier x with order inherited from P and map taking $p \in x$ to the equivalence class $\{p\}_\equiv$. Axioms (B) and (C) ensure that this realisation is extremal, via Lemma 16.10.

It follows from the remark that we define a map $\eta_P : P \to er(\mathcal{A})$ by sending $p \in P$ to the realisation with carrier $[p]$, ordered as in P, and function $[p] \to P_\equiv$

taking elements to their equivalence classes. The injectivity of η_P follows from (B). Moreover η_P reflects consistency because of axiom (A). We now only require its surjectivity to ensure η_P is an isomorphism.

We use (D) in showing that η_P is surjective. We show by induction on $n \in \omega$ that all extremal realisations with top of $col(P)$ of depth less than n are in the image of η_P. (Recall the depth of an event in an event structure is the length of a longest $\leq$-chain up to it; we take the depth of an event structure to be the maximum depth of its events.) Because η_P reflects consistency the induction hypothesis entails that all extremal realisations of depth less than n are (up to isomorphism) in the image under η_P of configurations of P.

Let $(R, \leq_R)$ of depth n with $\rho : R \to col(P)$ be an extremal realisation with top r, so $R = [r]_R$. Then its restriction $\rho' : [r)_R \to col(P)$ is an extremal realisation of lesser depth. By induction there is $x' \in \mathcal{C}(P)$ and an isomorphism of realisations $\theta' : \rho' \cong \eta_P x'$. Write $y =_{\mathrm{def}} \rho'[r)_R$, $z =_{\mathrm{def}} \rho[r]_R$. Then $y, z \in \mathcal{C}(col(P))$ and $y \xrightarrow{e}\!\!\subset z$ for some $e \in P_\equiv$. From the definition of $col(P)$, it follows fairly directly that there is some $t \in P$ s.t. $\{t\}_\equiv = e$ and $[t)_\equiv \subseteq y$. As η_P reflects consistency, $x' \cup [t] \in \mathcal{C}(P)$. We have

$$(x' \cup [t])_\equiv = x'_\equiv \cup \{\{t\}_\equiv\} = z\,.$$

By (D) there is some $p \in P$ s.t. $p \equiv t$ and $x' \cup \{p\} \in \mathcal{C}(P)$. The configuration $x =_{\mathrm{def}} x' \cup \{p\}$ with order inherited from P and map taking $p' \in x$ to $\{p'\}_\equiv$ is the realisation $\eta_P x$. Let θ be the function $\theta : R \to x$ extending θ' s.t. $\theta(r) = p$. Then $\theta : \rho \succeq \eta_P x$ is a map of realisations. But ρ is extremal ensuring $\theta : \rho \cong \eta_P x$, and that η_P is surjective. □

Corollary 16.22. *The adjunction from $\mathcal{ES}_\equiv$ to $\mathcal{GES}$ cuts down to a ***pseudo*** equivalence of categories between the subcategory of $\mathcal{ES}_\equiv$ satisfying axioms (A), (B), (C), (D) and the subcategory of $\mathcal{GES}$ comprising the replete general event structures.*

16.12 Coreflective subcategories of $\mathcal{ES}_\equiv$

Consider the following successively weaker axioms on $(P, \mathrm{Con}, \leq, \equiv)$:

Ax 0. $\{p_1, p_2\} \in \mathrm{Con}$ & $p_1 \equiv p_2 \implies p_1 = p_2$.

Ax 1. $p_1, p_2 \leq p$ & $p_1 \equiv p_2 \implies p_1 = p_2$.

Ax 2. $p_1 \leq p_2$ & $p_1 \equiv p_2 \implies p_1 = p_2$.

Ax 0 says that any two prime causes of disjunctive event are mutually exclusive. Ax 2 we have met as a consequence of a realisation being extremal (Lemma 16.9(i)) so it will always hold of any image under the construction er. Ax 1 forbids any prime cause from depending on two distinct prime causes of a common disjunctive event; while it does not hold of all extremal realisations (see Example 16.16) and so can fail in an image under the construction er, Ax 1 enforces a form atomicity on disjunctive events: whereas several prime causes of a disjunctive event may appear in a configuration, no other event is permitted to detect and react on the occurrence of a nontrivial conjunction of prime causes of the disjunctive event.

Restricting to the full subcategories of $\mathcal{ES}_\equiv$ satisfying these axioms we obtain $\mathcal{ES}^0_\equiv$, $\mathcal{ES}^1_\equiv$ and $\mathcal{ES}^2_\equiv$ respectively. The factorisation of maps we met for $\mathcal{ES}_\equiv$ is inherited by all the subcategories as their respective axioms are preserved by the projection operation. So all the subcategories support hiding.

The inclusion functors

$$\mathcal{ES}^0_\equiv \hookrightarrow \mathcal{ES}^1_\equiv \hookrightarrow \mathcal{ES}^2_\equiv \hookrightarrow \mathcal{ES}_\equiv$$

all have right adjoints so forming a chain of coreflections. Essentially the right adjoints work by restricting the structures to that part satisfying the stronger axiom. The adjunctions are enriched in the sense that the associated natural isomorphisms preserve and reflect the equivalence $\equiv$ between maps. (This would not be the case with relational maps.)

For example, $\mathcal{ES}^0_\equiv$ is the full subcategory of $\mathcal{ES}_\equiv$ in which objects

$$(P, \mathrm{Con}, \leq, \equiv)$$

satisfy the strongest axiom Ax 0. Consequently its maps are traditional maps of event structures which preserve equivalence. The inclusion functor $\mathcal{ES}^0_\equiv \hookrightarrow \mathcal{ES}_\equiv$ has a right adjoint $r : \mathcal{ES}_\equiv \to \mathcal{ES}^0_\equiv$ taking $Q = (Q, \mathrm{Con}_Q, \leq_Q, \equiv_Q)$ to $(Q', \mathrm{Con}', \leq', \equiv')$ where

Q' consists of all $q \in Q$ s.t. $q_1 \not\equiv_Q q_2$ for all $q_1, q_2 \leq_Q q$;
$X \in \mathrm{Con}'$ iff $X \subseteq Q'$ and $X \in \mathrm{Con}_Q$ and $q_1 \not\equiv_Q q_2$ for all $q_1, q_2 \in X$;
$\leq'$ and $\equiv'$ are the restrictions of $\leq_Q$ and $\equiv_Q$ to Q'.

The adjunction is enriched in the sense that the isomorphism

$$\mathcal{ES}^0_\equiv(P, r(Q)) \cong \mathcal{ES}^0_\equiv(P, Q)\,,$$

natural in $P \in \mathcal{ES}^0_\equiv$ and $Q \in \mathcal{ES}_\equiv$, preserves and reflects the equivalence $\equiv$ between maps.

As a consequence we obtain an adjunction from $\mathcal{ES}^0_\equiv$ to $\mathcal{GES}$.[2] The universality of counit is only up to $\equiv$.

The most important subcategory for us will be $\mathcal{ES}^1_\equiv$. The right adjoint to the inclusion

$$\mathcal{ES}^1_\equiv \hookrightarrow \mathcal{ES}_\equiv$$

on objects simply restricts them to those events which satisfy Ax 1. In general, within $\mathcal{ES}_\equiv$ we lose the local injectivity property that we're used to seeing for maps of event structures; the maps of event structures are injective from configurations, when defined. However for $\mathcal{ES}^1_\equiv$ we recover local injectivity w.r.t. prime configurations: If $f : P \to Q$ is a map in $\mathcal{ES}^1_\equiv$, then

$$p_1, p_2 \leq_P p \;\&\; f(p_1) = f(p_2) \implies p_1 = p_2\,.$$

[2] It was falsely claimed in [?] that the 'inclusion' of the category of prime event structures in that of general event structures had a right adjoint. The adjunction from $\mathcal{ES}^0_\equiv$ to $\mathcal{GES}$ corrects that originally incorrect idea; though the repair of that putative adjunction is at the cost of uniqueness up to $\equiv$.

In the composite adjunctions from $\mathcal{ES}^1_\equiv$ to $\mathcal{F}am_\equiv$, and from $\mathcal{ES}^1_\equiv$ to $\mathcal{GES}$, the right adjoint has the effect of restricting to those extremal realisations within which Ax 1 holds; recall that the prime extremal realisations of an equivalence family $\mathcal{A}$ correspond to the configurations of $er(\mathcal{A})$. Because such prime extremals are necessarily injective functions their carriers can be taken to be configurations of the equivalence family or general event structure of which they are realisations.

The coreflection from $\mathcal{ES}^0_\equiv$ to $\mathcal{ES}^1_\equiv$ is helpful in thinking about constructions like pullback and pseudo pullback in $\mathcal{ES}^1_\equiv$ as its right adjoint will preserve such limits. In the category $\mathcal{ES}^0_\equiv$, maps coincide with the traditional maps of labelled event structures, regarding events as labelled by their equivalence classes. Constructions such as pullback are already very familiar in $\mathcal{ES}^0_\equiv$. All that changes in the corresponding constructions in $\mathcal{ES}^1_\equiv$ is the manner of dealing with consistency.

The category $\mathcal{ES}^1_\equiv$ will be of special importance to us. Amongst the subcategories of $\mathcal{ES}_\equiv$ it is the smallest extension of prime event structures which supports parallel causes and hiding. It also has pullbacks and pseudo pullbacks, not the case for example in $\mathcal{ES}_\equiv$. It is within $\mathcal{ES}^1_\equiv$ that we shall develop probabilistic distributed strategies with parallel causes and be able to overcome the restrictions and difficulties explained in the introduction to this chapter. (Later objects of $\mathcal{ES}^1_\equiv$ will be renamed to *event structures with disjunctive causes* (edc's) and the category $\mathcal{ES}^1_\equiv$ to $\mathcal{EDC}$.)

16.13 A non-enriched coreflection

There is an obvious 'inclusion' functor from the category of event structures $\mathcal{ES}$ to the category $\mathcal{ES}^0_\equiv$; it takes an event structure to the same event structure but with the identity equivalence adjoined. Regarding $\mathcal{ES}^0_\equiv$ as a category, so dropping the enrichment by equivalence relations, the 'inclusion' functor

$$\mathcal{ES} \hookrightarrow \mathcal{ES}^0_\equiv$$

has a right adjoint, *viz.* the forgetful functor which simply drops the equivalence $\equiv$ from the ese. The adjunction is necessarily a coreflection because the inclusion functor is full. Of course it is no longer the case that the adjunction is enriched: the natural bijection of the adjunction cannot respect the equivalence on maps.

The adjunction

$$\mathcal{ES} \hookrightarrow \mathcal{ES}^1_\equiv$$

obtained as the composite of the adjunctions from $\mathcal{ES}$ to $\mathcal{ES}^0_\equiv$ and $\mathcal{ES}^0_\equiv$ to $\mathcal{ES}^1_\equiv$. If an edc P goes to event structure P_0 under the right adjoint, the configurations of P_0 are the unambiguous configurations of P. The adjunction is not enriched because that from $\mathcal{ES}$ to $\mathcal{ES}^0_\equiv$ isn't.

Despite this the adjunction from $\mathcal{ES}$ to $\mathcal{ES}^1_\equiv$ has many useful properties. Of importance for us is that the functor forgetting equivalence will preserve all limits and especially pullbacks. In composing strategies in edc's we shall only be involved with pseudo pullbacks of maps $f : A \to C$ and $g : B \to C$ in which C is essentially an event structure, *i.e.* an edc in which the equivalence is the identity relation. The construction of such pseudo pullbacks coincides with that of pullbacks. While this does not entail that composition of strategies is preserved by the forgetful functor—because the forgetful functor does not commute with hiding—it will give us a strong relationship, expressed as a map, between composition of strategies after and before applying the forgetful functor.

16.14 $\mathcal{ES}^1_\equiv$ and $\mathcal{SF}am_\equiv$—a coreflection

The closeness of $\mathcal{ES}^1_\equiv$ to prime event structures $\mathcal{ES}$ suggests a generalisation of stable families to aid with constructions such as product and pullback in $\mathcal{ES}^1$. The generalisation has in fact already appeared in Section 16.6. Recall, an equivalence family $\mathcal{A}, \equiv_A$, with underlying set of events A, is *stable* iff it satisfies

$$\forall x, y, z \in \mathcal{A}.\ x, y \subseteq z \ \&\ z \text{ is unambiguous} \implies x \cap y \in \mathcal{A} \text{ and}$$
$$\forall a \in A, x \in \mathcal{A}.\ a \in x \implies \exists z \in \mathcal{A}.\ z \text{ is unambiguous } \ \&\ a \in z \subseteq x\,.$$

(A configuration is unambiguous iff no two distinct elements are in the relation $\equiv$.) Given the other axioms of an ef, we can deduce the seemingly stronger property:

$$\emptyset \neq X \subseteq \mathcal{A}, z \in \mathcal{A}.\ (\forall x \in X.\ x \subseteq z)\ \&\ z \text{ is unambiguous} \implies \bigcap X \in \mathcal{A}\,.$$

We call $\mathcal{SF}am_\equiv$ the full subcategory of ef's with objects the stable ef's.

In effect a stable equivalence family $\mathcal{A}$ contains a stable subfamily $unamb\mathcal{A}$ of unambiguous configurations out of which all other configurations are obtainable as unions. There is an obvious 'inclusion' functor from the category of stable families $\mathcal{SF}am$ to $\mathcal{SF}am_\equiv$; it takes a stable family $\mathcal{A}$, with underlying set A, to the stable ef $(\mathcal{A}, \mathrm{id}_A)$. Its has *unamb* as a right adjoint:

$$\begin{tikzcd} \mathcal{SF}am \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathcal{SF}am_\equiv\,. \arrow[l, bend right, "unamb"'] \end{tikzcd}$$

As the 'inclusion' functor from $\mathcal{SF}am$ to $\mathcal{SF}am_\equiv$ is full the adjunction is a coreflection. The adjunction is not enriched in the sense that its natural bijection ignores the equivalence on maps present in $\mathcal{SF}am_\equiv$. As right adjoints preserve limits, the stable family of unambiguous configurations of the product, or pullback, of stable ef's is the product, respectively pullback, in stable families of the unambiguous configurations of the components.

Local to any unambiguous configuration there is a partial order on its events and we can extract an edc in $\mathcal{ES}^1_\equiv$ from a stable ef in the same way as we can extract an event structure from a stable family, though with a slight variation in the way consistency is determined. The construction appears as right adjoint to the 'inclusion' functor from $\mathcal{ES}^1_\equiv$ into the subcategory of stable equivalence families.

In more detail, the 'inclusion' functor from $\mathcal{ES}^1_\equiv$ takes an ese $(P, \leq, \mathrm{Con}, \equiv)$ to its ef $(\mathcal{C}^\infty(P), \equiv)$ with maps remaining the same partial function in translating from $\mathcal{ES}^1_\equiv$ to $\mathcal{SF}am_\equiv$. Its right adjoint takes a stable ef $(\mathcal{A}, \equiv_A)$ to the ese $(P, \leq, \mathrm{Con}, \equiv)$ where, making use of the fact that the subfamily of unambiguous configurations forms a stable family and the attendant notation,

$P = \{[a]_z \mid a \in z \in \mathcal{A}\ \&\ z$ is unambiguous$\}$, the prime unambiguous configurations;

$\leq$ is inclusion;

$X \in \mathrm{Con}$ iff $X \subseteq_{\mathrm{fin}} P$ and $\bigcup X \in \mathcal{A}$;

$p \equiv p'$ iff $p, p' \in P$ and $p = [a]_z$ and $p' = [a']_{z'}$ with $a \equiv_A a'$.

Recall from stable families that $[a]_z =_{\mathrm{def}} \bigcap\{x \in \mathcal{A} \mid a \in x\ \&\ x \subseteq z\}$ where in this case z is an unambiguous configuration in $\mathcal{A}$. We denote the right adjoint by Pr as it is a direct generalisation of the earlier construction we have seen providing a right adjoint to the 'inclusion' of prime event structures in stable families. As before, at a stable ef $\mathcal{A}$ the counit

$$(\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{A}, \equiv_A)), \equiv) \to (\mathcal{A}, \equiv_A)$$

takes a prime to its top element and consequently a configuration x of $\mathrm{Pr}(\mathcal{A}, \equiv_A)$ to $\bigcup x$, the configuration of $\mathcal{A}$ comprising the union of all the prime configurations x contains; the map need no longer be locally injective however as it need only reflect equivalence locally. The adjunction is enriched: the natural bijection between homsets preserves equivalence.

Compare the definition above with that of Pr on stable families. The significant difference is in the way that consistency is defined; in the construction on a stable ef the consistency is inherited not from the stable family of unambiguous configurations but from the ambient ef $\mathcal{A}$ in which configurations may not be unambiguous.

16.15 Constructions

Our major motivation in developing and exploring all the above categories was in order to extend strategies with parallel causes. The various subcategories of $\mathcal{ES}_\equiv$ have been designed to support the central operation of hiding. What about the other construction key to the composition of strategies, *viz.* pullback?

We first introduce the constructions of product and pullback of ef's; just as with prime event structures we cannot expect such constructions to be easily achieved directly on ese's. The pullback of stable ef's will be especially important. The constructions of product and pullback of ef's will reduce to product and pullback on families of configurations when we take the equivalences $\equiv$ to be the identity relation. On stable families they reduce to the product and pullback of stable families we have seen earlier.

The *product* of ef's is given as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be ef's with underlying sets A and B. Their product will have underlying set $A \times_* B$, the product of A and B in sets with partial functions with projections π_1 to A and π_2 to B.

We take $c \equiv c'$ in $A \times_* B$ iff $\pi_1 c \equiv \pi_1 c'$, or both are undefined, and $\pi_2 c \equiv \pi_2 c'$, or both are undefined. Define the configurations of the product by: $x \in \mathcal{A} \times \mathcal{B}$ iff

$x \subseteq A \times_* B$ s.t.
$\pi_1 x \in \mathcal{A}$ & $\pi_2 x \in \mathcal{B}$,
$\forall c, c' \in x.\ \pi_1(c) \equiv_A \pi_1(c')$ or $\pi_2(c) \equiv_B \pi_2(c') \implies c \equiv c'$ and
$\forall c \in x \exists c_1, \cdots, c_n \in x.\ c_n = c$ &
$\forall i \leq n.\ \pi_1\{c_1, \cdots, c_i\} \in \mathcal{A}$ & $\pi_2\{c_1, \cdots, c_i\} \in \mathcal{B}$.

We obtain the product in stable ef's by restricting to those configurations of the product of the stable ef's which are unions of unambiguous configurations. Notice that unambiguous configurations of the product of stable ef's are exactly the configurations in the product in stable families of the subfamilies of unambiguous configurations.

Restriction w.r.t. sets of events which are closed under $\equiv$ and synchronised compositions are defined analogously to before. In particular we obtain pullbacks and bipullbacks as restrictions of the product.

Pullbacks exist in general but we shall only need pullbacks of total maps. Let $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ be total maps of ef's. Assume $\mathcal{A}$ and $\mathcal{B}$ have underlying sets A and B. Define $D =_{\text{def}} \{(a, b) \in A \times B \mid f(a) = g(b)\}$ with projections π_1 and π_2 to the left and right components. On D, take $d \equiv_D d'$ iff $\pi_1(d) \equiv_A \pi_1(d')$ and $\pi_2(d) \equiv_B \pi_2(d')$. Define a family of configurations of the *pullback* to consist of $x \in \mathcal{D}$ iff

$x \subseteq D$ s.t.
$\pi_1 x \in \mathcal{A}$ & $\pi_2 x \in \mathcal{B}$, and
$\forall d \in x \exists d_1, \cdots, d_n \in x.\ d_n = d$ &
$\forall i \leq n.\ \pi_1\{d_1, \cdots, d_i\} \in \mathcal{A}$ & $\pi_2\{d_1, \cdots, d_i\} \in \mathcal{B}$.

The pullback in stable ef's is again obtained by restricting to those configurations which are unions of unambiguous configurations. The unambiguous configurations in the pullback of stable ef's are obtained as the pullback in stable families of the subfamilies of unambiguous configurations.

Given that maps are related by an equivalence relation it is sensible to broaden our constructions to pseudo pullbacks—the universal characterisation of pseudo pullback follows the concrete construction.

Pseudo pullbacks of total maps $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ of ef's are obtained in a similar way to pullbacks. Assume $\mathcal{A}$ and $\mathcal{B}$ have underlying sets A and B. Define $D =_{\text{def}} \{(a, b) \in A \times B \mid f(a) \equiv_C g(b)\}$ with projections π_1 and π_2 to the left and right components. On D, take $d \equiv_D d'$ iff $\pi_1(d) \equiv_A \pi_1(d')$ and $\pi_2(d) \equiv_B \pi_2(d')$. Define a family of configurations of the *pseudo pullback* to consist of $x \in \mathcal{D}$ iff

$x \subseteq D$ s.t.
$\pi_1 x \in \mathcal{A}$ & $\pi_2 x \in \mathcal{B}$, and
$\forall d \in x \exists d_1, \cdots, d_n \in x.\ d_n = d$ &
$\forall i \leq n.\ \pi_1\{d_1, \cdots, d_i\} \in \mathcal{A}$ & $\pi_2\{d_1, \cdots, d_i\} \in \mathcal{B}$.

When $\mathcal{A}$ and $\mathcal{B}$ are stable ef's we obtain their pseudo pullback by restricting to those configurations obtained as the union of unambiguous configurations.

Recall the universal property of a pseudo pullback of $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ (in this simple case). A pseudo pullback comprises two maps $\pi_1 : \mathcal{D} \to \mathcal{A}$ and $\pi_2 : \mathcal{D} \to \mathcal{B}$ such that $f\pi_1 \equiv g\pi_2$ with the universal property that given any two maps $p_1 : \mathcal{E} \to \mathcal{A}$ and $p_2 : \mathcal{E} \to \mathcal{B}$ such that $fp_1 \equiv gp_2$ there is a unique map $h : \mathcal{E} \to \mathcal{D}$ such that $p_1 = \pi_1 h$ and $p_2 = \pi_2 h$:

$$\begin{tikzcd}
 & \mathcal{E} \arrow[d, dashed, "h"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_2"] & \\
 & \mathcal{D} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[l, phantom, "="] \arrow[r, phantom, "="] & \\
\mathcal{A} \arrow[dr, "f"'] \arrow[rr, phantom, "\equiv"] & & \mathcal{B} \arrow[dl, "g"] \\
 & \mathcal{C} &
\end{tikzcd}$$

Pseudo pullbacks are defined up to isomorphism. Clearly pseudo pullbacks coincide with pullbacks when the maps involved have an event structure as their common codomain.

The right adjoint, *er*, a pseudo functor, of the pseudo adjunction from $\mathcal{ES}_{\equiv}$ to $\mathcal{F}am_{\equiv}$ will not preserve pullbacks and pseudo pullbacks in general; a pseudo pullback is only generally sent by the right adjoint to a bipullback, which satisfies a weaker condition, ensuring commutation and uniqueness only up to $\equiv$. Whereas $\mathcal{ES}_{\equiv}$ consequently has bipullbacks it does not have all pullbacks or all pseudo pullbacks. Bipullbacks have the drawback of not being defined up to isomorphism and only up to the equivalence on objects induced by the equivalence of maps.[3]

Fortunately we do have both pullbacks and pseudo pullbacks in the subcategory $\mathcal{ES}^1_{\equiv}$. This will be important later in characterising strategies based on maps in $\mathcal{ES}^1_{\equiv}$. The constructions of pullbacks and pseudo pullbacks in $\mathcal{ES}^1_{\equiv}$ can by-pass the complicated *er* construction and be done via the corresponding constructions in $\mathcal{SF}am_{\equiv}$ in the manner we're familiar with from event structures and stable families. This is because we have an adjunction from $\mathcal{ES}^1_{\equiv}$ to $\mathcal{SF}am_{\equiv}$ and moreover an adjunction which is enriched with respect the equivalence on homsets. So, for example, to form the (pseudo) pullback of ese's in $\mathcal{ES}^1_{\equiv}$ we regard their configurations as stable ef's, form the (pseudo) pullback in $\mathcal{SF}am_{\equiv}$ and take the image under the right adjoint Pr. Each stable ef includes a subfamily of unambiguous configurations and it is fortunate indeed that *e.g.* the subfamily of unambiguous configurations of the pullback of stable ef's $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ is got as the pullback in stable families of f and g between the subfamilies of unambiguous configurations.

[3]Two objects P and Q are equivalent iff there are two maps $f : P \to Q$ and $g : Q \to P$ such that $gf \equiv \mathrm{id}_P$ and $fg \equiv \mathrm{id}_Q$.

16.16 Summary

A figure summarising all the adjunctions:

$$
\begin{tikzcd}
\mathcal{ES}_\equiv \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\top"] & \mathcal{F}am_\equiv \arrow[l, bend right, "er"'] \arrow[r, bend right, "col"'] \arrow[r, phantom, "\top"] & \mathcal{GES} \arrow[l, bend right, "fam"'] \\
\mathcal{ES}^2_\equiv \arrow[u, bend right] \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & & \\
\mathcal{ES}^1_\equiv \arrow[u, bend right] \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathcal{SF}am_\equiv \arrow[l, bend right, "\mathrm{Pr}"'] \arrow[uu, hook] \arrow[dd, bend right, "unamb"'] \arrow[dd, phantom, "\vdash"] & \\
\mathcal{ES}^0_\equiv \arrow[u, bend right] \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & & \\
\mathcal{ES} \arrow[u, bend right] \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathcal{SF}am \arrow[l, bend right, "\mathrm{Pr}"'] \arrow[uu, bend right] &
\end{tikzcd}
$$

The adjunctions

$$
\begin{tikzcd}
\mathcal{ES} \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathcal{ES}^0_\equiv \arrow[l, bend right]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{SF}am \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathcal{SF}am_\equiv \arrow[l, bend right, "unamb"']
\end{tikzcd}
$$

are not enriched in the sense that the natural bijection does not respect the equivalence $\equiv$ on maps.

Ultimately our motivation has been to develop strategies with parallel causes. For this we would like an extension of general event structures which supports hiding and a pullback, or possibly a pseudo pullback. Although we could perhaps manage with bipullbacks the fact that these are only characterised up to equivalence would prevent us from a characterisation of strategies up to isomorphism of the kind we have seen earlier and will see again in the next chapter. We shall base strategies with parallel causes on maps in $\mathcal{ES}^1_\equiv$. The category supports hiding and has pullbacks, pseudo pullbacks and, it will turn out, probability, leading to a robust definition of probabilistic strategies with parallel causes. The category $\mathcal{ES}^1_\equiv$ is also much simpler to work with than $\mathcal{ES}_\equiv$ as, through the adjunction from $\mathcal{ES}^1_\equiv$ to $\mathcal{F}am_\equiv$, we can avoid the complicated construction of extremal realisations. Has something been lost? Quite possibly there will be a call for perhaps a more resource-conscious notion of strategy with parallel causes, one which requires strategies based on maps in $\mathcal{ES}_\equiv$ rather than $\mathcal{ES}^1_\equiv$. But first things first. The restrictions maps in $\mathcal{ES}^1_\equiv$ will impose implicitly on strategies do not seem unnatural.

