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Penalty Parameter Update Strategies in Progressive Hedging Algorithm

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Abstract. The progressive hedging algorithm is to date one of the most popular scenario decomposition methods in multi-stage stochastic programming. While it achieves full decomposition over scenarios, its efficiency remains sensitive to some implementation choices. In particular, the algorithm performance is highly sensitive to the penalty parameter value, and various authors have proposed different strategies to update it over iterations. We review some of the popular methods, and design a novel adaptive strategy that aims to better follow the algorithm process. Preliminaries numerical experiments on linear multi-stage stochastic test problems suggest that most of the existing techniques may exhibit premature convergence to a sub-optimal solution or converge to the optimal solution, but at a very slow rate. In contrast, the new strategy appears to be robust and efficient, converging to optimality in all our experiments and being the fastest in most of them.

Keywords: Stochastic programming, multi-stage programming, augmented Lagrangian, proximal methods, penalty parameter.

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1 Introduction

Optimization problems involving uncertainty occur in almost all areas of science and engineering. This stimulates interest of formulating, analyzing and solving such problems. Today, stochastic programming offers a variety of models to address the presence of random data in optimization problems, such as two- and multi-stage models (Kall and Wallace, 1994; Birge and Louveaux, 2011; Wallace and Ziemba, 2005). In scenario analysis, the uncertain parameters or components of the problems are modelled by a finite number of *scenarios* or possible representations of data (Rockafellar and Wets, 1991). Due to often very large dimensions of problems in scenario analysis, decomposition method is an option to solve this kind of problems and track their specific structures. Then, it is possible to split a large problem into manageable pieces corresponding scenarios, solve them and then come up with a good combined solution to the underlying large problem. Rockafellar and Wets (1991) proposed the *progressive hedging algorithm* (PHA) that achieves full decomposition over scenarios, and allows solving the scenario subproblems in parallel, while progressively enforcing the nonanticipativity (NA) constraints. The method uses a quadratic penalty term for the violation of NA constraints. Rockafellar and Wets (1991) briefly discussed the effect of the penalty parameter on the performance of PHA and the quality of the solution, but did not provide any indication on how to choose it. Various values for this parameter have been proposed, but they tend to differ with the application under consideration. The idea of updating the penalty parameter over the iterations was first introduced by Mulvey and Vladimirou (1991a,b), who noticed that the PHA convergence rate was extremely sensitive to the penalty parameter value. Hvattum and Løkketangen (2009) suggested a controlling approach for updating the parameter in order to ensure a stable progress toward both primal and dual convergence. In their method, if the progress in primal space is negative, they decrease the penalty parameter, and if the progress in dual space is negative, they increase it. More recently, Zéphyr et al. (2014) proposed to define the penalty parameter as a function of the solution state. To do so, they introduced an adaptive learning update with respect to the progress of optimality gap and also to the balance between NA gap and optimality gap in the PHA.

The choice of penalty parameter is therefore a recurrent question in the application of the PHA. We therefore aim to compare the various proposed strategies, and explore if we can improve them. The PHA maintains an approximation of the NA constraints, and can be viewed as a proximal point method. This suggests to adaptively update the penalty parameter based on the progress in both primal and dual spaces, increasing the parameter to enforce convergence when the NA constraints are properly represented, decreasing it when the approximation is not satisfactory. Consequently, we propose a new update strategy and numerically validate it on some test problems taken from the literature.

The paper is organized as follows. We briefly introduce multi-stage stochastic problems and the progressive hedging algorithm in Section 2. Then, we discuss about current strategies for updating the penalty parameter and introduce a novel adaptive approach in Section 3. Some numerical experiments are presented in Section 4 in order to compare the various approaches. We present some conclusions and research avenues in Section 5. We finally provide a short mathematical proof of the equality that motivates our the adaptive update and detailed numerical results in Appendix.

2 Progressive hedging algorithm

In many optimization problems, some data may be considered uncertain. Here, we assume that data uncertainty can be represented as a random vector $\boldsymbol{\xi}$ of finite support $\Xi \subseteq \mathbb{R}^q$, and we consider the optimization problem

$$\begin{aligned} &\min_x \; E_{\boldsymbol{\xi}}[f(x, \boldsymbol{\xi})] \\ &\text{s.t. } x \in \mathcal{X}(\boldsymbol{\xi}), \end{aligned} \tag{1}$$

where for all x, $f(x, .) : \Xi \to \mathbb{R}$ is a measurable objective function, and $\mathcal{X}(\cdot) : \Xi \to 2^{\mathbb{R}^n}$ is a measurable multivalued mapping representing (event-dependent) constraints. We denote a realization of $\boldsymbol{\xi}$ by ξ.

In a multi-stage stochastic problem, uncertainty is gradually revealed over time and decisions are made sequentially, at each stage where more information is available. The random vector vector $\boldsymbol{\xi}$ can be represented as $\boldsymbol{\xi} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_T\}$, where by convention $\boldsymbol{\xi}_1 = \xi_1$, as the first stage is deterministic. We also denote by $\bar{\xi}_t = (\xi_1, \xi_2, \dots, \xi_t)$ as the history of realizations up to stage t. The possible realizations of $\boldsymbol{\xi}$ are also called scenarios, and we denote a specific scenario as

$$s = \left(\xi_1^{(s)}, \xi_2^{(s)}, \cdots, \xi_T^{(s)}\right),$$

associated to the probability $p_s = P_{\boldsymbol{\xi}}[\xi = s]$. The set of scenarios is $\mathcal{S} = \{1, \dots, S\}$.

In contrast with (1), we now denote by $x^{(s)} = (x_1^{(s)}, \dots, x_T^{(s)})$ the sequence of decisions over time under scenario s, and by $\bar{x}_t^{(s)} = (x_1^{(s)}, \dots, x_t^{(s)})$ as the decision history up to stage t under scenario s. We now define x as $(x^{(1)}, x^{(2)}, \dots, x^{(S)})$. (1) can then be rewritten as

$$\begin{aligned} &\min_x \; \sum_{s \in \mathcal{S}} p_s f\left(x^{(s)}, \xi^{(s)}\right) \\ &\text{s.t. } x^{(s)} \in \mathcal{X}^{(s)}, \\ &\qquad x_t^{(s)} \text{ is nonanticipative, } t = 1, \dots, T. \end{aligned} \tag{2}$$

Mathematically speaking, the NA constraints can be expressed as $x_t^{(s)} = x_t^{(s')}$ if $\bar{\xi}_t^{(s)} = \bar{\xi}_t^{(s')}$, for all s, $s' \in \mathcal{S}$ and $t = 1, \dots, T$. In other terms, the decisions associated to two scenarios must coincide as long as the scenarios share the same history. The NA condition can be rewritten as

$$x_t^{(s)} = E\left[x_t^{(s')} \,\middle|\, s' \in \mathcal{S}_t^{(s)}\right], \tag{3}$$

where $\mathcal{S}_t^{(s)} = \left\{ s' \,\middle|\, \bar{\xi}_t^{(s)} = \bar{\xi}_t^{(s')} \right\}$. (3) can be reformulated as

$$x_t^{(s)} = \hat{x}_t^{(s)},$$

with

$$\hat{x}_t^{(s)} = \frac{\sum_{s' \in \mathcal{S}_t^{(s)}} p_{s'} x_t^{(s')}}{\sum_{s' \in \mathcal{S}_t^{(s)}} p_{s'}}. \tag{4}$$