Along with their future central role, objects of $\mathcal{ES}^1_\equiv$ will be rechristened to *event structures with disjunctive causes* (edc's) and the category $\mathcal{ES}^1_\equiv$ renamed to $\mathcal{EDC}$. In the next chapter we shall investigate probabilistic strategies based on maps within $\mathcal{EDC}$.

16.17 General event structures as edc's

Earlier in Section 16.11 we showed how to refine the pseudo adjunction from $\mathcal{ES}_\equiv$ to $\mathcal{GES}$ to a pseudo equivalence by imposing axioms on ese's and restricting to replete general event structures. By adapting these earlier results we can characterise those edc's which arise from families of configurations and obtain an analogous equivalence between a subcategory of edc's and replete general event structures. Recall the pseudo adjunction from edc's to families of configurations: the functor collapsing an edc to a replete general event structure, or equivalently to a family of configurations, has a right pseudo adjoint edc; the pseudo functor takes a family of configurations $\mathcal{A}$ to $er(\mathcal{A})$ its ese built form extremal realisations but cut down to those events which meet the axiom required of an edc.

Recall Proposition 16.17 expressing the order-isomorphism between the configurations of $er(\mathcal{A})$ and the order of extremal realisations of $\mathcal{A}$. With respect to this isomorphism, the configurations of the edc$(\mathcal{A})$ correspond to extremal realisations $(R, \leq_R), \rho)$ which satisfy

$$r_1, r_2 \leq_R r \ \&\ \rho(r_1) = \rho(r_2) \implies r_1 = r_2$$

—realisations which are *locally unambiguous*; just as we can say a realisation (R, ρ) is *unambiguous* when ρ is injective. In this context, where $\mathcal{A}$ does not itself carry a nontrivial equivalence, an *unambiguous* extremal realisations corresponds to an *unambiguous* configuration of the edc.

The characterising axioms on an edc $(P, \leq, \mathrm{Con}, \equiv)$ are:

(A) For X a finite down-closed subset of P,
$X \equiv y \ \&\ y \in \mathcal{C}(P) \implies X \in \mathcal{C}(P)\,;$

(B) For $p, q \in P$, $[p) = [q) \ \&\ p \equiv q \implies p = q\,;$

(C) For X a down-closed subset of P and $p \equiv q$,
$X \subseteq [p) \ \&\ [q)_\equiv \subseteq X_\equiv \implies X = [p)\,;$

(D_1) For x an unambiguous configuration in $\mathcal{C}(P)$ and $t \in P$,
$x \cup [t] \in \mathcal{C}(P) \ \&\ (x \cup [t])_\equiv = x_\equiv \cup \{\{t\}_\equiv\} \implies \exists p \in P.\ p \equiv t \ \&\ x \cup \{p\} \in \mathcal{C}(P)\,.$

In writing the axioms we have used expressions such as $X \equiv Y$, for subsets X and Y of P, to mean for any $p \in X$ there is $q \in Y$ with $p \equiv q$ and *vice versa*; and $X_\equiv$ to stand for the set of $\equiv$-equivalence classes $\{\{p\}_\equiv \mid p \in X\}$; so $X \equiv Y$ iff $X_\equiv = Y_\equiv$. Recall, a configuration x of an ese is unambiguous if

$$p_1, p_2 \in x \ \&\ p_1 \equiv p_2 \implies p_1 = p_2\,.$$

Axiom (D_1) may be replaced by

(D_1') For $x, y \in \mathcal{C}(P)$, with y unambiguous, and $t \in P$,
$x \overset{t}{\longrightarrow\!\!\!\subset} \ \&\ x \equiv y \implies \exists p \in P.\ p \equiv t \ \&\ x \cup \{p\} \in \mathcal{C}(P)\,.$

Assume (D_1) and, for $x, y \in \mathcal{C}(P)$, that $x \xrightarrow{t}\!\subset$ and $x \equiv y$ and y is unambiguous. Then, by (A), $y \cup [t] \in \mathcal{C}(P)$ as $y \cup [t] \equiv x \cup \{t\}$, clearly consistent; whence $y \cup \{p\} \in \mathcal{C}(P)$ for some p by (D_1). Conversely, assuming (D'_1) and $x \cup [t] \in \mathcal{C}(P)$ and $(x \cup [t])_\equiv = x_\equiv \cup \{\{t\}_\equiv\}$ and x unambiguous, in the case where $t \notin x$ we obtain $x \cup [t) \xrightarrow{t}\!\subset$ and $x \cup [t) \equiv x$; whence $x \cup \{p\} \in \mathcal{C}(P)$ for some p by (D'_1). This shows (D_1) follows from (D'_1) in the case when $t \notin x$; in the case when $t \in x$, axiom (D_1) is obvious.

Theorem 16.23. *Let P be an edc. Then, $P \cong \mathrm{edc}(\mathcal{A})$ for some equivalence family $\mathcal{A}$ iff P satisfies axioms (A), (B), (C) and (D_1) (or (D'_1)).*

Proof. The proof is essentially a slight refinement of the proof of Theorem 16.21.

As the axioms (A), (B), (C) hold of $er(\mathcal{A})$—Theorem 16.21—they certainly hold of its restriction to an edc. Given the correspondence between configurations of P and extremal realisations, axiom (D) of Section 16.11, without the assumption that x is unambiguous, expresses an obvious extension property of extremal realisations in general; the extra assumption that x is unambiguous ensures that the event $p \in P$, asserted to exist, satisfies the condition required of an edc.

Conversely, if an edc $P = (P, \mathrm{Con}, \leq, \equiv)$ satisfies (A), (B), (C), (D_1) then there is an isomorphism

$$\eta_P : P \cong \mathrm{edc}(\mathcal{A})$$

if we take the family of configurations so

$$\mathcal{A} = \mathcal{C}^\infty(\,col(\mathcal{C}^\infty(P), \equiv))\,.$$

Recall, from the remarks preceding the axioms, that the configurations of $\mathrm{edc}(\mathcal{A})$ correspond to extremal realisations of $col(\mathcal{C}^\infty(P), \equiv)$ which are locally unambiguous. In more detail, a configuration x of P determines a locally-unambiguous extremal realisation of $col(\mathcal{C}^\infty(P), \equiv)$: the realisation has carrier x with order inherited from P and map taking $p \in x$ to the equivalence class $\{p\}_\equiv$. Axioms (B) and (C) ensure that this realisation is extremal, via Lemma 16.10.

It follows that we define a map $\eta_P : P \to er(\mathcal{A})$ by sending $p \in P$ to the realisation with carrier $[p]$, ordered as in P, and function $[p] \to P_\equiv$ taking elements to their equivalence classes. The injectivity of η_P follows from (B). Moreover η_P reflects consistency because of axiom (A). We now only require its surjectivity to ensure η_P is an isomorphism.

We use (D_1) in showing that η_P is surjective. We show by induction on $n \in \omega$ that all unambiguous extremal realisations with top of $col(P)$ of depth less than n are in the image of η_P. Because η_P reflects consistency the induction hypothesis entails that all locally-unambiguous extremal realisations of depth less than n are (up to isomorphism) in the image under η_P of configurations of P; moreover, all unambiguous extremal realisations of depth less than n are (up to isomorphism) in the image under η_P of unambiguous configurations of P.

Let $(R, \leq_R)$ of depth n with $\rho : R \to col(P)$ be an unambiguous extremal realisation with top r, so $R = [r]_R$. Then its restriction $\rho' : [r)_R \to col(P)$ is an unambiguous extremal realisation of lesser depth. By induction there is an unambiguous $x' \in \mathcal{C}(P)$ and an isomorphism of realisations $\theta' : \rho' \cong \eta_P x'$. Write $y =_{\text{def}} \rho'[r)_R$, $z =_{\text{def}} \rho[r]_R$. Then $y, z \in \mathcal{C}(col(P))$ and $y \xrightarrow{e}\!\!\subset z$ for some $e \in P_{\equiv}$. From the definition of $col(P)$, it follows fairly directly that there is some $t \in P$ s.t. $\{t\}_{\equiv} = e$ and $[t)_{\equiv} \subseteq y$. As η_P reflects consistency, $x' \cup [t] \in \mathcal{C}(P)$. We have

$$(x' \cup [t])_{\equiv} = x'_{\equiv} \cup \{\{t\}_{\equiv}\} = z\,.$$

Because x' is unambiguous, by (D_1) there is some $p \in P$ s.t. $p \equiv t$ and $x' \cup \{p\} \in \mathcal{C}(P)$. The configuration $x =_{\text{def}} x' \cup \{p\}$ with order inherited from P and map taking $p' \in x$ to $\{p'\}_{\equiv}$ is the realisation $\eta_P x$. Let θ be the function $\theta : R \to x$ extending θ' s.t. $\theta(r) = p$. Then $\theta : \rho \geq \eta_P x$ is a map of realisations. But ρ is extremal ensuring $\theta : \rho \cong \eta_P x$, and that η_P is surjective. $\square$

Corollary 16.24. *The adjunction from $\mathcal{EDC}$ to $\mathcal{GES}$ cuts down to a ***pseudo*** equivalence of categories between the subcategory of $\mathcal{EDC}$ satisfying axioms (A), (B), (C), (D_1) and the subcategory of $\mathcal{GES}$ comprising the replete general event structures.*

16.18 Deterministic general event structures

Proposition 16.25. *(i) Let $\mathcal{A}$ be a family of configurations. Defining*

$$\mathcal{A}^0 =_{\text{def}} \{x \in \mathcal{A} \mid x \textit{ is finite}\}$$

we obtain a family of finite configurations which satisfy:

(a) $\varnothing \in \mathcal{A}^0$;

(b) If $x \subseteq x_1$ & $x \subseteq x_2$ & $x_1 \uparrow x_2$ *in* $\mathcal{A}^0$ *then* $x_1 \cup x_2 \in \mathcal{A}$;

(c) For all $x \in \mathcal{A}^0$, *there is a covering chain*

$$\varnothing \longrightarrow\!\!\!\subset x_1 \longrightarrow\!\!\!\subset \cdots \longrightarrow\!\!\!\subset x_n = x$$

in $\mathcal{A}^0$.

(ii) Conversely, if $\mathcal{A}^0$ is a family of finite sets satisfying axioms (a), (b) and (c) above then defining $\mathcal{A}$ to be the family consisting of unions of directed subfamilies of $\mathcal{A}^0$ we obtain a family of configurations.

Furthermore, the two operations described in (i) and (ii) are mutual inverses.

Remark Condition (c) above can be replaced by "coincidence-freeness" of earlier, by Exercise 3.6. In condition (b) it suffices to replace the supposed inclusions by the covering relation.

Definition 16.26. Let $\mathcal{A}$ be a family of configurations with underlying set of events A possessing a polarity function $pol : A \to \{+,-,0\}$. Say $\mathcal{A}, pol$ is *deterministic* iff

$$\forall a, a' \in S, x \in \mathcal{A}.\ x \overset{a}{\longrightarrow\!\!\subset} \ \&\ x \overset{a'}{\longrightarrow\!\!\subset} \ \&\ pol(a) \in \{+,0\} \implies x \cup \{a, a'\} \in \mathcal{A}.$$

[We shall sometimes leave the polarity information implicit.]

As earlier we shall write $\subseteq^+$, $\subseteq^-$ and $\subseteq^0$ to indicate inclusions in which all adjoined events have the specified parity; we shall use $\subseteq^p$ to indicate that the supplementary events of the inclusion all have +ve or neutral polarity.

Proposition 16.27. *A family of configurations is deterministic iff*

$$x \subseteq^p x_1 \ \&\ x \subseteq x_2 \ \mathit{in}\ \mathcal{A}^0 \implies x_1 \cup x_2 \in \mathcal{A}^0 .$$

Proof. (Idea) *"If"*: Straightforward. *"Only if"*: By repeated use of the definition of deterministic based on coverings, startng with covering chains from x to x_1 and from x to x_2. □

Deterministic general event structures support hiding. Let $\mathcal{A}, pol$ be a deterministic general event structure with underlying events A. Define its *visible* events to be

$$V =_{\text{def}} \{a \in A \mid pol(a) \in \{+,-\}\} .$$

Define the *projection*

$$\mathcal{A} \downarrow V =_{\text{def}} \{x \cap V \mid x \in A\} .$$

Lemma 16.28. *Let $\mathcal{A}^0$ be the finite configurations of a deterministic family of configurations $\mathcal{A}$. Then,*
(i) $\forall x, y \in \mathcal{A}^0.\ x \cap V \subseteq y \cap V \implies \exists y' \in A.\ y \subseteq^0 y' \ \&\ x \subseteq y' .$
(ii) $\forall x_2, x_2 \in \mathcal{A}^0.\ x_1 \cap V \uparrow x_2 \cap V \ \mathit{in}\ (\mathcal{A} \downarrow V)^0 \implies x_1 \uparrow x_2 \ \mathit{in}\ \mathcal{A}^0 .$

Proof. (i) Let $x, y \in \mathcal{A}^0$. Assume $x \cap V \subseteq y \cap V$. Define x_0 to be the largest subconfiguration of x such that $x_0 \subseteq y$. Choose a covering chain from x_0 to x:

$$\varnothing = x_0 \overset{a_1}{\longrightarrow\!\!\subset} x_1 \overset{a_2}{\longrightarrow\!\!\subset} \cdots \overset{a_n}{\longrightarrow\!\!\subset} x_n = x$$

We show by induction along the chain that for all i, $0 \le i \le n$,

$$x_i \subseteq y^{(i)} \text{ for some } y^{(i)} \supseteq^0 y .$$

Clearly, the basis of the induction, when $i = 0$, holds. Assume, inductively, for $i \le n$ that

$$x_i \subseteq y^{(i)} \text{ with } y^{(i)} \supseteq^0 y .$$

If $a_{i+1} \in y^{(i)}$ take $y^{(i+1)} =_{\text{def}} y^{(i)}$. Otherwise $a_{i+1} \notin V$, so has neutral polarity. In which case choose a covering chain

$$x_i {-\!\!\subset} \cdots {-\!\!\subset} y^{(i)} .$$

and notice

$$x_i \overset{a_{i+1}}{\longrightarrow\!\!\!\subset} x_{i+1}\,.$$

Now, by repeatedly using that $\mathcal{A}$ is deterministic, working along the chain we finally obtain $y_{(i+1)}$ with

$$x_{i+1} \subseteq y_{(i+1)} \;\&\; y_{(i)} \overset{a_{i+1}}{\longrightarrow\!\!\!\subset} y_{(i+1)}\,,$$

which with the induction hypothesis entails $y \subseteq^0 y_{(i+1)}$ —as required to maintain the induction hypothesis.

(ii) Assume $x_1 \cap V \uparrow x_2 \cap V$ in $(\mathcal{A} \downarrow V)^0$. Then

$$x_1 \cap V \subseteq y_1 \cap V \text{ and } x_2 \cap V \subseteq y_2 \cap V$$

for some $y_1, y_2 \in \mathcal{A}^0$ with $y_1 \cap V = y_2 \cap V$. Applying (i) to the former we obtain $y_1' \in \mathcal{A}^0$ with $y_1 \subseteq^0 y_1'$ and $x_1 \subseteq y_1'$. But $y_1' \cap V = y_1 \cap V = y_2 \cap V$, so

$$x_2 \cap V \subseteq y_1' \cap V\,,$$

which, by (i) again, yields $y'' \in \mathcal{A}^0$ with $y_1' \subseteq^0 y''$. Automatically we have both

$$x_1 \subseteq y'' \text{ and } x_2 \subseteq y''\,,$$

whence $x_1 \uparrow x_2$ in $\mathcal{A}^0$. □

Theorem 16.29. *When $\mathcal{A}$ is a deterministic family of configurations with visible events V its projection $\mathcal{A} \downarrow V$ is also a deterministic family of configurations.*

Proof. We use Proposition 16.25 to show $(\mathcal{A} \downarrow V)^0$ is a family of finite configurations. Properties (a) and (c) are straightforward. To show (b), use Proposition 16.27, on the assumption that $z_1 \uparrow z_2$ in $(\mathcal{A} \downarrow V)^0$. Then there are $x_1, x_2 \in \mathcal{A}^0$ such that

$$z_1 = x_1 \cap V \;\&\; z_2 = x_2 \cap V\,.$$

By Lemma 16.28 (ii),

$$x_1 \uparrow x_2 \text{ in } \mathcal{A}^0\,.$$

Therefore $x_1 \cup x_2 \in \mathcal{A}^0$ and

$$z_1 \cup z_2 = (x_1 \cup x_2) \cap V \text{ in } (\mathcal{A} \downarrow V)^0$$

ensuring condition (b).

To see that $\mathcal{A} \downarrow V$ is deterministic, assume

$$z \subseteq^+ z_1 \;\&\; z \subseteq z_2 \text{ in } (\mathcal{A} \downarrow V)^0\,.$$

Then there are $x_1, x_2, x \in \mathcal{A}^0$ such that

$$x_1 \cap V = z_1 \;\&\; x_2 \cap V = z_2 \;\&\; x \cap V = z\,.$$

By Lemma 16.28(i), w.l.o.g. we may assume

$$x \subseteq^p x_1 \;\&\; x \subseteq x_2\,.$$

As $\mathcal{A}$ is deterministic we obtain $x_1 \cup x_2 \in \mathcal{A}^0$. Clearly $(x_1 \cup x_2) \cap V = z_1 \cup z_2 \in (\mathcal{A} \downarrow V)^0$, as required to show $\mathcal{A} \downarrow V$ is deterministic. □

16.19 Strategies with general event structures

Prelims: rigid map of ef's : consec events go to concurrent implies they are concurrent. Same as lifting cond?

counits $\epsilon : \mathcal{C}^\infty(\text{ese})\mathcal{A} \to \mathcal{A}$ and $\epsilon : \mathcal{C}^\infty(\text{edc})\mathcal{A} \to \mathcal{A}$ rigid and reflect concurrency squares - via props of extremal realisations

Entails edc and ese of a rigid map of ef's are rigid: if f rigid so is esef ***RIGHT NAME?*** and edcf despite their being pseudo functors so characterised only up to $\equiv$

Warning: For edc's, ef's, ese's don't have rigidity respected by $\equiv$: obv two maps from $a < b$ to $a < b \| a \| b$

Partial strategies with replete general event structures, so families of configurations

A strategy based on families of configs (so equivalently replete general event structures) comprises a total map $\sigma : \mathcal{S} \to \mathcal{A}$ of families of configurations, where A is (the family of configurations of) an event structure with polarity, which is

receptive :***

innocent:****satn conds***

Same as via lifting conds?

Say deterministic when $\mathcal{S}$ is deterministic

copycat as before composition of det ges strategies by pb with hiding

Generalisation of partial strategies to those based on gen ev strs

under *col* a det edc partial strategy is sent to a det ges partial strategy

Lemma 16.30. *The composition (with or without hiding) of deterministic partial strategies based on general event structures is a deterministic partial strategy.*

Proof. I believe that the earlier proof of Lemma 5.8 goes through almost verbatim. □

an edc is *strongly deterministic* if deterministic and image under edc of a gen ev str

edc(σ) is a total map of edc's which satisfies **** (an edc strategy). Use edc(σ) = $\sigma\,\epsilon$ viewed IN ef's and facts about counit above

If σ is deterministic, then edc(σ) is strongly deterministic.

edc(σ) defined on the nose because A is an ev str and functorial action of edc defined up to $\equiv$ in the codomain, in this case the identity relation.

edc does not pres pbs of such maps, nor $\circledast$, but does preserve bipbs, as pseudo rt adjoint?

A first suggestion do the constructions with strongly det edc's:

But $\circledast$ does not pres strong det

and strong det isn't preserved by hiding as hiding of strong det edc's does not satis axioms of last section: the edc of after hiding two init events of the edc got from two parallel causes of an event doesn't satis (B)

Hiding on strongly det edc's wd require special treatment - closing up by ax (A) to (D) after trad edc hiding.

However all this fuss can be avoided I think if we work with edc strategies which are ≡-equivalent to strongly deterministic strategies because of:

For A, B objects in a ≡-category (***define?!***) say they are ≡-equivalent, and write $A \equiv B$, iff there are mutual inverses up to ≡ between them.

Note if $A \equiv B$ in $\mathcal{EDC}$ then $col(A) \cong col(B)$ as col is ≡-enriched.

Extend ≡-equivalence to strategies based on edc's in the obv way.

Lemma 16.31. *Let $\mathcal{F}$ be a deterministic family of configurations with underlying set of events A (with polarities $pol : A\{-,+,0\}$). Write $V =_{\rm def} \{a \in A \mid pol(a) \neq 0\}$ and $V' =_{\rm def} \{p \in er(\mathcal{F}) \mid pol(top(p)) \neq 0\}$ Then,*

$$er(\mathcal{F}) \downarrow V' \equiv er(\mathcal{F} \downarrow V)\,.$$

The ≡-equivalence restricts:

$$\mathrm{edc}(\mathcal{F}) \downarrow \mathrm{V}' \equiv \mathrm{edc}(\mathcal{F} \downarrow \mathrm{V})\,.$$

Proof. We obtain the map f from partial-total factorisation properties of the partial map $er(\mathcal{F}) \rightharpoonup V$ defined and acting as the identity function on V, where we have regarded V as the event structure comprising events V with trivial, identity causal dependency in which all finite subsets are consistent. The partial map factors as

$$er(\mathcal{F}) \rightharpoonup er(\mathcal{F}) \downarrow V' \to V\,.$$

Whereas from the analogous factorisation of $\mathcal{F} \rightharpoonup V$ we obtain

$$\mathcal{F} \rightharpoonup \mathcal{F} \downarrow V \to V\,,$$

so

$$er(\mathcal{F}) \rightharpoonup er(\mathcal{F} \downarrow V) \to V\,.$$

The existence of $f : er(\mathcal{F}) \downarrow V' \to er(\mathcal{F} \downarrow V)$ now follows from the universal property of the factorisation of $er(\mathcal{F}) \rightharpoonup V$. (Notice that the map $er(\mathcal{F}) \rightharpoonup er(\mathcal{F} \downarrow V)$ is obtained from the action of the *pseudo* functor er whose definition depends on the axiom of choice, as therefore does f—see *****.)

The other part of the ≡-equivalence,

$$g : er(\mathcal{F} \downarrow V) \to er(\mathcal{F}) \downarrow V'$$

is built by induction on depth of events making use of $\mathcal{F}$ being deterministic. Assume, inductively, that we have constructed a map g up to but not including depth n so that for all $r \in er(\mathcal{F} \downarrow V)$ of lesser depth

$$top(g(r)) = top(r)\,.$$

Let $p \in er(\mathcal{F} \downarrow V)$ have depth n. Then the configuration $[p)$ correponds to the extremal realisation

$$top : [p) \to \top[p)$$

in $\mathcal{F}\downarrow V$ in which the carrier $[p)$ is endowed with the order of $er(\mathcal{F}\downarrow V)$. The finite configuration $g[p)\in\mathcal{C}(er(\mathcal{F})\downarrow V')$ has down-closure $[g[p)]\in\mathcal{C}(er(\mathcal{F}))$. The down-closure corresponds to an extremal realisation

$$top:[g[p)]\to top[g[p)]$$

in $\mathcal{F}$, with it carrier $[g[p)]$ inheriting the order of $er(\mathcal{F})$. From the induction hypothesis,

$$(top[g[p)])\cap V = top[p)\,.$$

Observe that

$$top[p)\xrightarrow{a}\!\!\subset top[p]$$

where $a=_{\text{def}} top(p)\in V$. By Lemma 16.28(i), it follows that there is $x'\in\mathcal{F}$ such that $top[p)\subseteq x'$ and $x'\cap V = top[p]$. (This step relies on $\mathcal{F}$ being deterministic.) In other words the only events in x' additional to those in $top[p)$ are either the event a or neutral.

Extend the extremal realisation $top:[g[p)]\to top[g[p)]$ to a realisation $R\to x'$ where $[g[p)]\subseteq R$ and the order on R restricts to that on $[g[p)]$; one way to do this is to serialise the events in $R\smallsetminus[g[p)]$, making them all dependent on $[g[p)]$. By Lemma 16.4, there is a coarsening of this realisation that is extremal. This then restricts to a prime extremal realisation, defined to be $g(p)$, for which $top(g(p)) = a = top(p)$. (The definition of g also involves choice via the use of Lemma lem:existextr.)

Using Proposition 16.2, it is easy to check that g, defined inductively as above, is a map of ese's because the equivalences and consistency are inherited from equality and compatibility in $\mathcal{F}$; condition (ii) of the proposition, concerning causal dependency, is obvious from the way g is constructed. Similarly, that $gf\equiv \text{id}$ and $fg\equiv\text{id}$ falls back on the fact that equivalence stems from equality of events in $\mathcal{F}$.

Finally, the $\equiv$-equivalence restricts to the edc's because both the maps edc's f and g preserve unambiguous configurations so send a prime of $\text{edc}(\mathcal{F})$ to a prime of $\text{edc}(\mathcal{F}\downarrow \text{V})$ and *vice versa.* □

Corollary 16.32. *If $\sigma:S\to A\|N$ is a deterministic partial strategy of general event structures with σ_0 the result of hiding neutral events and $(\text{edc}(\sigma))_0$ the result of hiding neutral events in $\text{edc}(\sigma)$, then*

$$(\text{edc}(\sigma))_0\equiv\text{edc}(\sigma_0)\,.$$

Lemma 16.33. *(i) Let σ and τ be partial strategies of general event structures.*

$$\text{edc}(\sigma)\circledast\text{edc}(\tau)\equiv\text{edc}(\sigma\circledast\tau)\,.$$

(ii) Let σ and τ be deterministic strategies of general event structures.

$$\text{edc}(\sigma)\odot\text{edc}(\tau)\equiv\text{edc}(\sigma\odot\tau)\,.$$

Proof. (i) by the pseudo adjunction from edc's to *ges*'s, the pseudo right adjoint of which preserves bipullbacks and so composition without hiding, up to $\equiv$. (ii) By (i) and Corollary 16.32. $\square$

Although it is not the case that under edc composition of deterministic strategies is preserved, it is up to $\equiv$. Recall too, the converse, that if two deterministic edc strategies are $\equiv$-equivalent, then under *col* they are sent to isomorphic strategies of gen eve strs.

Thus working with deterministic edc strategies $\equiv$-equivalent to strongly deterministic strategies provides an alternative to working with deterministic strategies based on general event structures.