(2) would be separable by scenarios if the NA constraints, that link all the scenarios, were ignored. Rockafellar and Wets (1991) suggest to consider the augmented Lagrangian

$$L(x, \lambda, \rho) = E\left[f\left(x^{(s)}, \xi^{(s)}\right) + \sum_{t=1}^{T} \left(\lambda_t^{(s)\prime} \left(x_t^{(s)} - \hat{x}_t^{(s)} \right) + \frac{\rho}{2} \left\| x_t^{(s)} - \hat{x}_t^{(s)} \right\|^2 \right) \right],$$

where λ is the Lagrange multipliers vector associated to the NA constraints, and $\rho > 0$ is a penalty parameter. Given λ, the augmented Lagrangian program is

$$\begin{aligned} &\min_x \; L(x, \lambda, \rho) \\ &\text{s.t. } x^{(s)} \in \mathcal{X}^{(s)},\; s \in \mathcal{S}. \end{aligned} \tag{5}$$

In order to achieve full separability, Rockafellar and Wets (1991) moreover propose to fix $\hat{x}_t^{(s)}$ in (5), and repeatedly solve the program by updating the Lagrange multipliers vector and the value of $\hat{x}_t^{(s)}$ between consecutive resolutions. The resulting algorithm is known as the PHA, summarized below

Step 0. Set $\hat{x}^{(s),0} = \left(\hat{x}_1^{(s),0}, \ldots, \hat{x}_T^{(s),0} \right)$ and $k = 0$. Choose $\lambda^{(s),0} = \mathbf{0}$, $\rho^0 > 0$.

Step 1. Compute $x^{(s),k+1} = (x_1^{s,k+1}, \ldots, x_T^{s,k+1})$, $s = 1, \ldots, S$, by solving each scenario subproblem

$$\begin{aligned} &\min_{x^{(s)}} \; f\left(x^{(s)}, \xi^{(s)}\right) + \sum_{t=1}^{T} \left(\lambda_t^{(s)\prime} \left(x_t^{(s)} - \hat{x}_t^{(s),k} \right) + \frac{\rho^k}{2} \left\| x_t^{(s)} - \hat{x}_t^{(s),k} \right\|^2 \right) \\ &\text{s.t. } x^{(s)} \in \mathcal{X}^{(s)}. \end{aligned} \tag{6}$$

Step 2. For $s = 1, \ldots, S$, $t = 1, \ldots, T$, set

$$\hat{x}_t^{(s),k+1} = \frac{\sum_{s' \in \mathcal{S}_t^{(s)}} p_{s'} x_t^{(s'),k+1}}{\sum_{s' \in \mathcal{S}_t^{(s)}} p_{s'}}.$$

Step 3. Set ρ^{k+1} and

$$\lambda_t^{(s),k+1} = \lambda_t^{(s),k} + \rho^k \left(x_t^{(s),k+1} - \hat{x}_t^{(s),k+1} \right),\; t = 1, \ldots, T,\; s \in \mathcal{S}.$$

Step 4. Stop if convergence is achieved. Otherwise, set $k \leftarrow k+1$ and return to Step 1.

Several practical issues arise. A first question is the choice of the stopping criteria in Step 4. Rockafellar and Wets (1991) propose to stop if

$$\sqrt{\sum_{s \in \mathcal{S}} p_s \|x^{(s),k+1} - \hat{x}^{(s),k+1}\|_2^2 + \sum_{s \in \mathcal{S}} p_s \|\hat{x}^{(s),k+1} - \hat{x}^{(s),k}\|_2^2} \leq \varepsilon,$$

or equivalently if

$$\sqrt{\sum_{s \in \mathcal{S}} p_s \|x^{(s),k+1} - \hat{x}^{(s),k}\|_2^2} \leq \varepsilon, \tag{7}$$

as it can be easily proved (see Appendix A) that

$$\sum_{s \in \mathcal{S}} p_s \|x^{(s),k+1} - \hat{x}^{(s),k}\|_2^2 = \sum_{s \in \mathcal{S}} p_s \|x^{(s),k+1} - \hat{x}^{(s),k+1}\|_2^2 + \sum_{s \in \mathcal{S}} p_s \|\hat{x}^{(s),k+1} - \hat{x}^{(s),k}\|_2^2. \tag{8}$$