Chapter 17

Edc strategies

We mimic the work of earlier on developing the definition of strategy, based on pre-strategies which are left invariant under composition with copycat, but this time based on edc's rather than prime event structures. We shall make the simplifying assumption that games are represented by prime event structures (or, strictly speaking, the edc's which correspond to such, in which the equivalence is the identity relation); the copycat strategy is then defined as earlier. We characterise those edc pre-strategies which are left invariant under composition with copycat and take such as our definition of strategies based on edc's. We show that we can extend the probabilistic strategies of earlier to edc strategies.

17.1 Edc pre-strategies

We develop strategies in edc's in a similar way to that of strategies. But what is copy-cat on an edc? If games are edc's, shouldn't pullback be replaced by pseudo-pullback? To avoid such issues we assume that games are (the edc's of) prime event structures.

An *edc with polarity* comprises $(P, \leq, \mathrm{Con}, \equiv, pol)$, an edc $(P, \leq, \mathrm{Con}, \equiv)$ in which each element $p \in P$ carries a polarity $pol(p)$ which is + or −, according as it represents a move of Player or Opponent, and where the equivalence relation $\equiv$ respects polarity.

A *map* of edc's with polarity is a map of the underlying edc's which preserves polarity whenever the map is defined. The adjunctions of the previous chapter are undisturbed by the addition of polarity.

As before we can define the dual $A^{\perp}$ and simple parallel composition $A \| B$ of edc's with polarity; the additional equivalence relation of edc's stays passive in extending the former definitions on event structures.

A game is represented by an edc with polarity in which the edc is that of a prime event structure. A *pre-strategy* in edc's, or an *edc pre-strategy*, in a game A is a total map $\sigma : S \to A$ of edc's.

17.1.1 Constructions on edc's and stable equivalence-families

Recall the adjunction from $\mathcal{EDC}$ to $\mathcal{SF}am_{\equiv}$. *****

stable equivalence family *** stable ef ******

Π_1, Π_2 for the projections in edc's***

17.2 Composing edc pre-strategies

Because games are essentially event structures we can define the copy-cat strategy essentially as before; copycat associated with the game A is $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ defined as before but now regarding the event structures as edc's by associating them with the identity equivalence.

Given two edc pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can form the pullback in edc's:

$$
\begin{tikzcd}
 & T \circledast S \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, "\vee" near start] & \\
S \| C \arrow[dr, "\sigma \| C"'] & & A \| T \arrow[dl, "A \| \tau"] \\
 & A \| B \| C \,. &
\end{tikzcd}
$$

There is an obvious partial map of event structures $A \| B \| C \to A \| C$ undefined on B and acting as identity on A and C. The composite partial map $\tau \circledast \sigma$ from $T \circledast S$ to $A \| C$ given by following the diagram (either way round the pullback square)

$$
\begin{tikzcd}
 & T \circledast S \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, "\vee" near start] & \\
S \| C \arrow[dr, "\sigma \| C"'] & & A \| T \arrow[dl, "A \| \tau"] \\
 & A \| B \| C \arrow[d] & \\
 & A \| C &
\end{tikzcd}
$$

factors through the projection of $T \circledast S$ to those events at which the partial map is defined. Its defined part gives us the composition $\tau \odot \sigma : T \odot S \to A^\perp \| C$ once we reinstate polarities.

****SUGGESTION: EARLIER MAKE A DISTINCTION BETWEEN PARTIAL STRATEGIES AS $\tau \circledast \sigma$ ABOVE AND *EXPLICIT* PARTIAL STRATEGIES with total maps to an codomain extended with neutral events****

17.3 An alternative definition of composition

It is useful to have an alternative, if more laboured, definition of composition. It's the concrete definition we shall mainly use in proofs.

Consider two edc pre-strategies $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$ as spans:

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & & T \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ A^\perp & & B & B^\perp & & C\,. \end{tikzcd}$$

We form the product of stable ef's $(\mathcal{C}^\infty(S), \equiv_S) \times (\mathcal{C}^\infty(T), \equiv_T)$ with projections π_1 and π_2, and then form a restriction:

$$(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S) =_{\text{def}} (\mathcal{C}^\infty(S), \equiv_S) \times (\mathcal{C}^\infty(T), \equiv_T) \upharpoonright R$$

where

$$\begin{aligned} R \ =_{\text{def}}\ &\{(s, *) \mid s \in S \ \&\ \sigma_1(s) \text{ is defined}\} \cup \\ &\{(s,t) \mid s \in S \ \&\ t \in T \ \&\ \sigma_2(s) = \overline{\tau_1(t)} \text{ with both defined}\} \cup \\ &\{(*, t) \mid t \in T \ \&\ \tau_2(t) \text{ is defined}\}\,. \end{aligned}$$

I.e. the stable ef $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$ is the synchronized composition of the stable ef's $(\mathcal{C}^\infty(S), \equiv_S)$ and $(\mathcal{C}^\infty(T), \equiv_T)$ in which synchronizations are between elements of S and T which project, under σ_2 and τ_1 respectively, to complementary moves in B and $B^\perp$.

For this particular synchronized composition we can simplify the requirements to be a configuration:

Proposition 17.1. *A set x is a configuration of $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$ iff*

(i) $x \subseteq R$,

(ii) $\pi_1 x \in \mathcal{C}^\infty(S)$ & $\pi_2 x \in \mathcal{C}^\infty(T)$ *and*

(iii) $\forall c \in x \exists c_1, \cdots, c_n \in x.\ c_n = c\ \&$
$\forall i, j \le n.\ c_i \equiv c_j \implies i = j\ \&$
$\forall i \le n.\ \pi_1\{c_1, \cdots, c_i\} \in \mathcal{C}^\infty(S)\ \&\ \pi_2\{c_1, \cdots, c_i\} \in \mathcal{C}^\infty(T)\,.$

Proof. Suppose a set x satisfies the conditions above. We show it is a configuration of the product of stable ef's. Firstly we check it is a configuration of their product as ef's. The only nontrivial requirement we encounter is the third, that

$$\pi_1(c) \equiv_A \pi_1(c') \text{ or } \pi_2(c) \equiv_B \pi_2(c') \implies c \equiv c'\,,$$

and this only non-obvious in the situation where $c = (s,t)$ and $c' = (s', t')$. However, in this case if $\pi_1(c) = s \equiv_S s' = \pi_1(c')$ then $\sigma(s) = \sigma(s') = b \in B$ and $\tau(t) = \tau(t') = \overline{b} \in B^\perp$. But $t, t' \in \pi_2 x \in \mathcal{C}^\infty(T)$ so as τ is a map $t \equiv_T t'$ making $c \equiv c'$. A similar argument applies if $\pi_2(c) = \pi_2(c')$. For x to be a configuration of the product in stable ef's we require that each of its elements is in an unambiguous subconfiguration, but this follows from condition (iii) above.

Conversely, suppose x is a configuration of $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$. Conditions (i) and (ii) are obvious. Any $c \in x$ is in an unambiguous finite subconfiguration of x of which a serialization produces a chain $c_1, \cdots, c_n$ required for (iii). $\square$

The edc

$$T \circledast S =_{\mathrm{def}} \mathrm{Pr}((\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S))$$

is isomorphic to the pullback of the last section, so there is no conflict of notation—see Proposition 17.2. It represents the composition of pre-strategies, including internal, neutral elements arising from synchronizations.

To obtain the composition of pre-strategies we hide the internal elements due to synchronizations. The edc of the composition of pre-strategies is defined to be

$$T \odot S =_{\mathrm{def}} \mathrm{Pr}((\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)) \downarrow V\,,$$

the projection onto "visible" elements,

$$\begin{aligned} V =_{\mathrm{def}} \ & \{p \in \mathrm{Pr}((\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)) \mid \exists s \in S.\ top(p) = (s, *)\} \cup \\ & \{p \in \mathrm{Pr}((\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)) \mid \exists t \in T.\ top(p) = (*, t)\}\,. \end{aligned}$$

Finally, the composition $\tau \odot \sigma$ is defined by the span

$$\begin{tikzcd} & T \odot S \arrow[dl, "\upsilon_1"'] \arrow[dr, "\upsilon_2"] & \\ A^\perp & & C \end{tikzcd}$$

where υ_1 and υ_2 are maps of edc's, which on p of $T \odot S$ act so $\upsilon_1(p) = \sigma_1(s)$ when $top(p) = (s, *)$ and $\upsilon_2(p) = \tau_2(t)$ when $top(p) = (*, t)$, and are undefined elsewhere.

Proposition 17.2. *The stable ef* $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$ *with maps* f_1 *and* f_2 *is the pullback of* $\sigma \| C$ *and* $A \| \tau$, *where*

$$f_1(d) = \begin{cases} (1, \pi_1(d)) & \text{if } \pi_1(d) \text{ is defined,} \\ (2, \tau_2\pi_2(d)) & \text{otherwise} \end{cases}$$

and

$$f_2(d) = \begin{cases} (2, \pi_2(d)) & \text{if } \pi_2(d) \text{ is defined,} \\ (1, \sigma_1\pi_1(d)) & \text{otherwise.} \end{cases}$$

17.4 Edc strategies

We imitate [**?**] and provide necessary and sufficient conditions for a pre-strategy in edc's to be stable up to isomorphism under composition with copycat. Fortunately we can inherit a great deal from the proof of [**?**].

An edc strategy in a game A is an edc pre-strategy $\sigma : S \to A$ such that $\gamma_A \odot \sigma \cong \sigma$. In the next two sections we will show that an edc pre-strategy $\sigma : S \to A$ is an edc strategy if it is subject to the following axioms:

(1) *innocence:*
+-innocence: if $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$;
−-innocence: if $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

(2) *+-consistency:* $X \in \mathrm{Con}_S$ if $\sigma X \in \mathrm{Con}_A$ and $[X]^+ \in \mathrm{Con}_S$, for $X \subseteq_{\mathrm{fin}} S$. (Recall $[X]^+$ comprises the +ve elements in the downwards closure of X.)

(3) *≡-saturation:* $s_1 \equiv_S s_2$ if $\sigma(s_1) = \sigma(s_2)$.

(4) *∃-receptivity:* $\sigma x \overset{a}{\longrightarrow\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists s \in S.\ x \overset{s}{\longrightarrow\!\!\subset}$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$. (Note we no longer have uniqueness.)

(5) *non-redundancy:* $[s_1) = [s_2)$ & $s_1 \equiv_S s_2$ & $pol_S(s_1) = pol_S(s_2) = - \implies s_1 = s_2$.

The converse "only if" directions of axioms (2) and (3) are automatic. The new axiom (2) holds automatically for traditional strategies expressed as prime event structures. Reading (2) contrapositively, it says that any inconsistency derives from inconsistency in the underlying game or from prior moves of Player; so Player cannot impose additional consistency constraints on moves of Opponent. In the presence of axiom (3), the non-redundancy axiom (5), is equivalent to

$$[s_1]^+ = [s_2]^+ \ \&\ \sigma(s_1) = \sigma(s_2) \ \&\ pol_S(s_1) = pol_S(s_2) = - \implies s_1 = s_2$$

which says that the only distinctions an edc strategy makes between Opponent moves are those due to the game or prior distinctions between Player moves.

Proposition 17.3. *Axiom (2), +-consistency, is equivalent to*

$$\forall s, s' \in S, x \in \mathcal{C}(S).$$
$$x \overset{s}{\longrightarrow\!\!\subset} \ \&\ x \overset{s'}{\longrightarrow\!\!\subset} \ \&\ pol(s) = - \ \&\ \sigma x \cup \{\sigma(s), \sigma(s')\} \in \mathcal{C}(A) \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. The proof is very like that of Lemma 5.1.

Assuming axiom (2) it is easy to show the property above. Suppose $x \overset{s}{\longrightarrow\!\!\subset} x_1$ and $x \overset{s'}{\longrightarrow\!\!\subset} x_2$ with $pol(s) = -$ and $\sigma x_1 \uparrow \sigma x_2$. Taking $X =_{\mathrm{def}} x_1 \cup x_2$, axiom (2) yields the consistency of $x_1 \cup x_2$ ensuring $x_1 \cup x_2 \in \mathcal{C}(S)$.

To show the converse assume the property of the proposition's statement. Suppose both $[X]^+$ and σX are consistent. We have $[X]^+ \subseteq^- [X]$. Let z be a maximal configuration of S such that

$$[X]^+ \subseteq z \subseteq [X] .$$

Suppose, to obtain a contradiction, that $z \subsetneq [X]$. Then there is a $\leq$-minimal, necessarily $-$ve $s \in [X] \setminus z$. From the minimality of s we obtain $[s) \subseteq z$. Take a covering chain

$$[s) \overset{s_1}{\longrightarrow\!\subset} \cdots \overset{s_n}{\longrightarrow\!\subset} z .$$

As $[s) \overset{s}{\longrightarrow\!\subset} [s]$ and $\sigma\{s, s_1, \cdots, s_n\} \subseteq \sigma X \in \mathrm{Con}_A$, by repeated use of the property above in the proposition, we obtain a configuration $z' \subseteq [X]$ with $z \overset{s}{\longrightarrow\!\subset} z'$ —the desired contradiction. Hence $[X] = z$, ensuring X consistent, as required to establish axiom (2). □

17.4.1 Necessity

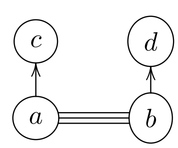
****HERE AND IN "Sufficiency" THERE ARE SOME NOTATIONAL INCONSISTENCIES WITH OTHER PARTS OF THE CHAPTER, USING $\mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S)$ FOR THE FINITE CONFIGS OF $(\mathcal{C}^\infty(\mathbb{C}_A), =) \circledast (\mathcal{C}^\infty(S), \equiv_S)$****

We shall exploit the close connection between pre-strategies in edc's and pre-strategies in event structures that we inherit from the adjunction from event structures $\mathcal{ES}$ to edc's $\mathcal{EDC}$—see Section 16.13. However we have to be careful.

While the right adjoint of the adjunction from event structures $\mathcal{ES}$ to edc's $\mathcal{EDC}$ preserves the composition of pre-strategies before hiding—because such composition is given by pullback—it is not the case that the right adjoint preserves their composition (with hiding). Hiding is not preserved by the right adjoint.

Recall the left adjoint from $\mathcal{ES}$ to $\mathcal{EDC}$ simply regards an event structure as an edc with the identity relation as its equivalence. To avoid clutter we shall identify an event structure A with its edc. Recall the right adjoint produces an event structure from an edc by simply making distinct elements in the equivalence of the edc inconsistent. Let us denote the action of the right adjoint by $E \mapsto E_0$. With these notational understandings the counit of the adjunction at an edc E is given essentially by the identity function (not the identity map) $E_0 \hookrightarrow E$; the identity function provides a rigid inclusion map from E to E' where all that changes is consistency.

Example 17.4. We illustrate why hiding is not preserved by the right adjoint. Consider the edc E



Let $V = \{a, b\}$. The event structure obtained after projection $(E \downarrow V)_0$ is

which is not isomorphic to the projection of the event structure obtained from the edc $E_0 \downarrow V$

$$c \sim\!\sim\!\sim d$$

The reason is that inconsistency (conflict) is always preserved upwards w.r.t. causal dependency while the equivalence relation of an edc need not. $\square$

Notice in the above example that there is an obvious map from $E_0 \downarrow V$ to $(E \downarrow V)_0$. Such a map exists in general and will help us relate the composition of edc pre-strategies to the composition of the event-structure strategies they are associated with. Suppose $f : E \to E'$ is a map of edc's. Let $V \subseteq E$ be the subset at which f is defined. Then f factors as

$$E \rightharpoonup E \downarrow V \to E'$$

which images under the right adjoint to

$$E_0 \rightharpoonup (E \downarrow V)_0 \to E'_0 .$$

Compare this with the factorisation of $f_0 : E_0 \to E'_0$, the image of f under the right adjoint; it is the same underlying partial function but now regarded as a map of event structures. This map has factorisation

$$E_0 \rightharpoonup E_0 \downarrow V \to E'_0 .$$

Its universal characterisation yields a total map

$$E_0 \downarrow V \to (E \downarrow V)_0$$

with the accompanying commutations. The map is the identity function id_V on events providing a rigid inclusion $E_0 \downarrow V \hookrightarrow (E \downarrow V)_0$.

We carry the same notation over to pre-strategies: an edc pre-strategy $\sigma : S \to A$ is sent to an event-structure pre-strategy $\sigma_0 : S_0 \to A_0 = A$, essentially got as the precomposition of $S_0 \hookrightarrow S$ with σ. Composition of edc pre-strategies is obtained from a pullback—preserved by the right adjoint to event structures—followed by hiding. Before hiding we have the following commuting diagram:

$$\begin{tikzcd}
T_0 \circledast S_0 = (T \circledast S)_0 \arrow[r, hook] \arrow[dr, "\tau_0 \circledast \sigma_0 = (\tau \circledast \sigma)_0"'] & T \circledast S \arrow[d, "\tau \circledast \sigma"] \\
& A^\perp \| C .
\end{tikzcd}$$

After hiding, for the reason above, there is a canonical map

$$\tau_0 \odot \sigma_0 \to (\tau \odot \sigma)_0$$

from composition of the event-structure pre-strategies of edc pre-strategies to the event-structure pre-strategy of the composition of the original edc pre-strategies; again the map has underlying function the identity on events and provides a rigid inclusion $T_0 \odot S_0 \hookrightarrow (T \odot S)_0$.

Despite the map $\tau_0 \odot \sigma_0 \to (\tau \odot \sigma)_0$ not being an isomorphism it is helpful to relate the two compositions of pre-strategies, between event-structure pre-strategies and between edc pre-strategies. The two compositions have much structure in common: they share the same set of events and the same causal dependency relation. What they don't share is a common consistency relation.

In this section we are specifically concerned with the composition $\gamma_A \odot \sigma$ of copycat with an edc pre-strategy and showing that it necessarily satisfies axioms (1)-(5). We pause to note a proposition that will be useful in its proof.

Proposition 17.5. *Let $z \in \mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S)$. Let $a \in A$ with $pol_A(a) = -$. Let u be an event of the family $\mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S)$. Then, $u \rightarrowtriangle_z (*, (2, a))$ iff $u = (*, (2, a'))$ for some $a \in A$ with $a' \rightarrowtriangle_A a$.*

Proof. *"Only if"*: Assume $u \rightarrowtriangle_z (*, (2, a))$. As u is an event of the family $\mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S)$ it must have the form $(s, (1, a'))$, wth $\sigma(s) = \overline{a'}$, or $(*, (2, a'))$. By Lemma 3.27 we must have either $(1, a') \rightarrowtriangle_{\mathbb{C}_A} (2, a)$ or $(2, a') \rightarrowtriangle_{\mathbb{C}_A} (2, a)$, but only the latter, with $a' \rightarrowtriangle_A a$, is possible as a is $-$ve. *"If"*: Conversely, if $a' \rightarrowtriangle_A a$ it can be checked that $z \cup \{(*, (2, a'))\} \in \mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S)$. □

Lemma 17.6. *Let $\sigma : S \to A$ be a pre-strategy in edc's. The composition $\gamma_A \odot \sigma$ satisfies axioms (1), (2), (3), (4) and (5).*

Proof. (1)*innocence*: By the remarks above $\gamma_A \odot \sigma_0$ and $\gamma_A \odot \sigma$ share the same events and causal dependency. Hence $\gamma_A \odot \sigma$ inherits innocence from that of $\gamma_A \odot \sigma_0$.

(2) *+-consistency*: We first prove the property

$$\text{(step)} \qquad \begin{aligned} & w \overset{u_1}{\longrightarrow\!\subset} z_1 \;\&\; w \overset{u_2}{\longrightarrow\!\subset} z_2 \;\&\; pol(u_1) = - \text{ in } \mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S) \\ & \&\; (\pi_2 z_1)_2 \cup (\pi_2 z_2)_2 \in \mathcal{C}(A) \\ & \implies z_1 \cup z_2 \in \mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S) \,. \end{aligned}$$

Let $z =_{\text{def}} z_1 \cup z_2$. We show $\pi_1 z$ is consistent in S and $\pi_2 z$ is consistent in $\mathbb{C}_A$. The finite set $\pi_2 z \subseteq \mathbb{C}_A$ is down-closed, being the union of (down-closed) configurations $\pi_2 z_1$ and $\pi_2 z_2$. Thus $\pi_2 z$ is consistent in $\mathbb{C}_A$ if the two components $(\pi_2 z)_1$ and $(\pi_2 z)_2$ are consistent in $A^\perp$ and A respectively.

In the case where $pol(u_2) \in \{+, -\}$ the configuration $\pi_1 z = \pi_1 w$, so is clearly consistent. In this case $(\pi_2 z)_1 = (\pi_2 w)_1$ is consistent as is $(\pi_2 z)_2 = (\pi_2 z_1)_2 \cup (\pi_2 z_2)_2$.

In the case where $pol(u_2) = 0$, we have $\pi_1 z = \pi_1 z_2$ which is consistent in S, while $(\pi_2 z)_1 = (\pi_2 z_2)_1$ and $(\pi_2 z)_2 = (\pi_2 z_1)_2$, both of which are consistent.

For z to be consistent we also require

$$\pi_1(u_1) \equiv_S \pi_1(u_2) \text{ or } \pi_2(u_1) = \pi_2(u_2) \implies u_1 \equiv u_2 \,.$$

As $\pi_1(u_1)$ is undefined the only nontrivial case is when $\pi_2(u_1) = \pi_2(u_2)$. But then we must have $u_1 = (*, (2, a)) = u_2$, for some $a \in A$.

To show +-consistency, assume that

$$
\begin{aligned}
& x \stackrel{p_1}{-\!\!\!\subset} y_1 \ \& \ x \stackrel{p_2}{-\!\!\!\subset} y_2 \ \& \ pol(p_1) = - \text{ in } \mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A \odot S) \\
& \& \ (\gamma_A \circledast \sigma) y_1 \cup (\gamma_A \circledast \sigma) y_2 \in \mathcal{C}(A)\,.
\end{aligned}
$$

Let p_1 have top u_1, $-$ve by assumption, and p_2 have top u_2. By Proposition 17.5 the local immediate causal predecessors of a visible $-$ve event in $\mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A \circledast S)$ must themselves be visible. It follows that

$$\bigcup x \stackrel{u_1}{-\!\!\!\subset} \bigcup y_1$$

in $\mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A) \circledast \mathcal{C}(S)$. Meanwhile

$$\bigcup x \stackrel{o}{-\!\!\!\subset} \cdot \cdots \cdot \stackrel{o}{-\!\!\!\subset} \cdot \stackrel{u_2}{-\!\!\!\subset} \bigcup y_2$$

in $\mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A) \circledast \mathcal{C}(S)$ a covering chain of synchronised events (signified by o) up to the occurrence of the visible event u_2. In addition $(\gamma_A \circledast \sigma) y_1 = (\pi_2 \bigcup y_1)_2$ and $(\gamma_A \circledast \sigma) y_2 = (\pi_2 \bigcup y_2)_2$ so $(\pi_2 \bigcup y_1)_2 \cup (\pi_2 \bigcup y_2)_2$ is a configuration of A. Now a straightforward induction using the property (step) above shows that $\bigcup y_1 \cup \bigcup y_2 \in \mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A) \circledast \mathcal{C}(S)$. The induction is illustrated below; each step of the "ladder" is an application of (step):

$$
\begin{tikzcd}
\bigcup y_1 \arrow[r, dashed] & \cdot \arrow[r, dashed] & \cdot & \cdots & \cdot \arrow[r, dashed] & \cdot \arrow[r, dashed] & \bigcup y_1 \cup \bigcup y_2 \\
\bigcup x \arrow[u, "u_1"] \arrow[r, "o"] & \cdot \arrow[u, dashed] \arrow[r] & \cdot \arrow[u, dashed] & \cdots & \cdot \arrow[u, dashed] \arrow[r, "o"] & \cdot \arrow[u, dashed] \arrow[r, "u_2"] & \bigcup y_2 \arrow[u, dashed]
\end{tikzcd}
$$

It follows that $y_1 \cup y_2$ is consistent so a configuration of $\mathbb{C}\!\!\!\mathbb{C}_A \odot S$, as required to show (2).

(3) $\equiv$-*saturated*: Suppose $\gamma_A \odot \sigma(q_1) = \gamma_A \odot \sigma(q_2) = a$, where $q_1, q_2 \in \mathbb{C}\!\!\!\mathbb{C}_A \odot S$. Then $top(q_1) = top(q_2) = (*, (2, a))$ which implies $q_1 \equiv q_2$ by definition.

(4) $\exists$-*receptivity*: Suppose $x \in \mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A \odot S)$ with $\gamma_A \odot \sigma x \stackrel{a}{-\!\!\!\subset}$ where $pol_A(a) = -$. Then $\bigcup x \in \mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A) \circledast \mathcal{C}(S)$ and $\gamma_A \circledast \sigma \bigcup x = \gamma_A \pi_2 \bigcup x \stackrel{a}{-\!\!\!\subset}$. There is an event $(*, (2, a))$ of $\mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A) \circledast \mathcal{C}(S)$. Recall from Proposition 17.5 that any immediate causal predecessor of $(*, (2, a))$ within any configuration takes the form $(*, (2, a'))$ with $a' \rightarrowtriangle_A a$. For this reason $z =_{\text{def}} \bigcup x \cup \{(*, (2, a))\}$ is a configuration in $\mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A) \circledast \mathcal{C}(S)$, as can be checked. Taking $q =_{\text{def}} [(*, (2, a))]_z$ we obtain $x \stackrel{q}{-\!\!\!\subset}$ and $\gamma_A \odot \sigma(q) = a$.

(5) *non-redundancy*: Suppose $[q_1) = [q_2)$ and $q_1 \equiv_S q_2$ for two $-$ve events in $\mathbb{C}\!\!\!\mathbb{C}_A \odot S$. Then q_1 and q_2 are prime configurations of $\mathcal{C}(\mathbb{C}\!\!\!\mathbb{C}_A) \circledast \mathcal{C}(S)$ with $top(q_1) = top(q_2) = (*, (2, a))$ for some $a \in A$ of $-$ve polarity. By Lemma 17.5

$$u \rightarrowtriangle_{q_1} (*, (2, a)) \text{ iff } u = (*, (2, a')) \text{ for some } a \in A \text{ with } a' \rightarrowtriangle_A a\,.$$

Because each $(*, (2, a'))$ is 'visible' (*i.e.* remains unhidden under composition),

$$[q_1) = \{[(*, (2, a'))]_{q_1} \mid a' \rightarrowtriangle_A a\}\,.$$

Analogous characterisations hold with q_2 in place of q_1. Consequently,

$$\begin{aligned} q_1 &= \{(*, (2, a))\} \cup \bigcup \{[(*, (2, a'))]_{q_1} \mid a' \rightarrowtriangle_A a\} \\ &= \{(*, (2, a))\} \cup \bigcup [q_1) \\ &= \{(*, (2, a))\} \cup \bigcup [q_2) \\ &= \{(*, (2, a))\} \cup \bigcup \{[(*, (2, a'))]_{q_2} \mid a' \rightarrowtriangle_A a\} \\ &= q_2 \end{aligned}$$

□

17.4.2 Sufficiency

Let $\sigma : S \to A$ be a pre-strategy in edc's which satisfies axioms (1), (2), (3), (4), (5).

Under the right adjoint of the adjunction from $\mathcal{ES}$ to $\mathcal{EDC}$ the edc (with polarity) S is sent to the event structure (with polarity) S_0. The counit provides a rigid inclusion map $S_0 \hookrightarrow S$; the configurations of S_0 are the unambiguous configurations of S. Through pre-composition with σ we obtain a map of event structures with polarity

$$\sigma_0 : S_0 \to A\,.$$

As a function on events σ_0 is exactly the same as σ.

Lemma 17.7. *The map $\sigma_0 : S_0 \to A$ is a strategy.*

Proof. The event structure S_0 shares the same causal dependency with S so innocence of σ_0 follows from innocence of σ. We obtain the existence part of receptivity for σ_0 directly from that of σ. We now only need verify the uniqueness part of receptivity for σ_0. Suppose $x \in \mathcal{C}(S_0)$ for which $x \overset{s_1}{\longrightarrow\!\!\subset}$ and $x \overset{s_2}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(S_0)$ where $\sigma_0(s_1) = \sigma_0(s_2)$ is $-$ve. Then $x \overset{s_1}{\longrightarrow\!\!\subset}$ and $x \overset{s_2}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(S)$ where moreover x is an unambiguous configuration of S. By axiom(2), $+$-consistency, $\{s_1, s_2\} \in \mathrm{Con}_S$. Consequently $s_1 \equiv s_2$ as $\sigma(s_1) = \sigma(s_2)$. From $\sigma(s_1) = \sigma(s_2)$ by $-$-innocence we deduce that $[s_1) = [s_2)$, essentially by a repetition of the argument for Lemma 4.4(i). Suppose $s \rightarrowtriangle s_1$. Then by $-$-innocence, $\sigma(s) \rightarrowtriangle \sigma(s_1)$. As $\sigma(s_1) = \sigma(s_2)$ and σ is a map of event structures there is $s' < s_2$ such that $\sigma(s') = \sigma(s)$. But s, s' both belong to the *unambiguous* configuration x, so $s = s'$ as σ is a map of edc's. Symmetrically, if $s \rightarrowtriangle s_2$ then $s < s_1$. It follows that $[s_1) = [s_2)$. Finally, by axiom (5), non-redundancy, we deduce that $s_1 = s_2$. □

Now that we know σ_0 is a strategy we can recall from Section **??** the isomorphism between strategies $\theta_0 : \gamma_A \odot \sigma_0 \to \sigma_0$. (In Section **??** we consider the isomorphism $\theta_0 : \sigma_0 \odot \gamma_A \to \sigma_0$ where $\sigma_0 : S \to A^\perp \| B$. Here we are considering the isomorphism obtained by duality in the special case where B is the empty game.) Let $p_0 : \mathcal{C}(\mathbb{CC}_A \odot S_0) \to \mathcal{C}(S_0)$ be the function $p_0(x) = \pi_1 \bigcup x$ for $x \in \mathcal{C}(\mathbb{CC}_A \odot S_0)$. Now the isomorphism is a map $\theta_0 : \mathbb{CC}_A \odot S_0 \to S_0$ such that $p_0(x) \subseteq^- \theta_0\, x$, for all $x \in \mathcal{C}(\mathbb{CC}_A \odot S)$, and $\sigma_0 \theta_0 = \gamma_A \odot \sigma_0$:

$$
\begin{tikzcd}
\mathbb{CC}_A \odot S_0 \arrow[r, dashed, "p_0"] \arrow[r, dotted, bend left=70, "\theta_0"] \arrow[r, phantom, bend left=35, "\subseteq^-"] \arrow[dr, "\gamma_A \odot \sigma_0"'] \arrow[dr, phantom, "\supseteq^-"] & S_0 \arrow[d, "\sigma_0"] \\
& A\,.
\end{tikzcd}
$$

The same underlying bijection as that of the map θ_0 will provide an isomorphism

$$\theta : \gamma_A \odot \sigma \to \sigma\,,$$

as will now be shown. For $x \in \mathcal{C}(\mathbb{CC}_A \odot S)$ define $p(x) =_{\text{def}} \pi_1 \bigcup x$, projecting to a configuration of S. Then, for $x \in \mathcal{C}(\mathbb{CC}_A \odot S)$,

$$p(x) = \bigcup \{ p_0(x_0) \mid x_0 \subseteq x \ \&\ x_0 \in \mathcal{C}(\mathbb{CC}_A \odot S_0) \}\,.$$

In other words p extends p_0 from the unambiguous finite configurations of $\mathbb{CC}_A \odot S$ to all the finite configurations. We tentatively extend θ_0 from unambiguous finite configurations of $\mathbb{CC}_A \odot S$ to all finite configurations by taking

$$\theta x =_{\text{def}} \bigcup \{ \theta_0 x_0 \mid x_0 \subseteq x \ \&\ x_0 \in \mathcal{C}(\mathbb{CC}_A \odot S_0) \}\,,$$

when $x \in \mathcal{C}(\mathbb{CC}_A \odot S)$. Clearly

$$p(x) \subseteq^- \theta x$$

because $p_0(x_0) \subseteq^- \theta_0 x_0$ for each unambiguous subconfiguration x_0 of x. From axiom (2), the +-consistency of σ, it follows that the rhs is consistent, so a configuration of S. It follows that we have a map of edc's $\theta : \mathbb{CC}_A \odot S \to S$ and moreover a map of edc strategies $\theta : \gamma_A \odot \sigma \to \sigma$ as the required $\sigma\theta = \gamma_A \odot \sigma$ is a direct consequence of $\sigma_0 \theta_0 = \gamma_A \odot \sigma_0$. It also follows that θ reflects as well as preserves equivalence.

It remains to show that θ reflects consistency. The proof depends on the function p reflecting consistency on special sets, those $\bigcup X$ for which $X \subseteq_{\text{fin}} \mathbb{CC}_A \odot S$ comprises primes of which all the top events are +ve (Lemma 17.9 below).

Lemma 17.8. *Let $q \in \mathbb{CC}_A \odot S$ with $top(q) \in \{+\}$. Then,*

$$\pi_2 q \subseteq \overline{\sigma \pi_1 q} \| \sigma \pi_1 q \in \mathcal{C}(\mathbb{CC}_A)$$

Proof. Let $q \in \mathbb{CC}_A \odot S$ with $top(q) = (*, c_0)$ where $pol_{\mathbb{CC}_A}(c_0) = +$. Firstly note that $\sigma\pi_1 q \in \mathcal{C}(A)$ being the image under the map $\sigma\pi_1$ of the configuration q of the family $\mathcal{C}(\mathbb{CC}_A) \circledast \mathcal{C}(S)$. Secondly note any member of q must have the form $(*, c)$, where $c \in \mathbb{CC}_A$ has the form $c = (2, a)$ for some $a \in A$, or (s, c), where

$s \in S$ and $c \in \mathbb{C}_A$ is of the form $c = (1, \overline{a})$ for some $a \in A$ such that $\sigma(s) = a$. These facts follow directly from the definition of $\mathbb{C}_A \odot S$ and that of the family $\mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S)$ on which its construction depends.

We show by induction on n that if $u \rightarrowtriangle_q^n (*, c_0)$ then

either $u = (*, c)$ and $\exists a \in A.\ c = (2, a)\ \&\ a \in \sigma\pi_1 q$

or $u = (s, c)$ and $\exists a \in A.\ c = (1, \overline{a})\ \&\ a \in \sigma\pi_1 q$.

In the basis case where $n = 0$ we must have $(s_0, \overline{c_0}) \rightarrowtriangle_q (*, c_0)$ for some $s_0 \in \pi_1 q$ because of the dependency $\overline{c_0} \rightarrowtriangle_{\mathbb{C}_A} c_0$ of copycat. Then $c_0 = (2, a_0)$ with $\sigma(s_0) = a_0$ ensuring $a_0 \in \sigma\pi_1 q$. For the induction step assume $n > 0$. Then

$$u \rightarrowtriangle_q u_1 \rightarrowtriangle_q^{(n-1)} (*, a_0) .$$

If u has the form (s, c) then we directly have $c = (1, \overline{a})$ and $\sigma(s) = a$ which combined with $s = \pi_1(u)$ yields $a \in \sigma\pi_1 q$. Otherwise $u = (*, c)$ and

either $(*, c) \rightarrowtriangle_q (s_1, c_1) = u_1$

or $(*, c) \rightarrowtriangle_q (*, c_1) = u_1$.

In the former case by Lemma 3.27, we must have $c \rightarrowtriangle_{\mathbb{C}_A} c_1$, while c and c_1 belong to different components of $\mathbb{C}_A$, ensuring that $c = \overline{c_1}$. Then $c = (2, a)$ and $c_1 = (1, \overline{a})$ for some $a \in A$. Inductively $a \in \sigma\pi_1 q$, maintaining the induction hypothesis.

In the latter case by Lemma 3.27, $c \rightarrowtriangle_{\mathbb{C}_A} c_1$, necessarily within the same rhs component of $\mathbb{C}_A$. Then $c = (2, a)$ and $c_1 = (2, a_1)$ with $a \rightarrowtriangle_A a_1$ in A. As inductively $a_1 \in \sigma\pi_1 q$ we deduce $a \in \sigma\pi_1 q$ as $\sigma\pi_1 q$ is a configuration of A, maintaining the induction hypothesis.

Having established the induction hypothesis we obtain

$$\pi_2 q \subseteq \overline{\sigma\pi_1 q} \| \sigma\pi_1 q$$

directly. As $\sigma\pi_1 q$ is a configuration of A and $\overline{\sigma\pi_1 q}$ its copy as a configuration of its dual $A^\perp$, we have that $\overline{\sigma\pi_1 q} \| \sigma\pi_1 q$ is a configuration of $\mathbb{C}_A$. □

Lemma 17.9. *Let $X \subseteq_{\text{fin}} \mathbb{C}_A \odot S$ with $topX \subseteq \{+\}$. Then.*

$$\pi_1 \bigcup X \in \text{Con}_S \implies \bigcup X \in \mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S) .$$

Proof. Suppose $X \subseteq_{\text{fin}} \mathbb{C}_A \odot S$ with $topX \subseteq \{+\}$. Assume $\pi_1 \bigcup X \in \text{Con}_S$. Then $\pi_1 \bigcup X \in \mathcal{C}(S)$ being the consistent union of configurations $\pi_1 q$ for $q \in X$. Thus its image under σ is a configuration $\sigma\pi_1 \bigcup X \in \mathcal{C}(A)$. Accordingly we obtain a configuration

$$\overline{\sigma\pi_1 \bigcup X} \| \sigma\pi_1 \bigcup X$$

of the copycat strategy.

We shall show $\pi_2 \bigcup X \in \mathcal{C}(\mathbb{C}_A)$. Note that $\pi_2 \bigcup X$ is down-closed being the union of $\pi_2 q$ for $q \in X$. To show that $\pi_2 \bigcup X$ is also consistent we observe

$$\forall q \in X.\ \pi_2 q \subseteq \overline{\sigma\pi_1 q} \| \sigma\pi_1 q \in \mathcal{C}(\mathbb{C}_A)$$

from Lemma 17.8, to derive

$$\pi_2 \bigcup X \subseteq \overline{\sigma\pi_1 \bigcup X} \| \sigma\pi_1 \bigcup X\,.$$

As the rhs is a configuration of $\mathbb{C}_A$ the set $\pi_2 \bigcup X$ is consistent and, being down-closed, also a configuration of $\mathbb{C}_A$.

Now we know $\pi_1 \bigcup X \in \mathcal{C}(S)$ and $\pi_2 \bigcup X \in \mathcal{C}(\mathbb{C}_A)$ it is a routine matter to verify that $\bigcup X \in \mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S)$, as required. □

It remains to show that θ reflects consistency. To this end, suppose $X \subseteq_{\text{fin}} \mathbb{C}_A \odot S$, the set X is down-closed and $\theta X \in \text{Con}_S$; so in fact θX is a configuration of S as it is down-closed. Then

$$\pi_1 \bigcup X = \bigcup_{q \in X} p([q]) \subseteq^- \theta X\,.$$

As θX is consistent, $\pi_1 \bigcup X$ is also consistent in S. *A fortiori* $\pi_1 \bigcup (X^+)$ is consistent in S. By Lemma 17.9, $\bigcup(X^+)$ is a configuration of $\mathcal{C}(\mathbb{C}_A) \circledast \mathcal{C}(S)$. From the construction of $\mathbb{C}_A \odot S$, it follows that X^+ is consistent in $\mathbb{C}_A \odot S$. Finally, because $\gamma_A \odot \sigma$ satisfies $+$-consistency, axiom (2), from Lemma 17.6, we deduce that X is consistent in $\mathbb{C}_A \odot S$, as required.

As θ is a bijection on events which preserves and reflects consistency, and equivalence and causal dependency (because θ_0 does) it is an isomorphism $\theta : \mathbb{C}_A \odot S \cong S$ of edc's.

We conclude:

Theorem 17.10. *Let $\sigma : S \to A$ be an edc pre-strategy. Then, $\sigma \cong \gamma_A \odot \sigma$ iff σ satisfies axioms (1)-(5).*

Corollary 17.11. *Let $\sigma : B \twoheadrightarrow C$ be an edc pre-strategy. Then, $\sigma \cong \gamma_C \odot \sigma \odot \gamma_B$ iff σ satisfies axioms (1)-(5).*

Proof. Write $A = B^\perp \| C$. The construction of $\gamma_C \odot \sigma \odot \gamma_B$ coincides with that of $\gamma_A \odot \sigma$. □

The new axiom (2) holds automatically for traditional strategies expressed as prime event structures. Reading (2) contrapositively, it says that any inconsistency derives from inconsistency in the underlying game or from prior moves of Player; so Player cannot impose additional consistency constraints on moves of Opponent. Axiom (4) says the only distinctions the strategy makes between Opponent moves are those due to the game or prior distinctions between Player moves.

We can derive a stronger form of receptivity for edc strategies.

Proposition 17.12. *In an edc strategy $\sigma : S \to A$ whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$, where $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(A)$, there is a maximum $x' \in \mathcal{C}(S)$ so that $x \subseteq x'$ & $\sigma x' = y$, i.e.*

$$\begin{tikzcd}
x \arrow[r, dotted, "\subseteq"] \arrow[d, mapsto, "\sigma"'] & x' \arrow[d, dotted, mapsto, "\sigma"] \\
\sigma x \arrow[r, phantom, "\subseteq^-"] & y\,.
\end{tikzcd}$$

Proof. Suppose $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$, where $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(A)$. Using $\exists$-receptivity, we obtain some $x' \in \mathcal{C}(S)$ such that $x \subseteq x'$ and $\sigma x' = y$. To see that we can obtain a maximum such x' notice by $+$-consistency that if $x \subseteq x_1$ and $x \subseteq x_2$ for $x_1, x_2 \in \mathcal{C}(S)$ with $\sigma x_1 = \sigma x_2 = y$, then $x_1 \cup x_2$ is consistent, so in $\mathcal{C}(S)$. □

The definition of *race-freeness* lifts directly from event structures with polarity to edc's.

Proposition 17.13. *In an edc strategy $\sigma : S \to A$ if the game A is race-free then so is S.*

Proof. Directly from $+$-consistency. Assume $x \overset{-}{\longrightarrow\!\!\!\!\!\subset} x_1$ and $x \overset{+}{\longrightarrow\!\!\!\!\!\subset} x_2$. Assuming A is race-free we obtain $\sigma x_1 \cup \sigma x_2$ is a configuration. Now, from axiom (2), $+$-consistency, taking $X =_{\text{def}} x_1 \cup x_2$ and remarking that $[X]^+ = x_2$ so consistent, we obtain that X is consistent, ensuring that $x_1 \cup x_2$ is a configuration of S, as required to show race-freeness. □

In considering the composition of edc strategies without hiding the following lemma is useful.

Lemma 17.14. *Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be edc strategies. Suppose $q \rightarrowtriangle p$ in $T \circledast S$.*

(i) If $pol(p) = -$ then $top(q) \in V$ (i.e. q is a visible event).

(ii) If $pol(q) = +$ then $top(p) \in V$ (i.e. p is a visible event).

Proof. We refer to the concrete construction of $T \circledast S$ in Section 17.3. We prove (i); the proof of (ii) is similar. Suppose $q \rightarrowtriangle p$ in $T \circledast S$ and $pol(p) = -$. Then $top(p)$ has the form $(s, *)$ with s $-$ve in S or $(*, t)$ with t $-$ve in T. Consider the case $top(p) = (s, *)$. Suppose $top(q)$ had the form (s', t') with $s' \in S$ and $t' \in T$. Then $s' \rightarrowtriangle_S s$ by Lemma 3.27. From the innocence of σ we obtain $\sigma(s') \rightarrowtriangle_{A^\perp \| B} \sigma(s)$ and hence $\sigma_1(s') \rightarrowtriangle_{A^\perp} \sigma_1(s)$. But then $(s', t') \notin R$, a contradiction. The case $top(p) = (*, t)$ similarly leads to a contradiction, using the innocence of τ. □

Our earlier treatment of (explicit) partial strategies generalises straightforwardly. An explicit partial strategy in edc's from game A to game B comprises a map of edc's $\sigma : S \to A^\perp \| N \| B$, where N consists solely of neutral events, satisfying exactly the same axioms, (1)-(5), as above, but where now events may also be neutral. The defined part of such a partial strategy in edc's is a strategy in edc's****CHECK***

17.5 A bicategory of edc strategies

Below we give an alternative description of an edc strategy as a function on events which restricts to a conventional strategy on unambiguous configurations, with some extra properties. The proposition uses the counit of the coreflection from $\mathcal{ES}$ to $\mathcal{EDC}$; it has components $S_0 \hookrightarrow S$ where essentially S_0 is the event structure obtained by making all distinct $\equiv$-equivalent event conflicting, so with configurations the unambiguous configurations of S.

Proposition 17.15. *An edc strategy to a game A corresponds to a function $\sigma : S \to A$ from events of an event structures S to those of A preserving polarities s.t.*

$fx \in \mathcal{C}(A)$ for all $x \in \mathcal{C}(S)$;

an edc $(S, \equiv)$ is obtained by defining $s \equiv s'$ iff $\sigma(s) = \sigma(s')$;

the restriction $\sigma_0 : S_0 \to A$ is a concurrent strategy as earlier;

and (+-consistency) for all $s, s' \in S, x \in \mathcal{C}(S)$

$$x \overset{s}{\longrightarrow\!\!\subset} \;\&\; x \overset{s'}{\longrightarrow\!\!\subset} \;\&\; pol(s) = - \;\&\; \sigma x \cup \{\sigma(s), \sigma(s')\} \in \mathcal{C}(A) \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

The corresponding edc strategy $\sigma : (S, \equiv) \to A$ is given by the function σ. Conversely from an edc strategy $\sigma : (S, \equiv) \to A$ we obtain such a function $\sigma : S \to A$.

Proof. Given an edc strategy $\sigma : (S, \equiv) \to A$ using Lemma 17.7 the function $f : S \to A$ clearly satisfies the conditions listed above. Conversely, given such a function $\sigma : S \to A$ we obtain an edc map $\sigma : (S, \equiv) \to A$ by taking $s_1 \equiv s_2$ iff $\sigma(s_1) = \sigma(s_2)$. By virtue of $\sigma_0 : S_0 \to A$ being a strategy $\sigma : (S, \equiv) \to A$ satisfies all conditions required to be an edc strategy but for +-consistency; and the latter is imposed directly as a condition on the function σ. □

Two cells comprise $f : \sigma \Rightarrow \sigma'$ where f is an edc map such that $\sigma = \sigma' f$. Note if f is a map of event structures with polarity then from the commutation $\sigma = \sigma' f$ it automatically preserves equivalence $\equiv$ and is thus a map of edc's; there are of course more edc maps f such that $\sigma = \sigma' f$ than those obtained as maps of event structures with polarity—see the following example.

Example 17.16. The game comprises a single Player move $\oplus$, the first strategy two parallel, and $\equiv$-equivalent, Player moves $\oplus \equiv \oplus$ and the second strategy a single Player move $\oplus$. The map is the obvious one collapsing the two parallel moves into one. □

Proposition 17.17. *Under the 'inclusion' functor taking event structures with polarity to edc's (endowing event structures with the identity as equivalence) a map which is a strategy becomes an edc strategy.*

Proof. If a total map $\sigma : S \to A$ is innocent and receptive then as a map of edc's it remains innocent and $\exists$-receptive, axioms (1) and (4). Axiom (3), $\equiv$-saturation is obvious, as is axiom (5), non-redundancy. It remains to establish axiom (2), +-consistency. Easiest is to prove the reformulation of +-consistency provided by Proposition 17.3. Suppose $x \xrightarrow{s}\!\!\subset x_1$ and $x \xrightarrow{s'}\!\!\subset x_2$ in $\mathcal{C}(S)$ with s −ve and that $\sigma x_1 \uparrow \sigma x_2$. Then $\sigma x_2 \xrightarrow{\sigma(s)}\!\!\subset$. By receptivity of σ there is $s'' \in S$ such that $x_2 \xrightarrow{s''}\!\!\subset$ and $\sigma(s'') = \sigma(s)$. But from the −-innocence of σ we derive $x \xrightarrow{s''}\!\!\subset$ and that $x \cup \{s''\} \uparrow x_2$ in $\mathcal{C}(S)$. Now both $x \xrightarrow{s}\!\!\subset$ and $x \xrightarrow{s''}\!\!\subset$ with $\sigma(s) = \sigma(s'')$. By the uniqueness part of receptivity, we immediately get $s'' = s$ ensuring that $x_1 \uparrow x_2$, as required. □

An edc strategy $\sigma : S \to A$ is *deterministic* iff S is deterministic in the old sense, forgetting about the equivalence $\equiv_S$, *i.e.*

$$\forall x \in \mathcal{C}(S), s_1, s_2 \in S.\ x \xrightarrow{s_1}\!\!\subset \ \&\ x \xrightarrow{s_2}\!\!\subset \ \&\ pol(s_1) = + \implies x \cup \{s_1.s_2\} \in \mathcal{C}(S)\,.$$

****

17.6 A language for edc strategies

duplication strategy $\sigma : A \to\!\!\!\!\!\!+\, A \| A$ is deterministic, if A is deterministic for Opponent, *i.e.* $A^\perp$ is deterministic as an event structure with polarity* As we now have parallel causes duplication strategy is more often, though not always, deterministic****

It seems (mathematically) sensible to say an edc strategy $\sigma : S \to A$ is deterministic if S is deterministic regarded as an event structure with polarity (*i.e.*, ignoring its equivalence), even though it may contain benign races between Player moves. When we adjoin probability later for a game A which is deterministic for Opponent we shall take $\delta_A : A \to\!\!\!\!\!\!+\, A \| A$ to have configuration-valuation assigning 1 to all finite configurations.****

Chapter 18

Probabilistic edc strategies

18.1 Probability with an Opponent

As before it will be convenient, to define a probabilistic stable ef in which some events are distinguished as Opponent events (where the other events may be Player events or "neutral" events due to synchronizations between Player and Opponent). Events which are not Opponent events we shall call p-events. For configurations x, y we shall write $x \subseteq^p y$ if $x \subseteq y$ and $y \setminus x$ contains no Opponent events; we write $x {-\!\!\subset}^p y$ when $x {-\!\!\subset} y$ and $x \subseteq^p y$; we continue to write $x \subseteq^- y$ if $x \subseteq y$ and $y \setminus x$ comprises solely Opponent events.

Definition 18.1. Let $\mathcal{F}$ be a stable ef $\mathcal{F}$ together with a specified subset of its events which are Opponent events. A *configuration-valuation* is a function $v : \mathcal{F} \to [0,1]$ for which $v(\varnothing) = 1$,

$$x \subseteq^- y \implies v(x) = v(y) \tag{1}$$

for all $x, y \in \mathcal{F}$, and satisfies the "drop condition"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0 \tag{2}$$

for all $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}$ with $y \subseteq^p x_1, \cdots, x_n$.

A *probabilistic equivalence family* a stable ef $\mathcal{F}$ together with a specified subset of Opponent events and a configuration-valuation $v : \mathcal{F} \to [0,1]$. The notion specialises to event structures with a distinguished subset of Opponent events.

In particular, a *probabilistic edc with polarity* comprises E an edc with polarity together with a configuration-valuation $v : \mathcal{C}(E) \to [0,1]$.

*****RACE-FREE WRT p and - moves? *****

Definition 18.2. Let A be (the edc of) a race-free event structure with polarity. A *probabilistic edc strategy* in A comprises a probabilistic edc S, v and an edc strategy $\sigma : S \to A$. [By Proposition 17.13, S will also be race-free.]

Let A and B be a race-free event structures with polarity. A *probabilistic edc strategy* from A to B comprises a probabilistic edc S, v and a strategy $\sigma : S \to A^\perp \| B$.

We remark that the configuration-valuation of an edc doesn't necessarily respect the equivalence of the edc; different prime causes of a common disjunctive event may well be associated with different probabilities.

Example 18.3. Recall the game of Section 16.2. ***the two watchers may be associated with probabilities $p \in [0,1]$ and $q \in [0,1]$ provided they form a configuration valuation *** diagram**** □

We extend the usual composition of edc strategies to probabilistic edc strategies. Assume probabilistic edc strategies $\sigma : S \to A^\perp \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^\perp \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$. We tentatively define their composition on stable ef's, taking v to be

$$v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$$

for x a finite configuration of $(\mathcal{C}(T), \equiv_T) \circledast (\mathcal{C}(S), \equiv_S)$.

Lemma 18.4. *Let $y, x_1, \cdots, x_n$ be finite configurations of $(\mathcal{C}(T), \equiv_T) \circledast (\mathcal{C}(S), \equiv_S)$ with $y {-\!\!\subset}^p x_1, \cdots, x_n$. Assume that $\pi_1 y {-\!\!\subset}^+ \pi_1 x_i$ when $1 \le i \le m$ and $\pi_2 y {-\!\!\subset}^+ \pi_2 x_i$ when $m+1 \le i \le n$. Then the drop function of $(\mathcal{C}(T), \equiv_T) \circledast (\mathcal{C}(S), \equiv_S)$ associated with v satisfies*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_v^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n]\,.$$

Proof. Under the assumptions of the lemma, by proposition 14.3,

$$d_v^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] = v_S(\pi_1 y) - \sum_{I_1} (-1)^{|I_1|+1} v_S(\bigcup_{i \in I_1} \pi_1 x_i)\,,$$

where I_1 ranges over sets satisfying $\varnothing \ne I_1 \subseteq \{1, \cdots, m\}$ s.t. $\{\pi_1 x_i \mid i \in I_1\}\uparrow$. Similarly,

$$d_v^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] = v_T(\pi_2 y) - \sum_{I_2} (-1)^{|I_2|+1} v_T(\bigcup_{i \in I_2} \pi_2 x_i)\,,$$

where I_2 ranges over sets satisfying $\varnothing \ne I_2 \subseteq \{m+1, \cdots, n\}$ s.t. $\{\pi_2 x_i \mid i \in I_2\}\uparrow$.