A second question concerns the initialization of the primal variables. As discussed in Chiche (2012), Chapter 6, various strategies have been considered but the most popular is to set $x^{(s),0}$, $s = 1, \ldots, S$ as the solution of the subproblem associated to s, without the NA constraints:

$$\begin{aligned} &\min_{x^{(s)}} f\left(x^{(s)}, \xi^{(s)}\right) \\ &\text{s.t. } x^{(s)} \in \mathcal{X}^{(s)}, \end{aligned}$$

and $\hat{x}^{s,0}$ is computed using (4). dos Santos et al. (2009) compared this procedure to other initializations, but did not find significant improvements.

3 Penalty parameter update

3.1 Existing strategies

Rockafellar and Wets (1991) analyzed the progressive hedging algorithm and established its convergence with a penalty parameter constant over the iterations. However, many authors have observed that in practice, the choice of penalty parameter value will greatly impact the numerical behavior of the algorithm. Considering the stopping criterion (7) Helgason and Wallace (1991) commented that the penalty parameter should be as small as possible but large enough to guarantee the convergence, more specifically to produce a monotone decrease in the criteria. Mulvey and Vladimirou (1991a,b) showed that the overall convergence rate of the PHA is particularly sensitive to the choice of ρ. Small values of the penalty parameter tend to produce a fast initial progress in primal sequence $\{\hat{x}^k\}$ with a slower progress in dual space, i.e. the sequence $\{\lambda^k\}$, while large values lead to an opposite behavior. They first suggested to consider a larger penalty parameter when the nonanticipavity constraints are more restrictive, and introduced the idea to dynamically update the parameter, increasing the value over the PHA iterations. However, they also noticed that a sudden increase in penalty parameter can drive the algorithm toward ill-conditioning or a suboptimal solution, suggesting to increase the penalty parameter smoothly. They also proposed to implement a sudden reduction to improve the convergence in primal space, if the dual convergence is already achieved. Chun and Robinson (1995) used two predefined values: they initialized the parameter with a large value and then changed it to a small value, if there was enough improvement in dual sequence. Jonsbråten (1998) decided to maintain the penalty parameter to zero, and defined a dynamically updated step size to compute the Lagrange multipliers. Some authors considered the possibility to use different penalty parameters, depending on the affected variables. In particular, Somervell (1998) suggested to use predefined fixed bundle-stage wise values, while Watson et al. (2007) proposed to set penalty parameters proportionally to the cost coefficient in the objective function, when dealing with linear functions. Fan and Liu (2010), inspired by Chun and Robinson (1995) and Watson et al. (2007), explored the use of two fixed values and cost-proportional values.

Following the idea of dynamic update, Reis et al. (2005) decreased the penalty over the iterations, while other authors, as Crainic et al. (2011); Carpentier et al. (2013), chose to increase the penalty parameter. Hvattum and Løkketangen (2009) suggested a controlling approach based on criteria (7) for updating the penalty parameter. They reduced ρ if $\sum_{s\in\mathcal{S}} p_s\|\hat{x}^{s,k+1} - \hat{x}^{s,k}\|_2^2$ does not decrease, and they increased ρ if $\sum_{s\in\mathcal{S}} p_s\|x^{s,k+1} - \hat{x}^{s,k+1}\|_2^2$ does not decrease. They also considered a node-cost proportional update. Inspired by them, Gul (2010) suggested to dynamically update the penalty parameter, increasing the parameter if no progress is observed for the dual variables, and decreasing it if no progress is observed for the primal variables. Gonçalves et al. (2012) used an increase factor proportional to the NA violation. They also insisted that the initial penalty parameter should be chosen small enough, for instance with a value of 10^{-4}. Zéphyr et al. (2014) dynamically updated the penalty parameter using coefficients based on optimality and NA indicators, allowing to increase or decrease the parameter.