We show that when $\varnothing \ne I_1 \subseteq \{1, \cdots, m\}$,

$$\{\pi_1 x_i \mid i \in I_1\}\uparrow \text{ in } \mathcal{C}(S) \text{ iff } \{x_i \mid i \in I_1\}\uparrow \text{ in } (\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)\,.$$

"*If*": obvious as the projection π_1 preserves consistency. "*Only if*": Assume $\bigcup_{i \in I_1} \pi_1 x_i$ is a configuration of S. We use Proposition 17.1 to show $\bigcup_{i \in I_1} x_i$ is a configuration of $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$. Conditions (i) and (iii) of Proposition 17.1 obviously hold of $\bigcup_{i \in I_1} x_i$. From the assumption, certainly $\pi_1 \bigcup_{i \in I_1} x_i$ a configuration of S, as π_1 distributes through unions. To verify the remaining condition (ii) we need to show $X =_{\mathrm{def}} \pi_2 \bigcup_{i \in I_1} x_i$ a configuration of T. Clearly X is down-closed being the union of configurations $\pi_2 x_i$. That it is consistent and so a configuration of T follows from the $+$-consistency of τ: notice that $\pi_2 y \subseteq^- X$ so $X^+ = \pi_2 y^+$ is consistent as is τX, being equal to the configuration $\overline{\sigma_2 \bigcup_{i\in I_1} \pi_1 x_i} \| \varnothing$; hence by the $+$-consistency of τ, the set X is consistent and, being down-closed, a configuration in $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$. Similarly it can be shown that when $\varnothing \neq I_2 \subseteq \{m+1, \cdots, n\}$,

$$\{\pi_2 x_i \mid i \in I_2\}\uparrow \text{ in } \mathcal{C}(T) \text{ iff } \{x_i \mid i \in I_2\}\uparrow \text{ in } (\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)\,.$$

Hence in the equations

$$\bigcup_{i\in I_1} \pi_1 x_i = \pi_1 \bigcup_{i\in I_1} x_i \quad \text{and} \quad \bigcup_{i\in I_2} \pi_2 x_i = \pi_2 \bigcup_{i\in I_2} x_i$$

we know, for instance in the first equation, that $\bigcup_{i\in I_1} \pi_1 x_i$ is a configuration in $\mathcal{C}(S)$ iff $\bigcup_{i\in I_1} x_i$ is a configuration in $\mathcal{C}(T) \circledast \mathcal{C}(S)$; a similar fact holds for the second equation.

Making these rewrites and taking the product

$$d_v^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_v^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n]\,,$$

we obtain

$$\begin{aligned} v_S(\pi_1 y) \times v_T(\pi_2 y) \; &- \sum_{I_2} (-1)^{|I_2|+1} \; v_S(\pi_1 y) \times v_T(\pi_2 \bigcup_{i\in I_2} x_i) \\ &- \sum_{I_1} (-1)^{|I_1|+1} \; v_S(\pi_1 \bigcup_{i\in I_1} x_i) \times v_T(\pi_2 y) \\ &+ \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|} \; v_S(\pi_1 \bigcup_{i\in I_1} x_i) \times v_T(\pi_2 \bigcup_{i\in I_2} x_i)\,. \end{aligned}$$

But at each index I_2,

$$v_S(\pi_1 y) = v_S(\pi_1 \bigcup_{i\in I_2} x_i)$$

as $\pi_1 y \subseteq^- \pi_1 \bigcup_{i\in I_2} x_i$. Similarly, at each index I_1,

$$v_T(\pi_2 y) = v_T(\pi_2 \bigcup_{i\in I_1} x_i)\,.$$

Hence the product becomes

$$\begin{aligned} v_S(\pi_1 y) \times v_T(\pi_2 y) \; &- \sum_{I_2} (-1)^{|I_2|+1} \; v_S(\pi_1 \bigcup_{i\in I_2} x_i) \times v_T(\pi_2 \bigcup_{i\in I_2} x_i) \\ &- \sum_{I_1} (-1)^{|I_1|+1} \; v_S(\pi_1 \bigcup_{i\in I_1} x_i) \times v_T(\pi_2 \bigcup_{i\in I_1} x_i) \\ &+ \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|} \; v_S(\pi_1 \bigcup_{i\in I_1} x_i) \times v_T(\pi_2 \bigcup_{i\in I_2} x_i)\,. \end{aligned}$$

To simplify this further, we observe that

$$\{x_i \mid i \in I_1\}\uparrow \;\&\; \{x_i \mid i \in I_2\}\uparrow \iff \{x_i \mid i \in I_1 \cup I_2\}\uparrow\,.$$

The "$\Leftarrow$" direction is clear. We show "$\Rightarrow$." Assume $\{x_i \mid i \in I_1\}\uparrow$ and $\{x_i \mid i \in I_2\}\uparrow$. We obtain $\{\pi_1 x_i \mid i \in I_1\}\uparrow$ and $\{\pi_1 x_i \mid i \in I_2\}\uparrow$ as the projection map π_1 preserves consistency. Hence $\bigcup_{i\in I_1} \pi_1 x_i$ and $\bigcup_{i\in I_2} \pi_1 x_i$ are configurations of S. Furthermore, by assumption,

$$\pi_1 y \subseteq^+ \bigcup_{i\in I_1} \pi_1 x_i \quad \text{and} \quad \pi_1 y \subseteq^- \bigcup_{i\in I_2} \pi_1 x_i \,.$$

As S, an edc strategy over the race-free game $A^\perp \| B$, is automatically race-free—Proposition 17.13—we obtain

$$\bigcup_{i\in I_1\cup I_2} \pi_1 x_i \in \mathcal{C}(S)$$

by Proposition 5.5. Similarly, because T is race-free, we obtain

$$\bigcup_{i\in I_1\cup I_2} \pi_2 x_i \in \mathcal{C}(T)\,.$$

By Proposition 17.1, together these entail

$$\bigcup_{i\in I_1\cup I_2} x_i \in \mathcal{C}(T) \circledast \mathcal{C}(S)\,,$$

i.e. $\{x_i \mid i \in I_1 \cup I_2\}\uparrow$, as required; condition (i) of Proposition 17.1 is obvious while its condition (iii) is inherited by $\bigcup_{i\in I_1\cup I_2} x_i$ from its holding for each x_i, $i \in I_1 \cup I_2$. Notice too that

$$\pi_1 \bigcup_{i\in I_1} x_i \subseteq^- \pi_1 \bigcup_{i\in I_1\cup I_2} x_i \quad \text{and} \quad \pi_2 \bigcup_{i\in I_2} x_i \subseteq^- \pi_2 \bigcup_{i\in I_1\cup I_2} x_i\,,$$

which ensure

$$v_S(\pi_1 \bigcup_{i\in I_1} x_i) = v_S(\pi_1 \bigcup_{i\in I_1\cup I_2} x_i) \quad \text{and} \quad v_T(\pi_2 \bigcup_{i\in I_2} x_i) = v_T(\pi_2 \bigcup_{i\in I_1\cup I_2} x_i)\,,$$

so that

$$v(\bigcup_{i\in I_1\cup I_2} x_i) = v_S(\pi_1 \bigcup_{i\in I_1} x_i) \times v_T(\pi_2 \bigcup_{i\in I_2} x_i)\,.$$

We can now further simplify the product to

$$\begin{aligned} v(y) &- \sum_{I_2} (-1)^{|I_2|+1}\ v(\bigcup_{i\in I_2} x_i) \\ &- \sum_{I_1} (-1)^{|I_1|+1}\ v(\bigcup_{i\in I_1} x_i) \\ &+ \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|}\ v(\bigcup_{i\in I_1\cup I_2} x_i)\,. \end{aligned}$$

Noting that any subset I for which $\varnothing \neq I \subseteq \{1, \cdots, n\}$ either lies entirely within $\{1, \cdots, m\}$, entirely within $\{m+1, \cdots, n\}$, or properly intersects both, we have finally reduced the product to

$$v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_I x_i)\,,$$

with indices those I which satisfy $\varnothing \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$, *i.e.* the product reduces to $d_v^{(n)}[y; x_1 \cdots, x_n]$ as required. $\square$

Corollary 18.5. *The assignment $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$ to finite configurations x in $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$ yields a configuration-valuation on the stable ef $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$.*

Proof. Clearly,

$$v(\varnothing) = v_S(\pi_1 \varnothing) \times v_T(\pi_2 \varnothing) = 1 \times 1 = 1\,.$$

Assuming $x \longrightarrow\!\!\subset^- y$ in $(\mathcal{C}^\infty(T), \equiv_T) \circledast (\mathcal{C}^\infty(S), \equiv_S)$, then either $x \xrightarrow{(s,*)}\!\!\subset y$, with s a $-$ve event of S, or $x \xrightarrow{(*,t)}\!\!\subset y$, with t a $-$ve event of T. Suppose $x \xrightarrow{(s,*)}\!\!\subset y$, with s $-$ve. Then $\pi_1 x \xrightarrow{s}\!\!\subset \pi_1 y$, where as s is $-$ve, $v_S(\pi_1 x) = v_S(\pi_1 y)$. In addition, $\pi_2 x = \pi_2 y$ so certainly $v_T(\pi_2 x) = v_T(\pi_2 y)$. Combined these two facts yield $v(x) = v(y)$. Similarly, $x \xrightarrow{(*,t)}\!\!\subset y$, with t $-$ve, implies $v(x) = v(y)$. As $x \subseteq^- y$ is obtained via the reflexive transitive closure of $\longrightarrow\!\!\subset^-$ it entails $v(x) = v(y)$, as required.

By Lemma 14.11(i) we need only verify requirement (2), the 'drop condition,' for p-covering intervals, which we can always permute into the form covered by Lemma 18.4—any p-event of $(\mathcal{C}(T), \equiv_T) \circledast (\mathcal{C}(S), \equiv_S)$ has a +ve component on one and only one side. The drop condition $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ of the composition is then inherited from the drop conditions $d_v^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \geq 0$ and $d_v^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] \geq 0$ of the components S and T with configuration-valuations v_S and v_T. $\square$

Lemma 18.6. *If $x \xrightarrow{p}\!\!\subset y$ with p $-$ve in $T \odot S$ then $\bigcup x \xrightarrow{top(p)}\!\!\subset \bigcup y$ with $top(p)$ $-$ve in $(\mathcal{C}(T), \equiv_T) \circledast (\mathcal{C}(S), \equiv_S)$.*

Proof. By Lemma 17.14, or copy of the proof of Lemma **??**. $\square$

We complete the definition of the composition of probabilistic edc strategies:

Lemma 18.7. *Let A, B and C be race-free event structure with polarity. Assume probabilistic edc strategies $\sigma : S \to A^\perp \| B$, with configuration-valuation v_S, and $\tau : T \to B^\perp \| C$ with configuration-valuation v_T. Assigning $v_S(\pi_1 \bigcup x) \times v_T(\pi_2 \bigcup x)$ to $x \in \mathcal{C}(T \odot S)$ yields a configuration-valuation on $T \odot S$ with which $\tau \odot \sigma : T \odot S \to A^\perp \| C$ forms a probabilistic strategy from A to C.*

Proof. (The proof copies that earlier for probabilistic strategies in Lemma 14.26.)

We need to show that the assignment $w(x) =_{\rm def} v_S(\pi_1 \bigcup x) \times v_T(\pi_2 \bigcup x)$ to $x \in \mathcal{C}(T \odot S)$ is a configuration-valuation on $T \odot S$. We know that $v(z) =_{\rm def} v_S \pi_1(z) \times v_T \pi_2(z)$, for z a finite configuration in $(\mathcal{C}(T), \equiv_T) \circledast (\mathcal{C}(S), \equiv_S)$, is a configuration-valuation.

Clearly

$$w(x) = v_S(\pi_1 \bigcup x) \times v_T(\pi_2 \bigcup x) = v(\bigcup x)\,.$$

Consequently,

$$w(\varnothing) = v(\bigcup \varnothing) = v(\varnothing) = 1\,.$$

The function w inherits requirement (1) to be a configuration-valuation from v because of Lemma 18.6. Suppose $x \xrightarrow{p}\!\!\subset y$ with p –ve in $T \odot S$. Then, by the lemma, $\bigcup x \xrightarrow{top(p)}\!\!\subset \bigcup y$ with $top(p)$ –ve in $(\mathcal{C}(T), \equiv_T) \otimes (\mathcal{C}(S), \equiv_S)$. Hence

$$w(x) = v(\bigcup x) = v(\bigcup y) = w(y)\,,$$

as required for (2).

In addition, w inherits requirement (2) from v, as w.r.t. w,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= w(y) - \sum_I (-1)^{|I|+1} w(\bigcup_{i \in I} x_i) \\ &= v(\bigcup y) - \sum_I (-1)^{|I|+1} v(\bigcup \bigcup_{i \in I} x_i) \\ &= v(\bigcup y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} (\bigcup x_i)) \\ &\geq 0\,, \end{aligned}$$

whenever $y \subseteq^p x_1, \cdots, x_n$ in $\mathcal{C}(T \odot S)$. (Above, the index I ranges over sets satisfying $\emptyset \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$.

□

18.2 A bicategory of probabilistic edc strategies

We obtain a bicategory of probabilistic edc strategies in which objects are race-free games, 1-cells (or maps) are probabilistic edc strategies and 2-cells are rigid 2-cells of edc strategies satisfying a constraint in the way that configuration-valuations are related. In detail, let $\sigma : S \to A^\perp \| B$ with configuration-valuation v and $v', \sigma' : S' \to A^\perp \| B$ with configuration-valuation v' be probabilistic edc strategies. A 2-cell from σ, v to σ', v' is a 2-cell $f : \sigma \Rightarrow \sigma'$ of edc strategies in which $f; S \to S'$ is a rigid map of event structures such that the push-forward fv satisfies

$$(fv)(x') \leq v'(x')\,,$$

for all configurations $x' \in \mathcal{C}(S')$. The statement relies on being able to push-forward a configuration-valuation across a rigid two cell. The following results ensure we can apply the earlier results of Section 14.3.

Lemma 18.8. *Let $f : \sigma \Rightarrow \sigma'$ be a rigid 2-cell between* edc strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ in a game A. Then, f is receptive.

Proof. We require receptivity, *i.e.* letting $x \in \mathcal{C}(S'), s' \in S'$ be such that $fx \xrightarrow{s'}\!\!\subset$ with s' -ve, there exists a unique $s \in S$ for which $x \xrightarrow{s}\!\!\subset$ with$f(s) = s'$.

We first show existence,*i.e.* the $\exists$-receptivity of f. By the rigidity of f the subconfiguration $[s') \subseteq fx$ determines a subconfiguration $z \subseteq x$ such that $fz = [s')$. Letting $a = \sigma'(s') \in A$,

$$\sigma z = \sigma' f z = \sigma' [s') \xrightarrow{a}\!\!\subset\,.$$

By the $\exists$-receptivity of σ, there is $s \in S$ with $z \overset{s}{\longrightarrow\!\!\subset}$ and $\sigma(s) = a$. As $[s) \subseteq z$ we have $f[s) \subseteq fz = [s')$ and, by rigidity, $[f(s)) = f[s)$. Hence

$$[f(s)) \subseteq [s') .$$

We now show the converse inclusion, so the equality $[f(s)) = [s')$. Consider an arbitrary $s_1 \rightarrowtriangle s'$. Then from the innocence of σ' we obtain $\sigma'(s_1) \rightarrowtriangle a$. Now $\sigma'(f(s)) = \sigma(s) = a$ and as σ' locally reflects dependency there is (for an edc S' a unique) $s_2 < f(s)$ for which $\sigma'(s_2) = \sigma'(s_1)$. But now $s_1 \equiv s_2$ because they share a common image under σ'. From $s_1, s_2 \in [s']$ and S' being an edc we obtain $s_1 = s_2$ so $s_1 \in [f(s))$. As s_1 was an arbitrary $s_1 \rightarrowtriangle s'$ we deduce the sought converse inclusion $[f(s)) \supseteq [s')$, so $[f(s)) = [s')$. Clearly $f(s) \equiv s'$ as they both become equal to a under σ'. Hence by the irredundancy of σ' we deduce $f(s) = s'$, as required for existence.

Now we show uniqueness. Suppose $x \overset{s_1}{\longrightarrow\!\!\subset}$ and $x \overset{s_2}{\longrightarrow\!\!\subset}$ and $f(s_1) = f(s_2) = s'$. As f is rigid,

$$f[s_1) = f[s_2) = [s') .$$

Because f is a map of event structures it is locally injective w.r.t. x. As $[s_1), [s_2) \subseteq x$ this implies $[s_1) = [s_2)$. Moreover, $\sigma(s_1) = \sigma' f(s_1) = \sigma' f(s_2) = \sigma(s_2)$ so $s_1 \equiv s_2$. As both s_1 and s_2 are $-$ve (the map f preserves polarity), by irredundancy, we deduce $s_1 = s_2$, as required to show uniqueness □

Theorem 18.9. *Let $f : \sigma \Rightarrow \sigma'$ be a 2-cell between edc strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ which is a rigid map of event structures. Let v be a configuration-valuation on S. Taking $v'(y) =_{\text{def}} \sum_{x: fx = y} v(x)$ for $y \in \mathcal{C}(S')$, defines a configuration-valuation, written fv, on S'.*

Proof. The push-forward results of earlier, Section 14.3, extend to rigid 2-cells of edc's as by Lemma 18.8 thay are automatically receptive. □

A probabilistic edc strategy is *deterministic* if its configuration-valuation assigns 1 to all finite configurations; its underlying edc strategy is then necessarily deterministic too.

18.3 A language of probabilistic edc strategies

duplication strategy $\sigma : A \twoheadrightarrow A \| A$ is deterministic, if A is deterministic for Opponent, *i.e.* $A^\perp$ is deterministic as an event structure with polarity, and we now have parallel causes* recursion simple now as I think is the relation of recursion with trace ****

a configuration-valuation on a general event structure with polarities (the earlier defns still apply) can be pulled backwards to a configuration-valuation on its edc

Chapter 19

Revisions/Extensions to edc-strategies

The concept of edc-rigid maps leads us to revise and generalise the earlier two chapters on edc-strategies and edc-strategies with probability. (The intention is to rewrite them to accommodate the greater generality here.) The main advantages are that we obtain a rigid image of edc maps (missing before), the ensuing rigid-image of edc-strategies, and more general 2-cells for probabilistic strategies via a stronger push-forward result.

19.1 Edc-rigid maps

Definition 19.1. *Let $f : A \to B$ be a total map of edc's. Say f is* edc-rigid *iff f preserves causal dependency, i.e.*

$$a \le a' \text{ in } A \implies f(a) \le f(a') \text{ in } B\,.$$

Below we characterise edc-rigidity in a similar way to rigidity on event structures, though notice that the existence asserted with edc's is not unique.

Lemma 19.2. *Let $f : A \to B$ be a total map of edc's. Then, f is rigid iff*

$$\forall x \in \mathcal{C}(A), y \in \mathcal{C}(B).\ y \subseteq fx \implies \exists x' \subseteq x.\ fx = y\,.$$

Proof. "if": Assume

$$y \subseteq fx \implies \exists x' \subseteq x.\ fx = y\,,$$

for all finite configurations x of A and y of B. Suppose to obtain a contradiction that $[f(a)]_B \subsetneq f[a]_A$. Then there a $\le$-maximal $b \in f[a]_A \setminus [f(a)]_B$, *i.e.* $b \in f[a]_A$ and $b \notin [f(a)]_B$. Then, by the maximailty of b, the set $y = f[a]_A \setminus \{b\}$ is a configuration for which $y \xrightarrow{b} f[a]_A$. From the assumption, there is a configuration x' such that $x' \subseteq [a]_A$ and $fx' = y$. As f is total and A an edc, it restricts to a bijection $f : [a]_A \cong f[a]_A$. The bijection restricts to a bijection between x' and $fx' = y \stackrel{b}{\longrightarrow\!\!\subset} f[a]_A$. From cardinality considerations there must be $a' \in A$ such that $x' \stackrel{a'}{\longrightarrow\!\!\subset} [a]_A$ and $f(a') = b$. But $x' \stackrel{a'}{\longrightarrow\!\!\subset} [a]_A$ implies $a' = a$. It follows that $f(a) = b$ so $b \in [f(a)]_B$, a contradiction. We conclude $[f(a)]_B = f[a]_A$ so that if $a' \leq a$ then $f(a') \leq f(a)$.

"Only if": Assume f is rigid, *i.e.* preserves causal dependency. Suppose $x \in \mathcal{C}(A)$ and $y \stackrel{b}{\longrightarrow\!\!\subset} fx$ in $\mathcal{C}(B)$. Suppose $a \in x$ and $f(a) = b$. Then a is $\leq$-maximal in x: Suppose $a \leq a'$ in x. Then $b = f9a) \leq f(a') \in fx$, from the assumption. But $y \stackrel{b}{\longrightarrow\!\!\subset} fx$ so $f(a') = f(a) = b$ and $a = a'$; otherwise we would not be able to "remove" b from fx. Hence $x' =_{\rm def} x \setminus \{a \in x \mid f(a) = b\} \in \mathcal{C}(A)$ with $x' \subseteq x$ and $fx' = y$. $\square$

Write $\mathcal{EDC}_t$ for the category of edc's with total maps and $\mathcal{EDC}_r$ for its subcategory of rigid maps. There is a right adjoint to $\mathcal{EDC}_r \hookrightarrow \mathcal{EDC}_t$ given by the following construction.

Let B be an edc. Define $aug(B)$ to comprise

- *events*, prime augmentations of finite unambiguous configurations p of B with top $top_B(p)$ and equivalence $p \equiv q$ iff $top_B(p) \equiv_B top_B(q)$;
- *causal dependency* given by rigid inclusion;
- *consistency*, $X \in \mathrm{Con}$ iff $top[X] \in \mathrm{Con}_B$.

We can develop rigid images in a way analogous to before. Via the adjunction any total map $f : A \to B$ of edc's factors through the counit $top_B : aug(B) \to B$ as the composite

$$A \xrightarrow{\overline{f}} aug(B) \xrightarrow{top_B} B$$

where $\overline{f}$ is edc-rigid. We take the rigid image of A to comprise: those events of B in the image of $\overline{f}$; with causal dependency that of B; with a finite set of its events consistent if they are the image of a consistent set in A; and two events equivalent if they are the image of equivalent events in A. There is a universal characterisation like that earlier.

19.2 Games as edc's

There are two ways to generalise edc-strategies to games which are proper edc's.

(1) This stays very close to the existing development—see below. Copycat is constructed in exactly the same way but for the addition of an equivalence $\equiv$ inherited from the game $A, \equiv_A$: two moves in $\mathbb{C}_A$ are $\equiv$-equivalent if their corresponding moves in $A^\perp \| A$ are $\equiv_A$-equivalent. Composition is achieved via pullback of edc's as before.

(2) This is a more radical departure from the existing approach. The copycat strategy associated with a game $A, \equiv_A$ now allows Player to copy $\equiv_A$-equivalent moves. Composition is now achieved via pseudo pullback.

For the moment we eschew approach (2) although it would sensible if the Player of copycat were unable to distinguish which parallel cause Opponent had applied in making their move. There are technical advantages in following (1). For instance, a 2-cell $f : \sigma \Rightarrow \sigma'$ between existing edc-strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ will if edc-rigid inherit the properties of a strategy developed under approach (1); note S' is generally a proper edc so to view f as a strategy in S' (as is useful in a proof below) requires the generalisation to games as edc's.

Definition 19.3. *(Strategies over games as edc's) Let $\sigma : S \to A$ be a total map of edc's with polarity, where A may be a proper edc with non-identity equivalence $\equiv_A$. Then $\sigma : S \to A$ is an edc strategy if it satisfies the following axioms:*

(1) innocence:
+-innocence: *if $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$;*
−-innocence: *if $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.*

(2) +-consistency: *$X \in \mathrm{Con}_S$ if $\sigma X \in \mathrm{Con}_A$ and $[X]^+ \in \mathrm{Con}_S$, for $X \subseteq_{\mathrm{fin}} S$. (Recall $[X]^+$ comprises the +ve elements in the downwards closure of X.)*

(3) ≡-saturation: *$s_1 \equiv_S s_2$ if $\sigma(s_1) \equiv_A \sigma(s_2)$.*

(4) ∃-receptivity: *$\sigma x \xrightarrow{a}\subset$ & $pol_A(a) = - \Rightarrow \exists s \in S.\ x \xrightarrow{s}\subset$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$. (Note we no longer have uniqueness.)*

(5) non-redundancy: *$[s_1) = [s_2)$ & $s_1 \equiv_S s_2$ & $pol_S(s_1) = pol_S(s_2) = - \implies s_1 = s_2$.*

Proposition 19.4. *Let $\sigma : S \to A$ be a total map of edc's with polarity where A is an edc. Then, σ is an edc-strategy as above iff*
$s_1 \equiv_S s_2 \iff \sigma(s_1) \equiv_A \sigma(s_2)$, for all $s_1, s_2 \in S$;
the image $\sigma_0 : S_0 \to A_0$ of σ (under the right adjoint to the inclusion of event structures in edc's) is a strategy of concurrent games, as earlier;
σ satisfies +-consistency.

CLAIM: The results of Chapters 17 and 18 carry over directly w.r.t. copycat $\gamma_A : \mathbb{C}_A \to A^\perp \| A$ obtained as before with equivalence inherited from the game $A^\perp \| A$. In particular, an edc strategy in a game A is a total map $\sigma : S \to A$ of edc's with polarity such that $\gamma_A \odot \sigma \cong \sigma$. In the following we shall refer to the existing theorems of Chapters 17 and 18 as if they apply in this more slightly general context.

Lemma 19.5. *Let $f : \sigma \Rightarrow \sigma'$ be a 2-cell between edc strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ (where A may be a proper edc). If f is edc-rigid then f is an edc strategy in S'.*

Proof. Under the right adjoint $(-)_0$ to the 'inclusion' functor $\mathcal{ES} \hookrightarrow \mathcal{EDC}$, the edc-rigid 2-cell f becomes a 2-cell $f_0 : \sigma_0 \Rightarrow \sigma_0'$ between the 'old' strategies. Thus f_0 is also an 'old' strategy. It is easy to show $\equiv$-saturation and $+$-consistency from the commutation associated with the 2-cell $f : \sigma \Rightarrow \sigma'$. $\square$

19.3 Push-forward across edc-rigid 2-cells

Results of the chapter on probability extend straightforwardly; the only change in the notion of strategy is a slight modification to $\equiv$-saturation.

Recall from Section 14.1 that we write *e.g.* $\bigvee Z$ for $\bigcup Z$ when a set of configurations Z is compatible, *i.e.* $Z\uparrow$, so $\bigcup Z$ is a configuration, and $\top$ otherwise.

Theorem 19.6. *Let $\sigma : S \to A$ be an edc-rigid edc-strategy. (A may be a proper edc.) Let v be a configuration-valuation for S. Then there is a push-forward configuration-valuation σv for A for which the value $(\sigma v)(y)$, at $y \in \mathcal{C}(A)$, is the supremum of*

$$\sum_{\varnothing \neq Z \subseteq X} (-1)^{|Z|+1} v(\bigvee Z)$$

as X ranges over finite subsets of $\{x \in \mathcal{C}(S) \mid y = \sigma x\}$.

Proof. W.r.t. $y \in \mathcal{C}(S)$, define a probablilistic counterstrategy $\tau_y : T_y \hookrightarrow A^\perp$ to σ as follows:

$$T_y =_{\text{def}} A^\perp \restriction (y \cup \{a \in A^\perp \mid pol_{A^\perp}(a) = -\})$$

with τ_y the inclusion map associated with $T_y \subseteq A^\perp$.

The strategy τ_y is deterministic and so can be associated with a configuration-valuation assigning 1 to each of its finite configurations. The composition without hiding $\tau_y \circledast \sigma$ is given by the pullback

$$\begin{tikzcd} & T_y \circledast S \arrow[dl, hook'] \arrow[dr] \arrow[dd, phantom, "\vee"] & \\ S \arrow[dr, "\sigma"'] & & T_y \arrow[dl, hook', "\tau_y"] \\ & A & \end{tikzcd}$$

where we can take advantage of the simple form of τ_y to describe $T_y \circledast S$ as a restriction of S,*viz.*

$$T_y \circledast S =_{\text{def}} S \restriction \{s \in S \mid \sigma(s) \in y \text{ or } pol_S(s) = +\}\,.$$

Then

$$x \in \mathcal{C}^\infty(T_y \circledast S) \text{ iff } x \in \mathcal{C}^\infty(S) \;\&\; y \cap \sigma x \subseteq^+ \sigma x\,.$$

The configuration-valuation of the composition is given by v_y the restriction of v to $\mathcal{C}(T_y \circledast S)$. The composition $T_y \circledast S$ consists purely of synchronisation (=neutral) events ensuring that v_y makes $T_y \circledast S$ into a probabilistic event structure.

Because $T_y \circledast S$ with v_y is a probabilistic event structure, v_y determines a continuous valuation w_y on the Scott-open sets of $\mathcal{C}^\infty(T_y \circledast S)$ in which

$$w_y(\widehat{x}) = v_y(x)$$

for all $x \in \mathcal{C}(T_y \circledast S)$; recall $\widehat{x}$ is the open set $\{z \in \mathcal{C}^\infty(T_y \circledast S) \mid x \subseteq z\}$.

For $y \in \mathcal{C}(A)$, define

$$\phi(y) =_{\text{def}} \{x \in \mathcal{C}^\infty(S) \mid y \subseteq^+ \sigma x\} .$$

From the construction of $T_y \circledast S$ it is seen that

$$\phi(y) = \{x \in \mathcal{C}^\infty(T_y \circledast S) \mid y \subseteq \sigma x\} ,$$

an open subset of $\mathcal{C}^\infty(T_y \circledast S)$. Take

$$(\sigma v)(y) =_{\text{def}} w_y(\phi(y)) ,$$

for $y \in \mathcal{C}(A)$. We show that σv is a configuration-valuation for A, the *push-forward* of v along σ. First, clearly

$$(\sigma v)(\varnothing) = w_y(\phi(\varnothing)) = w_y(\mathcal{C}^\infty(T_y \circledast S)) = 1 .$$

We show the 'drop' condition. Let $y_i \in \mathcal{C}(A)$ for $i \in I$, a finite set. Recall from earlier that we write $\bigvee_{i\in I} y_i$ for $\bigcup_{i\in I} y_i$ when $\{y_i \mid i \in I\}\uparrow$, *i.e.* the set of configurations is compatible, so $\bigcup_{i\in I} y_i \in \mathcal{C}(A)$, and $\top$ otherwise. Extend ϕ so $\phi(\top) = 0$. Observe that

$$\phi(\bigvee_{i\in I} y_i) = \bigcap_{i\in I} \phi(y_i) .$$

Now, supposing $y \subseteq^+ y_1, \cdots, y_n$ in $\mathcal{C}(A)$,

$$\begin{aligned} d^{(n)}_{(\sigma v)}[y; y_1, \cdots, y_n] =& w_y(\phi(y)) - \sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} w_y(\phi(\bigvee_{i\in I} y_i)) \\ =& w_y(\phi(y)) - \sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} w_y(\bigcap_{i\in I} \phi(y_i)) \\ =& w_y(\phi(y)) - w_y(\phi(y_1) \cup \cdots \cup \phi(y_n)) \end{aligned}$$

which is nonnegative by the monotone property of the continuous valuation w_y; clearly

$$\phi(y_1) \cup \cdots \cup \phi(y_n) \subseteq \phi(y) .$$

Remark Above, we have used the following: for a continuous valuation w, by virtue of its modular property, we can derive that for opens sets $U_1, \ldots, U_n$

$$w(U_1 \cup \cdots \cup U_n) = \sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} w(\bigcap_{i\in I} U_i) .$$

Suppose $y \subseteq^- y'$ in $\mathcal{C}(A)$. We shall show $(\sigma v)(y) = (\sigma v)(y')$, *i.e.*

$$w_y(\phi(y)) = w_{y'}(\phi(y')) .$$

We first show $w_y(\phi(y))) \le w_{y'}(\phi(y'))$. As a key step we observe

$$x \in \phi(y) \cap \mathcal{C}(S) \implies \exists x' \in \phi(y') \cap \mathcal{C}(S).\ x \subseteq^- x'\,. \tag{1}$$

To see this suppose $x \in \phi(y) \cap \mathcal{C}(S)$. Then $y \subseteq^+ \sigma x$ and $y \subseteq^- y'$ entail by the race-freeness of A that $y' \cup \sigma x \in \mathcal{C}(A)$ where $y' \subseteq^+ \subseteq y' \cup \sigma x$ and $\sigma x \subseteq^- y' \cup \sigma x$. But from the latter inclusion, by the $\exists$-receptivity of σ there is an x' such that $\sigma x' = y' \cup \sigma x$ with $x \subseteq^- x'$, which establishes (1).

The open set $\phi(y)$ of $\mathcal{C}^\infty(T_y \circledast S)$ is a directed union of basic open sets

$$\widehat{x_1} \cup \cdots \cup \widehat{x_n}$$

where $x_1, \cdots, x_n \in \phi(y) \cap \mathcal{C}(S)$. The value $w_y(\phi(y))$ is the supremum of the values

$$w_y(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) = \sum_{\varnothing \ne I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i\in I} x_i)$$

of the basic open sets in the directed union—because w_y is a *continuous* valuation. By (1), for each x_i, with $1 \le i \le n$, there is $x'_i \in \phi(y') \cap \mathcal{C}(S)$ with $x_i \subseteq^- x'_i$. Hence, once we have shown that, for $\varnothing \ne I \subseteq \{1, \cdots, n\}$,

$$\{x_i \mid i \in I\}\!\uparrow \iff \{x'_i \mid i \in I\}\!\uparrow \tag{2}$$

we are assured that

$$v(\bigvee_{i\in I} x_i) = v(\bigvee_{i\in I} x'_i)\,,$$

as, when configurations, $\bigvee_{i\in I} x_i \subseteq^- \bigvee_{i\in I} x'_i$. Then it will follow that

$$w_y(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) = w_y(\widehat{x'_1} \cup \cdots \cup \widehat{x'_n})$$

because both will expand to the same sum of values.

We now show (2). We have $\{x_i \mid i \in I\}\uparrow$ iff $\bigcup_{i\in I} x_i \in \mathrm{Con}_S$. By the $+$-consistency of σ,

$$\bigcup_{i\in I} x_i \in \mathrm{Con}_S \text{ iff } (\bigcup_{i\in I} x_i)^+ \in \mathrm{Con}_S \;\&\; \bigcup_{i\in I} \sigma x_i \in \mathrm{Con}_A\,.$$

As each $x'_i \supseteq^- x_i$, clearly

$$\bigcup_{i\in I} \sigma x'_i \in \mathrm{Con}_A \implies \bigcup_{i\in I} \sigma x_i \in \mathrm{Con}_A\,.$$

The converse also holds. If $\bigcup_{i\in I} \sigma x_i \in \mathrm{Con}_A$ then $\bigcup_{i\in I} \sigma x_i \in \mathcal{C}(A)$ with both

$$y \subseteq^+ \bigcup_{i\in I} \sigma x_i \text{ and } y \subseteq^- y'\,.$$

Because A is race-free, $y' \cup \bigcup_{i\in I} \sigma x_i \in \mathcal{C}(A)$. But $y' \cup \bigcup_{i\in I} \sigma x_i$ equals $\bigcup_{i\in I} \sigma x'_i$ which is therefore consistent. Just as

$$\{x_i \mid i \in I\}\!\uparrow \text{ iff } (\bigcup_{i\in I} x_i)^+ \in \mathrm{Con}_S \;\&\; \bigcup_{i\in I} \sigma x_i \in \mathrm{Con}_A\,.$$

so

$$\{x'_i \mid i \in I\}\uparrow \text{ iff } (\bigcup_{i \in I} x'_i)^+ \in \mathrm{Con}_S \ \& \ \bigcup_{i \in I} \sigma x'_i \in \mathrm{Con}_A \,.$$

But clearly $(\bigcup_{i \in I} x_i)^+ = (\bigcup_{i \in I} x'_i)^+$ and we have just shown $\bigcup_{i \in I} \sigma x_i \in \mathrm{Con}_A$ iff $\bigcup_{i \in I} \sigma x'_i \in \mathrm{Con}_A$. Hence (2), as required.

The value $w_{y'}(\phi(y'))$ is obtained as the supremum of contributions

$$w_{y'}(\widehat{x'_1} \cup \cdots \cup \widehat{x'_n})$$

where $x'_1, \cdots, x'_n \in \phi(y') \cap \mathcal{C}(S)$. We now obtain $w_y(\phi(y)) \leq w_y(\phi(y'))$ as any contribution to the supremum determining $w_y(\phi(y))$ is matched by a contribution to the supremum determining $w_{y'}(\phi(y'))$.

We also need $w_{y'}(\phi(y'))) \leq w_y(\phi(y)$. This is the one place in the proof where edc-rigidity plays a role—see Example 19.8 below for comments on the necessity of rigidity.

Write

$$\mu(y') =_{\mathrm{def}} \{x \in \mathcal{C}(S) \mid y' = \sigma x\} \,.$$

Observe that

$$x' \in \mu(y') \implies \exists x \in \phi(y) \cap \mathcal{C}(S).\ x \subseteq^- x' \tag{3}$$

follows directly from the edc-rigidity of σ: if $x' \in \mu(y')$ then $y' = \sigma x'$ which with $y \subseteq^- y'$ entails $y = \sigma x$ for some $x \subseteq^- x'$. Because σ is edc-rigid

$$\forall z' \in \phi(y') \exists x' \in \mu(y').\ x' \subseteq^+ z' \,.$$

Hence the open subset $\phi(y')$ of $\mathcal{C}^\infty(T_y \circledast S)$ is a directed union of

$$\widehat{x'_1} \cup \cdots \cup \widehat{x'_n}$$

where $x'_1, \cdots, x'_n \in \mu(y')$. The value $w_{y'}(\phi(y'))$ is the supremum of the values

$$w_{y'}(\widehat{x'_1} \cup \cdots \cup \widehat{x'_n}) = \sum_{\varnothing \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x'_i) \,.$$

Let $\varnothing \neq I \subseteq \{1, \cdots, n\}$. By (3), for each $i \in I$ there is some x_i such that $x_i \subseteq^- x'_i$. As above,

$$\{x_i \mid i \in I\}\uparrow \text{ iff } \{x'_i \mid i \in I\}\uparrow$$

and

$$v(\bigvee_{i \in I} x'_i) = v(\bigvee_{i \in I} x_i) \,.$$

Hence

$$\begin{aligned} w_{y'}(\widehat{x'_1} \cup \cdots \cup \widehat{x'_n}) &= \sum_{\varnothing \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x'_i) \\ &= \sum_{\varnothing \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \\ &= w_y(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \,. \end{aligned}$$

The value $w_y(\phi(y))$ is obtained as the supremum of contributions

$$w_y(\widehat{x_1} \cup \cdots \cup \widehat{x_n})$$

where $x_1, \cdots, x_n \in \phi(y) \cap \mathcal{C}(S)$. Hence $w_{y'}(\phi(y')) \leq w_y(\phi(y))$ as required.

Thus σv is indeed a configuration-valuation for A. To complete the characterisation of σv stated in the theorem, let $y \in \mathcal{C}(A)$ and write

$$\mu(y) =_{\text{def}} \{x \in \mathcal{C}(S) \mid y = \sigma x\}\,.$$

As above, $(\sigma v)(y)$ is obtained as the supremum of values

$$w_y(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) = \sum_{\varnothing \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i)$$

over $x_1, \cdots, x_n \in \mu(y)$. A slight reformulation gives the statement of the theorem. □

We recover the earlier Theorem 14.35 as a special case:

Corollary 19.7. *(***Theorem** *14.35) Let $\sigma : S \to A$ be a rigid, receptive map between event structures with polarity S and A. Let v be a configuration-valuation for S. Then, taking*

$$(\sigma v)(y) =_{\text{def}} \sum_{x : \sigma x = y} v(x)$$

for $y \in \mathcal{C}(A)$, defines a configuration-valuation σv for A, the push-forward of v.

Proof. In the case where σ is such a map it can be identified with an edc-rigid edc-strategy (the edc equivalence is taken to be the identity) so Theorem 19.6 applies—see Proposition 17.17. In this case however distinct x and x' for which $\sigma x = \sigma x' = y$ are incompatible and the complicated sum of Theorem 19.6 simplifies to the above: each finite $X \subseteq \{x \in \mathcal{C}(S) \mid y = \sigma x\}$ is associated with summand $\sum_{x \in X} v(x)$. (Recall an infinite sum of non-negative reals is the supremum of its finite summands.) □

Example 19.8. It is intriguing that rigidity is only needed for one part of the proof of Theorem 19.6. We comment on the necessity of σ being rigid in the proof.

Consider S comprising $\ominus \rightarrowtriangle \oplus$ and A comprising the two moves $\ominus, \oplus$ concurrent with each other. Suppose S carries a configuration valuation v which takes value p on the configuration $\{\ominus, \oplus\}$; it is 1 elsewhere. The map $\sigma : S \to A$ is the only total map respecting polarity; it is clearly not rigid. In this case the constructions of the proof of Theorem 19.6 would give $\phi(y) = \varnothing$ when $y = \{\oplus\}$, as there is no configuration x of S such that $y \subseteq^+ \sigma(x)$ and consequently $(\sigma v)(y) = 0$, which can be seen to be impossible for a configuration-valuation for A unless $p = 0$.

In the light of the above example it might be thought that one could modify the definition of $\phi(y)$ in the proof so that

$phi(y) = \{x \in \mathcal{C}^\infty(S) \mid y \subseteq \sigma x\}$—so not insisting that $y \subseteq^+ \sigma x$. However this suggestion is foiled by the following example. Let both A and S comprise $\ominus \rightsquigarrow\!\!\!\!\leftarrow \ominus$ consisting of two conflicting Opponent moves and let σ be the identity function. Let S carry configuration-valuation v, the only one possible assigning 1 to all configurations. According to the modified definition the push forward σv would give value $(\sigma v)(\varnothing) = 2$, clearly impossible for a configuration-valuation.

There may of course be more subtle ways in which to push forward configuration-valuations across more general 2-cells between strategies, though experimentation has suggested that they are, at the very least, quite complicated. From the proof of Theorem 19.6 it can be seen that a modified form of configuration-valuation v in which $y \subseteq^- y'$ only implies $v(y) \leq v(y')$ would be preserved by arbitrary 2-cells; I can't presently understand the intuition behind such a generalisation or if it is useful.

In the light of this result it is sensible to take 2-cells $f : (\sigma, v) \Rightarrow (\sigma', v')$ between probabilistic edc strategies to be 2-cells $f : \sigma \Rightarrow \sigma'$ which are edc-rigid maps for which the push forward fv is pointwise less than or equal to v'.

Now we have edc-rigid images of total maps of edc's we can develop rigid-images of edc strategies analogously to earlier.

Chapter 20

Disjunctive causes via symmetry

20.1 Games with symmetry

In this chapter we shall refer to the paper "Symmetry in concurrent games" [**?**]. However there are a few extra definitions and results on which we shall rely, so they are included here.

Recall *weak* maps $f : \sigma \Rightarrow \sigma'$ between pre-strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ are maps $f : S \to S'$ for which $\sigma \sim f\sigma'$. When $f : \sigma \Rightarrow \sigma'$ and $g : \sigma' \Rightarrow \sigma$ satisfy $gf \sim \mathrm{id}$ and $fg \sim \mathrm{id}$ we say f, g forms a weak equivalence between pre-strategies σ and σ' and write $\sigma \approx \sigma'$ and sometimes $f : \sigma \approx \sigma'$ or even $f : \sigma \approx \sigma' : g$, though note g is determined up to symmetry, *i.e.*, up to $\sim$, by f.

It is convenient to work with a $\sim$-bicategory of games with symmetry and weak strategies (pre-strategies weakly equivalent to strategies) considered up to weak equivalence $\approx$. It comprises

objects, which are games with symmetry;

****DOESN'T SEEM NEC TO IMPOSE THINNESS THO THE GAMES AND STRATS WE OBTAIN IN CONSIDERING IMAGES OF EDC STRATS WILL BE THIN I THINK - IS $?A$ THIN IF A IS THIN ???

PIERRE AND SIMON, THERE SHD BE A RESULT:

WHEN GAME IS THIN ITS THIN COPYCAT IS WEAK EQUIV TO ITS FAT COPYCAT ????**** FOR THIS WE PROBABLY JUST NEED THAT THE *ESSP* A IS THIN AND l_A PRESERVES RACES ???

maps (1-cells) which are weak strategies;

2-cells $f : \sigma \Rightarrow \sigma'$, which are *rigid* weak maps between strategies;

Vertical composition is that of maps of event structures with symmetry and horizontal composition that of weak strategies ****using bipullbacks ****. Note that $\mathbf{Strat}(A, B)$, the category of strategies from A to B with maps 2-cells between them is enriched in setoids; the maps are between event structures with symmetry and bear the equivalence relation $\sim$****.

The bicategory laws hold, though only up to $\sim$ *****

****without restricting to rigid 2-cells would later get $?^s$ a functor****

Need composition before hiding presented explicitly for *wis* to be a pseudo functor *** think this needs pseudo pbs Thm 20.18 ****

IS THE ROLE OF THE EXTRA SUBSYMM CONDITIONS ON A GAME THERE TO ENSURE PULLBACKS ARE BIPULLBACKS ??? SORRY FORGOTTEN.

The following lemma presents a sufficient condition for a prestrategy to be a weak strategy:

Lemma 20.1. *Let A be a game with symmetry. A pre-strategy $\sigma : S \to A$ which is innocent and strong-receptive is a weak strategy; it is weakly equivalent to the strategy $Sat(\sigma)$ obtained as the saturation of σ.* ****

Throughout this chapter assume event structures A are consistent-countable, *i.e.* there is a consistent-enumeration $\chi : A \to \omega$ such that

$$\{a, a'\} \in \mathrm{Con}_S \ \&\ \chi(a) = \chi(a') \implies a = a' .$$

A consistent-enumeration need only be injective on consistent sets.

20.2 A pseudo monad

Let $A = (A, \le_A, \mathrm{Con}_A, pol_A)$ be an event structure with polarity. Define

$$?A = (?A, \le, \mathrm{Con}, pol)$$

to comprise

- events $(a, \alpha) \in ?A$ where $a \in A$ and $\alpha : [a]_A \to \omega$ such that $\alpha(a) = 0$ if $pol_A(a) = -$ with $pol((a, \alpha)) = pol_A(a)$;
- causal dependency,
$$(a', \alpha') \le (a, \alpha) \iff a' \le_A a \ \&\ \alpha' = \alpha \upharpoonright [a']_A ;$$
- consistency,
$$X \in \mathrm{Con} \iff X \subseteq_{\mathrm{fin}} ?A \ \&\ \{a \mid \exists \alpha.\ (a, \alpha) \in X\} \in \mathrm{Con}_A .$$

We extend a symmetry on A to a symmetry on $?A$ via the following construction on isomorphism families: for $x, y \in \mathcal{C}(?A)$,

$$\begin{aligned} \theta : x \cong_{?A} y \iff & \theta \text{ is a bijection respecting } \le \text{ and} \\ & \{(a, a') \mid \exists \alpha, \alpha'.\ \theta(a, \alpha) = (a', \alpha')\} \text{ is in the isomorphism family of } A . \end{aligned}$$

Idea: the different copies (a, α), variants of events of A, correspond to different parallel causes of event a.

Except for Section 20.4, we shall almost exclusively consider $?A$, a game with symmetry, when A is game, with trivial symmetry. Then $\theta : x \cong_{?A} y$ holds of two finite configurations of $?A$ iff θ is a bijection respecting $\leq$ and the underlying moves of the game A.

Lemma 20.2. *Let A be an event structure with polarity. For events (a,α) and (a',α') in $?A$, write*

$$(a,\alpha) \equiv_{?A} (a',\alpha') \text{ iff } a = a' .$$

The function $d_A : (?A, \equiv_{?A}) \to A$ taking $(a,\alpha) \in ?A$ to $a \in A$ is an edc strategy in A. It satisfies the equation

$$\sigma = d_A \circ wis(\sigma) .$$

Proof. The properties required for d_A to be an edc strategy are shown straightforwardly: innocence and $\exists$-receptivity because indices do not disturb the underlying causal dependency of A; non-redundancy because we always index $-$ve events by 0; the property $\equiv$-saturation by definition; and finally $+$-consistency because d_A reflects consistency. Directly from the definitions, $\sigma x = d_A(wis(\sigma)\, x)$, for any $x \in \mathcal{C}(S)$, establishing the equation. □

The operation ? forms a pseudo monad. Its unit $\eta^?_A : A \to ?A$ takes a to $(a,)_a)$ where $0_a : [a] \to \omega$ is constantly 0. To define its multiplication $\mu^?_A : ??A \to ?A$ we use an injective pairing $\langle m, n\rangle$ of natural numbers in natural numbers. Define ***** $\mu^?_A(([a],\alpha),\beta) = (a,\gamma)$ where $\gamma(a') = \langle \alpha(a'), \beta([a'],\alpha')\rangle$ where α' is the restriction of α to $[a']$ for $a' \leq a$. **** ***a pseudo monad****

We shall also use a pseudo monad ! defined by

$$!A =_{\text{def}} (?(A^\perp))^\perp$$

which in contrast to ? duplicates Opponent events.

20.3 Edc strategies as strategies

Edc strategies $\sigma : S \to A$ in a game A can be viewed as Kleisli maps $\sigma' : S' \to ?A$. The Kleisli maps are weak strategies in the game with symmetry $?A$.

An edc strategy $\sigma : S \to A$ determines a Kleisli map $wis(\sigma) : S' \to ?A$: the event structure with polarity and symmetry S' is obtained from the edc S by simply dropping the equivalence $\equiv_S$ and imposing the identity symmetry while the map $wis(\sigma)$ chooses a copy of the event $\sigma(a)$ in an appropriately coherent way, using the consistent-countability of S. Consistent countability of S provides a function $\chi : S \to \omega$ which is injective on consistent sets. With it define

$$wis(\sigma)(s) = (\sigma(s), \alpha)$$

where $\alpha : [\sigma(s)]_A \to \omega$ is defined so $\alpha(a) = 0$ if a is $-$ve while

$$\alpha(a) = \chi(s') \text{ for that unique } s' \leq s \text{ such that } \sigma(s') = a$$

if a is +ve.

Of course, the construction of $wis(\sigma) : S' \to ?A$ from an edc strategy $\sigma : S \to A$ is w.r.t. a choice of enumeration χ of S. However, in this way we do determine a weak strategy—by Proposition 20.3—which is weakly equivalent to any other obtained via a different choice of enumeration.

Proposition 20.3. *Given an edc strategy σ, the function $wis(\sigma)$ is a weak strategy $\sigma' : S' \to ?A$ which is*

(i) innocent and receptive, for which

(ii) S' has the trivial identity symmetry and

(iii) $s_1, s_2 \le s$ & $\sigma'(s_1) = (a, \alpha)$ & $\sigma'(s_2) = (a, \alpha') \implies s_1 = s_2$.

If two edc strategies σ_1 and σ_2 are isomorphic, then $wis(\sigma_1)$ and $wis(\sigma_2)$ are weakly equivalent weak strategies.

Proof. That properties (ii), (iii) and the innocence of (i) hold of $wis(\sigma)$ is obvious. We show the receptivity of $\sigma' =_{\text{def}} wis(\sigma)$ required by (i).

Let $x \in \mathcal{C}(S)$. Suppose $\sigma'(x) \overset{(a,\alpha)}{\longrightarrow\!\!\subset}$ where $pol_a(a) = -$. Define

$$x_0 = [\{s' \in x \mid \exists (a', \alpha') < (a, \alpha).\ \alpha'(a') = \chi(s')\}]_S .$$

Then x_0 is the minimum subconfiguration of x for which $\sigma' x_0 \overset{(a,\alpha)}{\longrightarrow\!\!\subset}$. Consequently, $\sigma x_0 \overset{a}{\longrightarrow\!\!\subset}$. As σ is $\exists$-receptive, there is some s such that $x_0 \overset{s}{\longrightarrow\!\!\subset}$ and $\sigma(s) = a$. Thus $\sigma[s)_S \supseteq [a)_A$. It follows from the definition of $wis(\sigma)$ that $\sigma'[s)_S \supseteq [(a, \alpha))$ with $[s)_S = x_0$, and hence that $\sigma'(s) = (a, \alpha)$. This yields $x \overset{s}{\longrightarrow\!\!\subset}$ with $\sigma'(s) = (a, \alpha)$. To show its uniqueness suppose $x \overset{s_1}{\longrightarrow\!\!\subset}$ and $x \overset{s_2}{\longrightarrow\!\!\subset}$ with $\sigma'(s_1) = \sigma'(s_2) = (a, \alpha)$. Then the causal predecessors $s_1' <_S s_1$ share the same enumeration index as the causal predecessors $s_2' <_S s_2$ within the configuration x. Hence $[s_1) = [s_2)$ with $\sigma(s_1) = \sigma(s_2)$ and s_1 and s_2 −ve. By the "non-redundancy" of σ we obtain $s_1 = s_2$, so uniqueness.