Therefore, many strategies for updating the penalty parameter have been considered, highlighting the sensitivity of the PHA to its value, but no clear consensus exists to date. We summarize the main approaches in Table 1. More recently, Chiche (2012) highlighted that a dynamic update strategy can even lead to a complete failure of the PHA. She provided an example where an apparently genuine choice of the penalty parameter results in a cyclic behavior of the PHA between two solutions, none of them satisfying the NA constraints. In her example, the penalty parameter oscillates between two inverse values. Therefore, while the use of a fixed penalty parameter is usually associated with a slow convergence, care must be exercised when designing a dynamic update strategy if we want to improve the PHA performance. In the following, we propose a novel approach that aims to learn from the algorithm process, while remaining simple and independent of the application.

3.2 Adaptive penalty parameter update

As pointed by Takriti and Birge (2000), the PHA is a proximal point method producing the contraction of a sequence of primal-dual pairs $\{(\hat{x}^{s,k}, \lambda^{s,k})\}$ around an optimal saddle point. The primal convergence can be monitored by considering the expectation of the changes between consecutive NA solution $\sum_{s \in \mathcal{S}} p_s \|\hat{x}^{s,k+1} - \hat{x}^{s,k}\|_2$ while, from Step 3. of the PHA, $\sum_{s \in \mathcal{S}} p_s \|x^{s,k+1} - \hat{x}^{s,k+1}\|_2$ gives the order of the change in dual variables. Recall that from (8),

$$\sum_{s \in \mathcal{S}} p^s \|x^{s,k+1} - \hat{x}^{s,k}\|_2^2 = \sum_{s \in \mathcal{S}} p^s \|\hat{x}^{s,k+1} - \hat{x}^{s,k}\|_2^2 + \sum_{s \in \mathcal{S}} p^s \|x^{s,k+1} - \hat{x}^{s,k+1}\|_2^2,$$

i.e. the PHA aggregates the primal and dual changes. In constrast to many papers that monitors the values of $\sum_{s \in \mathcal{S}} p^s \|x^{s,k+1} - \hat{x}^{s,k+1}\|_2^2$ to decide to increase or decrease the penalty parameter ρ, we first check the changes in $\{\hat{x}^k\}$, i.e. $\sum_{s \in \mathcal{S}} p^s \|\hat{x}^{s,k+1} - \hat{x}^{s,k}\|_2^2$. The main motivation is to avoid to enforce the NA constraints when we have not yet identified the correct NA solution, or in other terms the approximation of the NA is not accurate enough. Otherwise, we could converge to a suboptimal solution, as also noticed by Chun and Robinson (1995). Therefore, if the primal variables significantly change, we avoid to increase the penalty parameter. Simultaneously, we aim to keep a balance between the Lagrangian function and the quadratic penalty. If the NA solution seems to stabilize, but we observe larger NA constraints violations, we slightly increase the penalty parameter if the new violations are significantly more important, otherwise, we keep the penalty parameter fixed. We do not expect this case to often happen, but if it occurs, we try to stabilize the process. Finally, if none of the previous situations is appearing, we deduce that we have achieved convergence in the primal space, so we force convergence in the dual space by increasing the penalty parameter value. The procedure is summarized below.

Step 0. Set γ_1, γ_2, $\gamma_3 \in (0,1)$, α, ν, $\sigma \in (0,1)$, $1 < \theta < \beta < \eta$.