As S' has the trivial identity symmetry, the receptivity of $wis(\sigma)$ ensures its strong-receptivity. Then by Lemma 20.1 we obtain that $wis(\sigma)$ is a weak strategy in the game $?A$. It is easy to check that the choices of enumeration for two isomorphic edc strategies will take corresponding configurations to results within the isomorphism family of $?A$. □

Conversely, recalling the edc strategy $d_A : ?A \to A$ of Lemma 20.2,

Proposition 20.4. *Given a weak strategy $\sigma' : S' \to ?A$ which satisfies conditions (i), (ii), (iii) of Proposition 20.3, there is an edc strategy $wos(\sigma') : S \to A$: define S to be the edc obtained from S' by dropping its trivial symmetry and endowing it with equivalence $\equiv_S$ where*

$$s_1 \equiv_S s_2 \text{ iff } d_A\sigma'(s_1) = d_A\sigma'(s_2)$$

and define $wos(\sigma')$ to be the function $d_A\sigma' : S' \to A$.

If weak strategies σ'_1 and σ'_2 satisfy conditions (i), (ii), (iii) of Proposition 20.3 and are weakly equivalent, then $wos(\sigma'_1)$ and $wos(\sigma'_2)$ are isomorphic edc strategies.

Proof. The weak strategy σ' is in particular a strategy so can be identified with an edc strategy. Composed with the edc strategy d_A we obtain $wos(\sigma')$, which is thus an edc strategy. Weak equivalence is sent to isomorphism under wos because the event structures involved carry the trivial identity equivalence. $\square$

Theorem 20.5. *Let σ be an edc strategy. Then, $wos \circ wis(\sigma) = \sigma$.*

Let $\sigma' : S' \to ?A$ be a weak strategy satisfying conditions (i), (ii), (iii) of Proposition 20.3. The weak strategy $wis \circ wos(\sigma')$ is weakly equivalent to the weak strategy σ'.

Proof. That $wos \circ wis(\sigma) = \sigma$ is easy to see. That $wis \circ wos(\sigma')$ is weakly equivalent to σ' follows straightforwardly using the local injectivity of σ'. $\square$

Of course we should check that the operation of converting an edc strategy to a strategy respects composition. We must first settle the question of how to compose strategies $\sigma : S \to ?(A^\perp \| B)$ and $\tau : T \to ?(B^\perp \| C)$. Notice that, *e.g.*,

$$?(A^\perp \| B) = ?(A^\perp) \| ?B = (!A)^\perp \| ?B\,.$$

Consequently,

$$\sigma : !A \mathrel{+\!\!\!\twoheadrightarrow} ?B \quad \text{and} \quad \tau : !B \mathrel{+\!\!\!\twoheadrightarrow} ?C\,.$$

As we shall see in the next section, strategies of this form and a specified composition arise in a double Kleisli construction [14].

20.4 Composition

THE FOLLOWING NEEDS TO BE CHECKED*

The operations ? and ! are (pseudo) monads up to symmetry on event structures with polarity and symmetry with units and multiplication $\eta^?$, $\mu^?$ and $\eta^!$, $\mu^!$, respectively.

They lift to (pseudo) monads and, by the duality of strategies, to comonads on strategies on games with symmetry. We first lift ? and ! to (pseudo) functors on strategies.

Within event structures with polarity, a total map $\sigma : S \to A^\perp \| B$ determines a total map $?^s\sigma : ?^s S \to (?A)^\perp \| ?B$. We first describe $?^s S$ and $?^s\sigma$ on event structure with polarity $S = (S, \leq_S, \mathrm{Con}_S, pol_S)$ without symmetry, which we shall treat later.

Define

$$?^s S = (?^s S, \leq, \mathrm{Con}, pol)$$

to comprise:

- Events $(s,\alpha) \in ?^s S$ if $s \in S$ and $\alpha : [s]_S \to \omega$ is such that $\alpha(s) = 0$ if $\sigma_1(s)$ is defined and $pol_A(a) = +$ or $\sigma_2(s)$ is defined and $pol_A(a) = -$. The polarity of (s,α) is that of s. The function

$$?^s\sigma : ?^s S \to (?A)^\perp \| ?B$$

acts so $?^s\sigma((s,\alpha)) =_{\mathrm{def}} (\sigma(s),\beta)$. The function β has domain $[\sigma(s)]$, the down-closure of $\sigma(s)$ in $(?A)^\perp \| ?B$; it sends $c \le \sigma(s)$ to $\beta(c) = \alpha(s')$ where s' is the unique event $s' \le s$ such that $\sigma(s') = c$.

- Causal dependency,

$$(s',\alpha') \le (s,\alpha) \iff s' \le_S s \;\&\; \alpha' = \alpha \upharpoonright [s']_S\,.$$

- Consistency,

$$\begin{gathered} X \in \mathrm{Con} \iff X \subseteq_{\mathrm{fin}} ?^s S \;\&\; \{s \mid \exists \alpha.\ (s,\alpha) \in X\} \in \mathrm{Con}_S \;\& \\ \forall (s,\alpha_1),(s_1,\alpha_2) \in X.\ ?^s\sigma((s_1,\alpha_1)) = ?^s\sigma((s_2,\alpha_2)) \implies (s_1,\alpha_1) = (s_2,\alpha_2) \;\& \\ \forall s \in S^+, \alpha_1,\alpha_2.\ (s,\alpha_1),(s,\alpha_2) \in X \;\&\; [(s,\alpha_1))^- = [(s,\alpha_2))^- \implies \alpha_1 = \alpha_2\,. \end{gathered}$$

[****USING $s \in S^+$ TO SIGNIFY s HAS +VE POLARITY IN S****EARLIER??*****]

We extend a symmetry on S to a symmetry on $?^s S$ via the following construction on isomorphism families: for $x, y \in \mathcal{C}(?^s S)$,

$$\begin{aligned} \theta : x \cong_{?^s S} y \iff & \theta \text{ is a bijection respecting } \le \text{ and} \\ & \{(s,s') \mid \exists \alpha,\alpha'.\ \theta(s,\alpha) = (s',\alpha')\} \text{ is in the isomorphism family of } S\,. \end{aligned}$$

The first two clauses in the definition ensure that $?^s\sigma$ is a map of event structures. The final clause is more odd. Without it we could not show that $?^s$ preserves copycat or, later, satisfies the monad laws. The final clause says that consistent distinct variants of a +ve event causally depend on distinct −ve variants. By the following remark we can drop the insistence on the varaints on which the two +ve variants depend being −ve. The consistency condition in the definition of $?^s S$ may equivalently be replaced by

$$\begin{gathered} X \in \mathrm{Con} \iff X \subseteq_{\mathrm{fin}} ?^s S \;\&\; \{s \mid \exists \alpha.\ (s,\alpha) \in X\} \in \mathrm{Con}_S \;\& \\ \forall (s,\alpha_1),(s_1,\alpha_2) \in X.\ ?^s\sigma((s_1,\alpha_1)) = ?^s\sigma((s_2,\alpha_2)) \implies (s_1,\alpha_1) = (s_2,\alpha_2) \;\& \\ \forall s \in S^+, \alpha_1,\alpha_2.\ (s,\alpha_1),(s,\alpha_2) \in X \;\&\; [(s,\alpha_1)) = [(s,\alpha_2)) \implies \alpha_1 = \alpha_2\,. \end{gathered}$$

This is by virtue of the following proposition.

Proposition 20.6. *Let X be a down-closed finite subset of events of $?^s S$ for an event structure with polarity S. The following are equivalent*

(i) $\forall s \in S^+, \alpha_1,\alpha_2.\ (s,\alpha_1),(s,\alpha_2) \in X \;\&\; [(s,\alpha_1))^- = [(s,\alpha_2))^- \implies \alpha_1 = \alpha_2\,;$

(ii) $\forall s \in S^+, \alpha_1,\alpha_2.\ (s,\alpha_1),(s,\alpha_2) \in X \;\&\; [(s,\alpha_1)) = [(s,\alpha_2)) \implies \alpha_1 = \alpha_2\,.$

Proof. (i) ⇒ (ii) is obvious. To show (ii) ⇒ (i), assume (ii). Suppose $(s,\alpha_1),(s,\alpha_2) \in X$ and $[(s,\alpha_1))^- = [(s,\alpha_2))^-$ where s is +ve. Suppose $[(s,\alpha_1)) \neq [(s,\alpha_2))$, to obtain a contradiction. Then, for some $\le$-minimal +ve $s' \le s$ we have $\alpha_1(s') \neq \alpha_2(s')$. Write α'_1 and α'_2 for the restrictions of α_1 and α_2 to $[s']$. Then $(s',\alpha'_1),(s',\alpha'_2) \in X$, as X is down-closed, and

$$[(s,\alpha'_1)) = [(s,\alpha'_1))^- = [(s,\alpha'_2))^- = [(s,\alpha'_2))\,,$$

by the minimality of s'. Hence by (ii), $\alpha'_1 = \alpha'_2$ making $\alpha_1(s') = \alpha_2(s')$ — a contradiction. Now, as $[(s,\alpha_1)) = [(s,\alpha_2))$, we obtain $\alpha_1 = \alpha_2$ by (ii), as required. □

We extend $?^s$ to 2-cells. Suppose $f : \sigma \Rightarrow \sigma'$ is a rigid 2-cell between pre-strategies $\sigma : S \to A^\perp \| B$ and $\sigma' : S' \to A^\perp \| B$. We describe the rigid 2-cell $?^s f : ?^s\sigma \Rightarrow ?^s\sigma'$. For $(s,\alpha) \in ?^s S$ we define $?^s f(s,\alpha) = (f(s),\alpha')$ where for $s' \le f(s)$ we take $\alpha'(s') = \alpha(s_1)$ for s_1 that unique $s_1 \le s$ for which $f(s_1) = s'$. Because we restrict to rigid 2-cells we so obtain a functor from $\mathbf{Strat}(A,B)$ to $\mathbf{Strat}(?A,?B)$; the functor preserves $\sim$, the equivalence of maps up to symmetry.

Proposition 20.7. *The operation $?^s$ sends any weak strategy to a weak strategy.*

Proof. Because $?^s$ is a functor from $\mathbf{Strat}(A,B)$ to $\mathbf{Strat}(?A,?B)$ which preserves $\sim$, the operation $?^s$ preserves weak equivalence between pre-strategies: if $\sigma \approx \sigma'$, for pre-strategies σ,σ', then $?^s\sigma \approx ?^s\sigma'$. Any weak strategy is weak equivalent to a weak strategy which is innocent and strong receptive—Lemma 20.1. It can be checked **** that if σ is innocent and strong receptive then so is $?^s\sigma$ and hence also a weak strategy, again by Lemma 20.1. Consequently, the operation $?^s$ sends any weak strategy to a weak strategy. □

The operation $?^s$ yields a (pseudo) functor, which must preserve copycat and composition up to weak equivalence of strategies.

Lemma 20.8. *If $\gamma_A : A \twoheadrightarrow A$ then $?^s\gamma_A \cong \gamma_{?^s A}$.*

Proof. Sketch: The causal dependency of $\mathbb{C}_A$ ensures that the down-closure of a +ve event of $\mathbb{C}_A$ consists of $[\overline{a}]\|[a]$, for $a \in A$, where $\theta : [\overline{a}] \cong_A [a]$ is in the isomorphism family of A; if a is +ve then a is the top event and otherwise the top is $\overline{a}$. Consequently the +ve events of $?^s\mathbb{C}_A$ correspond to bijections $\theta : [\overline{a}] \cong_A [a]$ together with functions $\alpha_1 : [\overline{a}] \to \omega$ and $\alpha_2 : [a] \to \omega$, for $a \in A$. The consistency condition on $?^s\mathbb{C}_A$ ensures that configurations of $?^s\mathbb{C}_A$ are isomorphic to those of $\mathbb{C}_{?A}$. □

The following propositions show the close relationship between configurations of $?^s S$, with $\sigma : S \to A^\perp \| B$, and those of an event structure with symmetry S —useful in demonstrating that $?^s$ preserves composition.

Proposition 20.9. *Let S be an event structure with polarity $\sigma : S \to A^\perp \| B$. Say $z \in \mathcal{C}(?^s S)$ is* unambiguous *iff*

$$(s,\alpha),(s,\alpha') \in z \implies \alpha = \alpha'\,.$$

Let $x \in \mathcal{C}(?^s S)$. Let X consist of the $\subseteq$-maximal unambiguous subconfigurations of x. Then,

$$\begin{aligned}
&\bigcup X = x\,,\\
&z \in X \ \&\ z \subseteq z' \subseteq x \ \&\ z' \text{ unambiguous} \implies z = z'\,, \text{ and}\\
&y, z \in X \ \&\ (?^s\sigma)_1 y = (?^s\sigma)_1 z \ \&\ d_S y = d_S z \implies y = z\,.
\end{aligned}$$

Above, d_S is a function from $?^s S$ to S acting so $d_S(s\alpha) = s$; it takes a configuration y of $?^s S$ to the configuration $d_S y = \{s \mid \exists \alpha.\ (s, \alpha) \in y\}$.

Above, each maximal unambiguous subconfiguration z of a configuration of $?^s S$ corresponds to a configuration of S. Hence:

Proposition 20.10. *Let S be an event structure with polarity $\sigma : S \to A^\perp \| B$. The finite configurations of $?^s S$ correspond to finite families W of pairs*

$$v, l_v : v \to \omega$$

where $v \in \mathcal{C}(S)$ and $l_v(s) = 0$ if $\sigma_1(s)$ is defined and $pol_A(a) = +$ or $\sigma_2(s)$ is defined and $pol_A(a) = -$ for $s \in v$ and

$$\forall (v, l_v), (w, l_w) \in W.\quad \begin{aligned} &l_v \upharpoonright v_1 = l_w \upharpoonright w_1 \implies v = w \ \&\\ &l_v \upharpoonright v \cap w = l_w \upharpoonright v \cap w \implies (v, l_v) = (w, l_w)\,. \end{aligned}$$

Above, for instance, $w_1 =_{\text{def}} \{s \in w \mid \sigma s \text{ is defined}\}$.

The correspondence takes a finite configuration x of $?^s S$ to the family consisting of pairs, one for each maximal unambiguous subconfiguration z of x; the pair for z comprises the configuration

$$d_s z = \{s \mid \exists \alpha.\ (s, \alpha) \in z\} \in \mathcal{C}(S)$$

and the function taking s in this set to $\alpha(s)$ where, because z is unambiguous, α is the necessarily unique α such that $(s, \alpha) \in z$.

Lemma 20.11. *Let $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$ be weak strategies. Then $?^s\tau \odot ?^s\sigma$ and $?^s(\tau \odot \sigma)$ are weakly equivalent strategies.*

Proof. Idea: As $?^s$ preserves weak equivalence of strategies and any strategy is weak equivalent to an innocent, strong receptive strategy, w.l.o.g. we may assume that σ and τ are innocent, strong receptive strategies. In this case we can show that $?^s\tau \odot ?^s\sigma$ and $?^s(\tau \odot \sigma)$ are isomorphic strategies from which the claim follows. The idea is to use Proposition 20.10 to relate the secured bijections involved in the definition of $?^s\tau \odot ?^s\sigma$ to those in $?^s(\tau \odot \sigma)$. $\square$

Corollary 20.12. *The operation $?^s$ is a pseudo endofunctor on the $\sim$-bicategory of strategies on games with symmetry, provided 2-cells are restricted to rigid maps.*

To lift the monad structure we use the fact that an affine map $f : A \to B$ of event structures with polarity lifts forwards to a strategy $f_! : A \twoheadrightarrow B$ and backwards to a strategy $f^* : B \twoheadrightarrow A$ and that this also applies when the event structures carry symmetry *****the metalanguage extends to games with symmetry****. In fact, there is also a less direct way in which we can lift f to a strategy from B to A: first form the map $f^s : A^\perp \to B^\perp$ got as f but with a switch of polarities; lift this to a strategy $f^s_! : A^\perp \twoheadrightarrow B^\perp$; then form the dual strategy $(f^s_!)^\perp : B \twoheadrightarrow A$. However this coincides with the direct backwards lift $f^* : B \twoheadrightarrow A$, *viz.* $f^* = (f^s_!)^\perp : B \twoheadrightarrow A$. There is an unfortunate clash of notation with ! both representing an operation duplicating Opponent events and the forwards lift. In this chapter we shall from now on write $f_* : A \twoheadrightarrow B$ for the forwards lift of $f : A \to B$.

The forwards lifts of the original units and multiplications of the monad associated with ? provide us with the units and multiplications of the monad ? on strategies which, overloading notation we shall write as $\eta^?$ and $\mu^?$. The backwards lifts of the original units and multiplications of ? provide us with counit $\epsilon^?$ and comultiplication $\delta^?$ of the comonad ? on strategies. Analogously, we obtain a monad and comonad by lifting the monad associated with ! to strategies, with *e.g.* the counit and comultiplication being written as $\epsilon^!$, $\delta^!$. We should also verify that ? is a (pseudo) monad with unit $\eta^?$ and multiplication $\mu^?$; duality will then ensure analogous results for the remaining putative monads and comonads.

The (pseudo) functor $!^s$ on strategies is defined in a dual fashion:

$$!^s(\sigma) = (?^s(\sigma^\perp))^\perp ,$$

for $\sigma : A \twoheadrightarrow B$.

It will be useful later to observe the simple form that $\eta^?_A$ takes when A has the trivial identity strategy.

Proposition 20.13. *Assume A has trivial identity symmetry. The strategy $\eta^?_A : A \twoheadrightarrow ?A$ comprises $\eta^?_A : E_A \to A^\perp \| ?A$. The events E_A are the subset $E_A \subseteq !A^\perp \| ?A$ comprising*

$$E_A = \{(1, \overline{a}) \mid a \in A\} \cup \{(2, (a, 0_a) \mid a \in A\}$$

where 0_a denotes the constantly 0 function from $[a]_A$. The causal dependency $\leq$ of E_A is the least transitive relation including that from $A^\perp \| ?A$ and

$$(1, \overline{a}) \leq (2, (a, 0_a))$$

when $pol_A(a) = +$, and

$$(2, (a, 0_a)) \leq (1, \overline{a})$$

when $pol_A(a) = -$. The map $\eta^?_A$ is the inclusion function on events. The symmetry on E_A is the trivial identity symmetry.

The (co)monad laws for the (co)units and (co)multiplications of $?^s$ and $!^s$ lift from the original (co)monads ? and !. However, their naturality has to be verified separately.

Theorem 20.14. *$?^s$ and $!^s$ are pseudo monads on the $\sim$-bicategory of strategies.*

Proof. Because of duality it suffices to verify naturality just for the unit and multiplication of $?^s$. For a weak strategy $\sigma : A \nrightarrow B$ we need to verify that

$$
\begin{tikzcd}
A \arrow[r, "\eta^?_A"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\approx"] & ?A \arrow[d, "?^s\sigma"] \\
B \arrow[r, "\eta^?_B"'] & ?B
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
??A \arrow[r, "\mu^?_A"] \arrow[d, "?^s?^s\sigma"'] \arrow[dr, phantom, "\approx"] & ?A \arrow[d, "?^s\sigma"] \\
??B \arrow[r, "\mu^?_B"'] & ?B
\end{tikzcd}
$$

commute up to weak equivalence $\approx$. ******* □

Images of edc strategies $\sigma : A \nrightarrow B$ are maps $\sigma : !A \nrightarrow ?B$ of the double Kleisli construction w.r.t. comonad ! and monad ?. In such situations maps $\sigma : !A \nrightarrow ?B$ and $\tau : !B \nrightarrow ?C$ standardly compose as

$$!A \xrightarrow{\delta^!_A} !!A \xrightarrow{!\sigma} !?B \xrightarrow{d_B} ?!B \xrightarrow{?\tau} ??C \xrightarrow{\mu^?_C} ?C\,.$$

with the help of a distributive law $d_B : !?B \nrightarrow ?!B$.

However both $!?B$ and $?!B$ are isomorphic to $‽B$ defined as $?B$ and $!B$ but allowing arbitrary indices on events of both polarities:
For $A = (A, \leq_A, \mathrm{Con}_A, pol_A)$ an event structure with polarity, define

$$‽A = (‽A, \leq, \mathrm{Con}, pol)$$

to comprise

- events $(a, \alpha) \in ‽A$ where $a \in A$ and $\alpha : [a]_A \to \omega$ with $pol((a, \alpha)) = pol_A(a)$;
- causal dependency,

$$(a', \alpha') \leq (a, \alpha) \iff a' \leq_A a \ \&\ \alpha' = \alpha \upharpoonright [a']_A\,;$$

- consistency,

$$X \in \mathrm{Con} \iff X \subseteq_{\mathrm{fin}} ‽A \ \&\ \{a \mid \exists \alpha.\ (a, \alpha) \in X\} \in \mathrm{Con}_A\,.$$

We extend a symmetry on A to a symmetry on $‽A$ via the following construction on isomorphism families: for $x, y \in \mathcal{C}(‽A)$,

$$
\begin{aligned}
\theta : x \cong_{‽A} y \iff\ & \theta \text{ is a bijection respecting } \leq \text{ and} \\
& \{(a, a') \mid \exists \alpha, \alpha'.\ \theta(a, \alpha) = (a', \alpha')\} \text{ is in the isomorphism family of } A\,.
\end{aligned}
$$

As $!?B \cong ‽B \cong ?!B$ the distributive law from $!?B$ to $?!B$ is trivial and composition of $\sigma : !A \nrightarrow ?B$ and $\tau : !B \nrightarrow ?C$ can be given as the composite strategy

$$!A \xrightarrow{\delta^!_A} !!A \xrightarrow{!^s\sigma} !?B \cong ?!B \xrightarrow{?^s\tau} ??C \xrightarrow{\mu^?_C} ?C\,.$$

The identity at a game A is given by the composite strategy

$$!A \xrightarrow{\epsilon^!_A} A \xrightarrow{\eta^?_A} ?A\,.$$

Proposition 20.15. *When A has trivial identity symmetry, the composite strategy* $!A \xrightarrow{\epsilon^!_A} A \xrightarrow{\eta^?_A} ?A$. *is isomorphic to the strategy* $\kappa_A : K_A \to (!A)^\perp \| ?A$. *The events K_A form a subset of $(!A)^\perp \| ?A$ and comprise*

$$K_A = \{(1, (\overline{a}, 0_{\overline{a}}) \mid a \in A\} \cup \{(2, (a, 0_a) \mid a \in A\}$$

where 0_a denotes the constantly 0 function from $[a]_A$. The causal dependency $\le$ of K_A is the least transitive relation including that from $(!A)^\perp \| ?A$ and

$$(1, (\overline{a}, 0_{\overline{a}})) \le (2, (a, 0_a))$$

when $pol_A(a) = +$, and

$$(2, (a, 0_a)) \le (1, (\overline{a}, 0_{\overline{a}}))$$

when $pol_A(a) = -$. The map κ_A is the inclusion function on events. The symmetry on K_A is the trivial identity symmetry.

Proof. We use Proposition 20.13 which characterises the strategy $\eta^?_A$. We obtain an analogous simple characterisation of $\epsilon^!_A$ by duality. Their composition is then seen to take the form described. □

We now show that wis, the operation taking an edc strategy $\sigma : A \rightarrow\!\!\!\!+\, B$ to a double-Kleisli map, a strategy $wis(\sigma) :!A \rightarrow\!\!\!\!+\, ?B$ is a pseudo functor. We need to check that wis preserves identities and that the image of the composition of edc strategies coincides to within weak equivalence of strategies with composition in double Kleisli maps of their images.

Lemma 20.16. *Let A be a game. Then, $wis(\gamma_A) :!A \rightarrow\!\!\!\!+\, ?A$ is the identity for composition of strategies in the double-Kleisli construction.*

Proof. Consider the image under wis of the copycat strategy $\gamma_A : \mathbb{C}_A \to A^\perp \| A$. Its image $wis(\gamma_A)$ is isomorphic to the identity $!A \xrightarrow{\epsilon^!_A} A \xrightarrow{\eta^?_A} ?A$ in the double-Kleisli construction by the characterisation of Proposition 20.15.
****
□

Lemma 20.17. *Let $\sigma : S \to A \rightarrow\!\!\!\!+\, B$ and $\tau : B \rightarrow\!\!\!\!+\, C$ be edc strategies. Then,*

$$wis(\tau \odot \sigma) \approx wis(\tau) \odot wis(\sigma) .$$

Proof. Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be edc strategies. Write $wis(\sigma) : S' \to ?(A^\perp \| B)$ and $\tau : T' \to ?(B^\perp \| C)$. Let $\tau \circledast \sigma : T \circledast S \to A^\perp \| B \| C$ be the partial edc strategy before hiding. Write $(T \circledast S)'$ for the event structure obtained by dropping its equivalence. We show that $(T \circledast S)'$ is isomorphic to $?^s T \circledast !^s S$ obtained as the pullback of $?^s \sigma \| C$ and $A \| !^s \tau$; because of thinness of the games and strategies involved the pullback is a bipullback. ****

□

Theorem 20.18. *The operation wis is a pseudo functor from edc strategies, in which 2-cells are rigid, to strategies in the double-Kleisli construction.*

Chapter 21

Probabilistic programming

By specialising to games in which all moves are those of Player we obtain a monoidal-closed sub-bicategory of probabilistic strategies that can serve as a foundation for probabilistic programming with discrete probability distributions. The restriction to discrete probability distributions is a consequence of the fact that configuration-valuations correspond to continuous valuations on domains of configurations.

21.1 Stable spans

Over the years there have been many solutions to giving a compositional semantics to nondeterministic dataflow (see [?] for fuller references). But they all hinge on some form of generalized relation, to distinguish the different ways in which output is produced from input. A compositional semantics can be given using *stable spans* of event structures, an extension of Berry's stable functions to include nondeterminism [?, ?]. A process of nondeterministic dataflow, with input type given by an event structure A and output by an event structure B, is captured by a pair of maps (a span)

$$\begin{tikzcd} & E \arrow[dl, "dem"'] \arrow[dr, "out"] & \\ A & & B \end{tikzcd}$$

where E is also an event structure. The map $out : E \to B$ is a *rigid* map, *i.e.* a total map of event structures as in Section **??** which preserves the relation of causal dependency, or equivalently, a total map with the property that for a configuration x of E if y is a subconfiguration of $out\, x$ then there is a (necessarily unique) subconfiguration x' of x such that $out\, x' = y$:

$$\begin{tikzcd} x' \arrow[r, dotted, phantom, "\subseteq"] \arrow[d, dotted, mapsto] & x \arrow[d, mapsto] \\ y' \arrow[r, phantom, "\subseteq"] & out\, x \end{tikzcd}$$

The map $dem : E \to A$, associated to input, is of a different character. It is a *demand* map, *i.e.* a function from $\mathcal{C}(E)$ to $\mathcal{C}(A)$ which preserves finite configurations and unions; $dem\, x$ is the minimum input for x to occur and is the union of the demands of its events. The occurrence of an event e in E demands minimum input $dem\,[e]$ and is observed as the output event $out(e)$. *Deterministic* stable spans, where consistent demands in A lead to consistent behaviour in E, correspond to Berry's stable functions.

The stable span $A \xleftarrow{dem} E \xrightarrow{out} B$ determines a *profunctor* $\widetilde{E}$ from the finite configurations p of A to the finite configurations q of B:

$$\widetilde{E}(p,\, q) = \{x \in \mathcal{C}(E) \mid dem\, x \subseteq p \;\&\; out\, x = q\}\,,$$

the set of ways the input-output pair (p, q) is realized.

Stable spans can be composed one after the other (essentially by a pullback construction, as rigid maps extend to special demand maps between configurations)—their composition coincides with the composition of their profunctors. They also have a nondeterministic sum, and compose in parallel, and most significantly allow a feedback operation [**?**]. Stable spans form a bicategory; their two cells are rigid maps *****

In fact, stable spans were first discovered explicitly as a way to represent, and give operational meaning to, the profunctors that arose as denotations of terms in affine-HOPLA, an affine Higher Order Process LAnguage [**?**, **?**]. The spans helped explain the tensor of affine-HOPLA as the parallel juxtaposition of event structures and a form of entanglement which appeared there as patterns of consistency and inconsistency on events. The use of stable spans in nondeterministic dataflow came later as a representation of the profunctors used in an earlier semantics [**?**, **?**].

Consider the sub-bicategory of games and strategies in which all moves are those of Player. This sub-bicategory is equivalent to the bicategory of *stable spans*. In this case, a strategy $\sigma : S \to A^\perp \| B$ corresponds to a *stable span*:

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & & \longleftrightarrow & & S^+ \arrow[dl, "\sigma_1^-"'] \arrow[dr, "\sigma_2^+"] & \\ A^\perp & & B & & A & & B\,, \end{tikzcd}$$

where S^+ is the projection of S to its +ve events; σ_2^+ is the restriction of σ_2 to S^+, necessarily a rigid map by innocence; σ_2^- is a *demand map* taking $x \in \mathcal{C}(S^+)$ to $\sigma_1^-(x) = \sigma_1[x]$; here $[x]$ is the down-closure of x in S. Composition of stable spans coincides with composition of their associated profunctors—see [16, 17, 3]. If we further restrict strategies to be deterministic (and, strictly, event structures to be countable) we obtain a bicategory equivalent to Berry's *dI-domains and stable functions* [3].

Let A and B be games in which all moves are +ve, *i.e.* those of Player. We construct its stable function space by first describing a stable family. The stable family $\mathcal{F}$ comprises those $F \subseteq_{\text{fin}} \mathcal{C}(A) \times B$ for which

(i) $\bigcup\{x \mid \exists b.\ (x,b) \in F\} \in \mathcal{C}(A)$,

(i) $\forall x_0 \in \mathcal{C}(A).\ \{b \mid \exists x \subseteq x_0.\ (x,b) \in F\} \in \mathcal{C}(B)$ and

(i) $\forall (x,b),(x',b) \in F.\ x = x'$.

It can be checked that $\mathcal{F}$ is a stable family. Define $(A \Rightarrow B) =_{\text{def}} \Pr(\mathcal{F})$.

***There is a stable span

$$\begin{tikzcd} & A \Rightarrow B \arrow[dl, "d_0"'] \arrow[dr, "r_0"] & \\ A & & B, \end{tikzcd}$$

where

$$d_0(z) = \bigcup\{x \mid \exists b.\ (x,b) \in \bigcup z\},$$

for $z \in \mathcal{C}(A \Rightarrow B)$, and

$$r_0(p) = b \text{ if } top(p) = (x,b) \text{ for some } x,$$

for $p \in \Pr(A \Rightarrow B)$.

Lemma 21.1. *For each stable span* $B \xleftarrow{dem} E \xrightarrow{out} C$ *there is a unique rigid map* $f : E \to (B \Rightarrow c)$ *such that* $dem = d \circ f$ *and* $out = r \circ f$*; the map* f *takes* $e \in E$ *to* $(dem(e), out(e))$.

Lemma 21.2. ***** monoidal-closed in the sense that there is a bijection ****

Proof. Via Lemma 21.1, there is a bijection between stable spans $A\|B \xleftarrow{dem} E \xrightarrow{out} C$ and stable spans $\varnothing \xleftarrow{\varnothing} E \xrightarrow{f} B \Rightarrow C$. *** □

RIGID IMAGE*

21.2 Probability

with probabilityLOSE MONOIDAL CLOSURE WRT ‖*****

Note though that probability distributions are discrete in that they correspond to *continuous* valuations on open sets. *****

Assume that games A and B comprise solely +ve moves. A probablistic strategy $v, \sigma : S \to A^\perp \| B$ corresponds to a probabilistic stable span

$$\begin{tikzcd} & S^+ \arrow[dl, "\sigma_1^-"'] \arrow[dr, "\sigma_2^+"] & \\ A & & B \end{tikzcd},$$

in which S^+ is endowed with a configuration-valuation v^+ to make it into a probabilistic event structure: take $v^+(x) =_{\text{def}} v([x])$ for $x \in \mathcal{C}(S^+)$. It is easy to check that v^+ is a configuration valuation for S^+.

Generally for a configuration-valuation v on an event structure with polarity S whenever $y \subseteq^+ x$ we can read the conditional probability $\text{Prob}(x \mid y) = v(x)/v(y)$. Consequently *** for a probabilistic stable span, with configuration-valuation v we can read v as giving

$$v(x) = \text{Prob}(x \mid x^-)\,,$$

the probability of $x \in \mathcal{C}(S)$ conditional on its Opponent moves x^-.

21.3

Consider now the sub-bicategory of games and *edc* strategies in which all moves are those of Player. ****the fn space for product now seems to need an equivalence $\equiv$ forcing all games to be edc's. But then what is copycat? **** Think the fn space is as above but with

$$(x_1, b_1) \equiv (x_2, b_2) \text{ iff } x_1 \equiv_A x_2 \ \&\ b_1 \equiv_B b_2\,,$$

where $x_1 \equiv_A x_2$ means x_1 and x_2 determine the same equivalence $\equiv$-classes, *i.e.* $x_{1\equiv_A} = x_{2\equiv_A}$.

*** can the earlier work on edc strategies be generalised to allow proper edc's as games? ***** Appears so with copycat which allows 'cross-overs" between $\equiv$-equivalenet events and composition based on pseudo pullback. ***only can characterise edc strategies now up to $\equiv$***

Acknowledgments

Thanks to Aurore Alcolei, Samy Abbes, Nathan Bowler, Simon Castellan, Pierre Clairambault, Pierre-Louis Curien, Marcelo Fiore, Mai Gehrke, Julian Gutierrez, Jonathan Hayman, Martin Hyland, Alex Katovsky, Tamas Kispeter, Marc Lasson, Paul-André Melliès, Samuel Mimram, Hugo Paquet, Gordon Plotkin, Silvain Rideau, Frank Roumen, Sam Staton and Marc de Visme for helpful discussions. The support of Advanced Grant ECSYM of the European Research Council is acknowledged with gratitude.

Bibliography

[1] Nielsen, M., Plotkin, G., Winskel, G.: Petri nets, event structures and domains. Theoretical Computer Science **13** (1981) 85–108

[2] Winskel, G., Nielsen, M.: Models for concurrency. In Abramsky, S., Gabbay, D., eds.: Semantics and Logics of Computation. OUP (1995)

[3] Saunders-Evans, L., Winskel, G.: Event structure spans for nondeterministic dataflow. Electr. Notes Theor. Comput. Sci. 175(3): 109-129 (2007)

[4] Winskel, G.: Event structure semantics for CCS and related languages. In: ICALP'82. Volume 140 of LNCS., Springer (1982)

[5] Rideau, S., Winskel, G.: Concurrent strategies. In: LICS 2011, IEEE Computer Society (2011)

[6] Joyal, A.: Remarques sur la théorie des jeux à deux personnes. Gazette des sciences mathématiques du Québec, 1(4) (1997)

[7] Winskel, G.: Event structures with symmetry. Electr. Notes Theor. Comput. Sci. 172: 611-652 (2007)

[8] Laird, J.: A games semantics of idealized CSP. Vol 45 of Electronic Books in Theor. Comput. Sci. (2001)

[9] Ghica, D.R., Murawski, A.S.: Angelic semantics of fine-grained concurrency. In: FOSSACS'04, LNCS 2987, Springer (2004)

[10] Melliès, P.A., Mimram, S.: Asynchronous games : innocence without alternation. In: CONCUR '07. Volume 4703 of LNCS., Springer (2007)

[11] Katovsky, A.: Concurrent games. First-year report for PhD study, Computer Lab, Cambridge (2011)

[12] Curien, P.L.: On the symmetry of sequentiality. In: MFPS. Number 802 in LNCS, Springer (1994) 29–71

[13] Hyland, M.: Game semantics. In Pitts, A., Dybjer, P., eds.: Semantics and Logics of Computation. Publications of the Newton Institute (1997)

[14] Harmer, R., Hyland, M., Melliès, P.A.: Categorical combinatorics for innocent strategies. In: LICS '07, IEEE Computer Society (2007)

[15] Melliès, P.A.: Asynchronous games 2: The true concurrency of innocence. Theor. Comput. Sci. 358(2-3): 200-228 (2006)

[16] Nygaard, M.: Domain theory for concurrency. PhD Thesis, Aarhus University (2003)

[17] Winskel, G.: Relations in concurrency. In: LICS '07, IEEE Computer Society (2005)

[18] Abramsky, S., Melliès, P.A.: Concurrent games and full completeness. In: LICS '99, IEEE Computer Society (1999)

[19] Hyland, J.M.E., Ong, C.H.L.: On full abstraction for PCF: I, II, and III. Inf. Comput. 163(2): 285-408 (2000)

[20] Abramsky, S., Jagadeesan, R., Malacaria, P.: Full abstraction for PCF. Inf. Comput. 163(2): 409-470 (2000)

[21] Varacca, D., Völzer, H., Winskel, G.: Probabilistic event structures and domains. Theor. Comput. Sci. 358(2-3): 173-199 (2006)

[22] Hyland, M.: Some reasons for generalising domain theory. Mathematical Structures in Computer Science **20**(2) (2010) 239–265

[23] Cattani, G.L., Winskel, G.: Profunctors, open maps and bisimulation. Mathematical Structures in Computer Science **15**(3) (2005) 553–614

[24] Curien, P.L., Plotkin, G.D., Winskel, G.: Bistructures, bidomains, and linear logic. In: Proof, Language, and Interaction, essays in honour of Robin Milner, MIT Press (2000) 21–54

[25] Abramsky, S.: Semantics of interaction. In Pitts, A., Dybjer, P., eds.: Semantics and Logics of Computation. Publications of the Newton Institute (1997)

[26] Martin, D.A.: Borel determinacy. Annals of Mathematics **102**(2) (1975) 363–371

[27] Claire Jones, G.P.: A probabilistic powerdomain of valuations. In: LICS '89, IEEE Computer Society (1989)

[28] Varacca, D.: Probability, nondeterminism and concurrency. PhD Thesis, Aarhus University (2003)

[29] M Alvarez-Manilla, A Edalat, N.S.D.: An extension result for continuous valuations. Journal of the London Mathematical Society **61**(2) (2000) 629–640

Appendix A

Exercises

On event structures and stable families

Recommended exercises: 1, 3, 4, 5 (Harder), 6, 7, 10.

Exercise A.1. *Let* $(A, \leq_A, \mathrm{Con}_A), (B, \leq_B, \mathrm{Con}_B)$ *be event structures. Let* $f : A \rightharpoonup B$. *Show* f *is a map of event structures,* $f : (A, \leq_A, \mathrm{Con}_A) \to (B, \leq_B, \mathrm{Con}_B)$, *iff*

(i) $\forall a \in A, b \in B.\ b \leq_B f(a) \implies \exists a' \in A.\ a' \leq_A a\ \&\ f(a') = b$, *and*

(ii) $\forall X \in \mathrm{Con}_A.\ fX \in \mathrm{Con}_B\ \&\ \forall a_1, a_2 \in X.\ f(a_1) = f(a_2) \implies a_1 = a_2$.

□

Exercise A.2. *Show a map* $f : A \rightharpoonup B$ *of* $\mathcal{E}$ *is mono if the function* $\mathcal{C}(A) \to \mathcal{C}(B)$ *taking configuration* x *to its direct image* fx *is injective. [Recall a map* $f : A \to B$ *is mono iff for all maps* $g, h : C \to A$ *if* $fg = fh$ *then* $g = h$*.] Show the converse does not hold, that it is possible for a map to be mono but not injective on configurations. Taking* B *to be the event structure comprising two concurrent events, can you find an event structure* A *and an example of a total map* $f : A \to B$ *of event structures which is both mono and where* f *is not injective as a function on events?* □

Exercise A.3. *Verify that the finite configurations of an event structure form a stable family.* □

Exercise A.4. *Say an event structure* A *is* tree-like *when its concurrency relation is empty (so two events are either causally related or inconsistent). Suppose* B *is tree-like and* $f : A \to B$ *is a total map of event structures. Show* A *must also be tree-like, and moreover that the map* f *is rigid, i.e. preserves causal dependency.*

Exercise A.5. *Let $\mathcal{F}$ be a nonempty family of finite sets satisfying the Completeness axiom in the definition of stable families. Show $\mathcal{F}$ is coincidence-free iff*

$$\forall x, y \in \mathcal{F}.\ x \subsetneq y \implies \exists x_1, e_1.\ x \overset{e_1}{\longrightarrow\!\subset} x_1 \subseteq y\,.$$

[Hint: For 'only if' use induction on the size of $y \setminus x$.] □

Exercise A.6. *Prove Proposition 3.14: Let $f : \mathcal{F} \to \mathcal{G}$ be a map of stable families. Let $e, e' \in x$, a configuration of $\mathcal{F}$. Show if $f(e) \leq_{fx} f(e')$ (with both $f(e)$ and $f(e')$ defined) then $e \leq_x e'$.*

Exercise A.7. *Prove the two propositions 3.7 and 3.10.* □

Exercise A.8. *(From Section 3.2) For an event structure E, show $\mathcal{C}^\infty(E) = \mathcal{C}(E)^\infty$.* □

Exercise A.9. *(From Section 3.2) Let $\mathcal{F}$ be a stable family. Show $\mathcal{F}^\infty$ satisfies:*

Completeness: $\forall Z \subseteq \mathcal{F}^\infty.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}^\infty$ *;*
Stability: $\forall Z \subseteq \mathcal{F}^\infty.\ Z \neq \varnothing \ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}^\infty$*;*
Coincidence-freeness: *For all $x \in \mathcal{F}^\infty$, $e, e' \in x$ with $e \neq e'$,*

$$\exists y \in \mathcal{F}^\infty.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y)\,;$$

Finiteness: *For all $x \in \mathcal{F}^\infty$,*

$$\forall e \in x \exists y \in \mathcal{F}.\ e \in y \ \&\ y \subseteq x \ \&\ y \text{ is finite}\,.$$

Show that $\mathcal{F}$ consists of precisely the finite sets in $\mathcal{F}^\infty$. □

Exercise A.10. *Let A be the event structure consisting of two distinct events $a_1 \leq a_2$ and B the event structure with a single event b. Following the method of Section 3.3.1 describe the product of event structures $A \times B$.* □

On strategies

Recommended exercises: 11, 12, 13, 14, 15, 17.

Exercise A.11. *Consider the empty map of event structures with polarity $\varnothing \to A$. Is it a strategy? Is it a deterministic strategy? Consider now the identity map $\mathrm{id}_A : A \to A$ on an event structure with polarity A. Is it a strategy? Is it a deterministic strategy?* □

Exercise A.12. *For each instance of total map σ of event structures with polarity below say whether σ is a strategy and whether it is deterministic. In each case give a short justification for your answer. (Immediate causal dependency within the event structures is represented by an arrow $\rightarrowtriangle$ and inconsistency, or conflict, by a wiggly line $\sim\!\sim\!\sim$.)*

(*i*)

$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \ominus \arrow[r, -Triangle] \arrow[d, dotted, |->] & \oplus \arrow[d, dotted, |->] \\ A & \ominus & \oplus \end{tikzcd}$$

(*ii*)

$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \oplus \arrow[r, -Triangle] \arrow[d, dotted, |->] & \ominus \arrow[d, dotted, |->] \\ A & \oplus & \ominus \end{tikzcd}$$

(*iii*)

$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \oplus \arrow[r, -Triangle] \arrow[d, dotted, |->] & \oplus \arrow[d, dotted, |->] \\ A & \oplus & \oplus \end{tikzcd}$$

(*iv*)

$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \ominus \arrow[r, -Triangle] \arrow[d, dotted, |->] & \ominus \arrow[d, dotted, |->] \\ A & \ominus & \ominus \end{tikzcd}$$

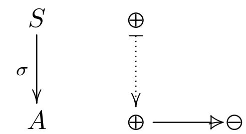
(*v*)

$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \ominus \arrow[d, dotted, |->] & \\ A & \ominus \arrow[r, -Triangle] & \oplus \end{tikzcd}$$

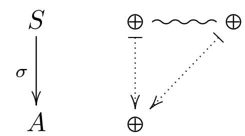
(*vi*)

$$\begin{tikzcd} S \arrow[d, "\sigma"'] & \oplus \arrow[d, dotted, |->] & \\ A & \oplus \arrow[r, -Triangle] & \oplus \end{tikzcd}$$

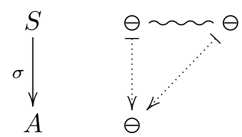
(vii)



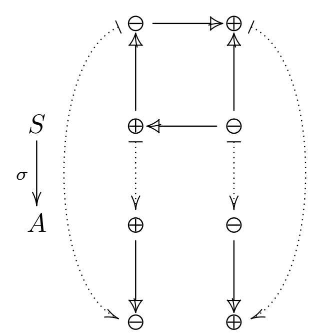
$(viii)$



(ix)



(x)



$\square$

Exercise A.13. *Let* $\mathrm{id}_A : A \to A$ *be the identity map of event structures, sending an event to itself. Show the identity map forms a strategy in the game A. Is it deterministic in general?* $\square$

Exercise A.14. *Show any strategy* $\sigma : A \mathrel{+\!\!\!\rightarrow} B$ *has a dual strategy* $\sigma^\perp : B^\perp \mathrel{+\!\!\!\rightarrow} A^\perp$*. In more detail, supposing* $\sigma : S \to A^\perp \| B$ *is a strategy show* $\sigma^\perp : S \to (B^\perp)^\perp \| A^\perp$ *is a strategy where*

$$\sigma^\perp(s) = \begin{cases} (1, b) & \text{if } \sigma(s) = (2, b) \\ (2, a) & \text{if } \sigma(s) = (1, a) \,. \end{cases}$$

$\square$

Exercise A.15. *Let* B *be the event structure consisting of the two concurrent events* b_1*, assumed* $-$*ve, and* b_2*, assumed* $+$*ve in* B *. Let* C *consist of a single* $+$*ve event* c*. Let the strategy* $\sigma : \varnothing \mathrel{+\!\!\!\rightarrow} B$ *comprise the event structure* $s_1 \rightarrowtriangle s_2$

with s_1 *−ve and* s_2 *+ve,* $\sigma(s_1) = b_1$ *and* $\sigma(s_2) = b_2$*. In* $B^\perp$ *the polarities are reversed so there is a strategy* $\tau : B \twoheadrightarrow\!\!\!\!\!+\; C$ *comprising the map* $\tau : T \to B^\perp \| C$ *from the event structure* T*, with three events* t_1 *and* t_3 *both +ve and* t_2 *−ve so* $t_2 \rightarrowtriangle t_1$ *and* $t_2 \rightarrowtriangle t_3$*, which acts so* $\tau(t_1) = \overline{b}_1$*,* $\tau(t_2) = \overline{b}_2$ *and* $\tau(t_3) = c$*. Describe the composition* $\tau \odot \sigma$*.* □

Exercise A.16. *Say an event structure is* set-like *if its causal dependency relation is the identity relation and all pairs of distinct events are inconsistent. Let* A *and* B *be games with underlying event structures which are set-like event structures. In this case, can you see a simpler way to describe deterministic strategies* $A \twoheadrightarrow\!\!\!\!\!+\; B$*? What does composition of deterministic strategies between set-like games corresponds to? What do strategies in general between set-like games correspond to? What does composition of strategies between set-like games corresponds to? [No proofs are required.]* □

Exercise A.17. *By considering the game* A *comprising two concurrent events, one +ve and one −ve, show there is a nondeterministic pre-strategy* $\sigma : S \to A$ *such that* $s \rightarrowtriangle s'$ *in* S *without* $\sigma(s) \rightarrowtriangle \sigma(s')$*. Could you find such a counterexample were* σ *deterministic? Explain.* □

Exercise A.18. *Let* $G =_{\mathrm{def}} (A, W)$ *be a game with winning conditions. Say a pre-strategy* $\sigma : S \to A$ *is winning iff* $\sigma x \in W$ *for all +-maximal configurations* $x \in \mathcal{C}^\infty(S)$*. Show that if* G *has a winning* receptive *pre-strategy, then the dual game* $G^\perp$ *has no winning strategy (use Corollary 9.3.) Show that* G *may have a winning pre-strategy (necessarily not receptive) while* $G^\perp$ *has a winning strategy.* □

Appendix B

Projects

The projects are quite ambitious and to some extent open-ended. You can achieve a good grade, even in the more technical questions, without completing every part. You may use any results from the notes provided you state them clearly.

Project 1. Stable families with coincidence. There are possibly good reasons to investigate event structures and stable families in which the causal dependency relation is a pre-order rather than a partial order (*cf.* the work on "round abstraction" in circuits of Ghica and Menaa). In particular, investigate stable families but without the axiom of coincidence-freeness; what are their maps, what are their products, how do they relate to event structures? [My ICALP 1982 paper and report on "Event structure semantics of CCS and related languages," available from my Cambridge homepage, might be helpful for proofs.]

Project 2. Strategies from maps of event structures. In this project you are guided part of the way to showing that $f : A \to B$, a partial map between event structures with polarity, can be regarded as a (special) strategy $\sigma : A \rightarrow\!\!\!\!\!\!\!+\!\!\!\!> B$ in such a way that composition and identities are respected.

For $f : A \to B$, a partial map of event structures with polarity, we construct a strategy $\sigma(f) : S \to A^\perp \| B$. The event structure S is built as $\Pr(\mathcal{S})$ from a stable family $\mathcal{S}$. The family $\mathcal{S}$ consists of subsets

$$\{1\} \times \overline{x} \;\cup\; \{2\} \times y\,,\ \text{abbreviated to } (\overline{x}, y)\,,$$

where $x \in \mathcal{C}(A)$, $y \in \mathcal{C}(B)$, which satisfy

$$\begin{aligned} &\overline{a} \in \overline{x} \ \&\ pol_{A^\perp}(\overline{a}) = + \implies f(a) \in y \ \text{ and} \\ &b \in y \ \&\ pol_B(b) = + \implies \exists a \in x.\ f(a) = b\,. \end{aligned}$$

(1) Show, for $(\overline{x}, y) \in \mathcal{S}$,

(i) $\forall x_0 \in \mathcal{C}(A).\ x_0 \subseteq x \implies (\overline{x}_0,\ (f x_0) \cap y) \in \mathcal{S}$

(ii) $\forall y_0 \in \mathcal{C}(B).\ y_0 \subseteq y \implies (\overline{x} \cap [f^{-1} y_0],\, y_0) \in \mathcal{S}$.

(2) Show $\mathcal{S}$ is a stable family.

With $S =_{\mathrm{def}} \Pr(\mathcal{S})$, define

$$\sigma(f)(s) = \begin{cases} \overline{a} & \text{if } top(s) = (1, \overline{a})\,, \\ b & \text{if } top(s) = (2, b)\,. \end{cases}$$

(3) Show $\sigma(f)$ is a total map of event structures $\sigma(f) : S \to A^{\perp} \| B$ which respects polarity.

(4) Show $\sigma(f)$ is a strategy $\sigma(f) : A \twoheadrightarrow B$.

(5) Show, in the case where f is the identity map $\mathrm{id}_A : A \to A$, that $\sigma(\mathrm{id}_A) = \gamma_A$, the copy-cat strategy.

(6) Suppose now $f : A \to B$ and $g : B \to C$ are maps of event structures with polarity. Can you show that $\sigma(gf) \cong \sigma(g) \odot \sigma(f)$? (Hard)

(7) Is $\sigma(f)$ always a deterministic strategy for all maps f of event structures with polarity? If not can you see what properties are required of f for $\sigma(f)$ to be deterministic?

Project 3. Winning strategies with neutral positions. A natural generalisation of the games with winning conditions of Chapter 9 is to games (A, W, L) comprising an event structure with symmetry A and disjoint subsets W and L of $\mathcal{C}^{\infty}(A)$ which specify the winning and losing configurations without necessarily having that one is the complement of the other—configurations in $\mathcal{C}^{\infty}(A) \setminus (W \cup L)$ would be *neutral* positions. Imitate the constructions on games and winning conditions of Chapter 9 in this broader framework. Adopt the same definition of winning strategy as before. For the new dual operation and parallel composition take

$$G^{\perp} = (A, L_G, W_G) \quad \text{and} \quad G \| H = (A \| B,\ W_G \| \mathcal{C}^{\infty}(B) \cup \mathcal{C}^{\infty}(A) \| W_H),\ L_G \| L_H)\,,$$

where $G = (A, W_G, L_G)$ and $H = (B, W_H, L_H)$—the notation of Chapter 9 is being used here. In the new parallel composition to win is to win in either component and to lose is to lose in both. What is the unit of $\|$? What are the winning and losing configurations of $G^{\perp} \| H$? As before, a winning strategy from G to H is a winning strategy in $G^{\perp} \| H$. It is important that you try to show that the composition of winning strategies is winning (follow the pattern of the proof in Chapter 9), and that for suitable games copy-cat is winning.

Project 4. An essay on strategies in logic. Write an essay explaining to your best friend in humanities why logicians and philosophers are interested in games and strategies. The papers of Johan van Bentham provide a good start.

Project 5. Games in other models. Take a favourite model, *e.g.* transition systems, languages, some variety of Petri nets, Mazurkiewicz trace languages, and try to imitate the constructions on games there. You might find it convenient to allow "internal" events, which are neither moves of Opponent or Player, for instance in defining composition of strategies in your model.