Step 1. If the change in $\{\hat{x}^k\}$ is large enough, i.e. if

$$\frac{E\left[\left\|\hat{x}^{(s),k+1} - \hat{x}^{(s),k}\right\|_2^2\right]}{\max\left\{E\left[\left\|\hat{x}^{(s),k+1}\right\|_2^2\right], E\left[\left\|\hat{x}^{(s),k}\right\|_2^2\right]\right\}} \geq \gamma_1,$$

or if the quadratic penalty term is important compared to the Lagrangian function, i.e. if

$$\rho^k E\left[\left\|x^{(s),k+1} - \hat{x}^{(s),k+1}\right\|_2^2\right] \geq \sigma E\left[\left|f\left(x^{(s),k+1}, \xi(s)\right) + \lambda^{(s),k\prime}\left(x^{(s),k+1} - \hat{x}^{(s),k}\right)\right|\right],$$

then

Strategy	Penalty parameter update
Fixed value (Rockafellar and Wets, 1991)	$\rho = a > 0$
Dynamic increase with possibly a sudden reduction (Mulvey and Vladimirou, 1991a)	$\rho^{k+1} = (\tau_\rho \rho^k)^\mu$, $\tau_\rho \geq 1$, $0 < \mu \leq 1$, $\rho^{k+1} = \rho_\varepsilon$, $\rho_\varepsilon > 0$, small
Fixed to two predefined positive values with reduction (Chun and Robinson, 1995)	$\rho = \alpha > 0$, if the convergence in dual space is not achieved, otherwise $\rho = \beta > 0$, with $\beta < \alpha$
Bundle-stage wise value (Somervell, 1998)	$\rho_t^{(s')}$, for $s' \in \mathcal{S}_t^{(s)}$ and $t = 1, \cdots, T$
Cost-proportional value (Watson et al., 2007)	$\rho(i) = \frac{c(i)}{\max_s\{x^{(s),0} - \min_s x^{(s),0} + 1\}}$ or $\rho(i) = \frac{c(i)}{\max\{\sum_s p_s \lvert x^{(s),0} - \hat{x}^0 \rvert, 1\}}$
Decreasing values (Reis et al., 2005)	$\rho^k = \frac{1}{\alpha + \beta k}$, $\alpha, \beta \in (0,1)$
Increasing values (Crainic et al., 2011)	$\rho^{k+1} = \tau_\rho \rho^k$, $\tau_\rho \geq 1$
Dynamic update, convergence- and cost-proportional node-wise values (Hvattum and Løkketangen, 2009)	$\rho_v = c_v \delta(v) \rho^{k+1}$ for node $v \in \mathcal{V}$, where $\rho^{k+1} = \tau_{inc} \rho^k$ with $\tau_{inc} > 1$, if the progress toward dual convergence is negative, $\rho^{k+1} = \tau_{dec} \rho^k$ with $1 > \tau_{dec} > 0$, if the progress toward primal convergence is negative, for some discount factor $\delta(v)$
Dynamic update, convergence-proportional values by predefined multipliers (Gul, 2010)	$\rho^{k+1} = \delta \rho^k$, if the progress toward dual convergence is negative, $\rho^{k+1} = \frac{1}{\delta} \rho^k$, if the progress toward primal convergence is negative, with $\delta > 1$
Increasing values, with NA-proportional multiplier (Gonçalves et al., 2012)	$\rho^{k+1} = \rho^k \left\{ \alpha E \left[\sum_t \left(\frac{\lVert x_t^{(s),k} - \hat{x}^k \rVert^2}{x_{t,\max}^{(s),k} - x_{t,\min}^{(s),k} + 1} \right) \right] + 1 \right\}$, $\alpha > 1$,
Dynamic update, bounded values based on optimality and NA indicators (Zéphyr et al., 2014)	$\rho^{k+1} = \max\{0.01, \min\{100, q^{k+1} \rho^k\}\}$, where $q^{k+1} = \left(\max\{g^{k+1}, h^{k+1}\}\right)^{\frac{1}{1+0.01(k-1)}}$, where g^{k+1} and h^{k+1} are optimality and NA indicators, respectively

Table 1: Penalty parameter updates

a) if the change in $\{\hat{x}^k\}$ is dominating the change in $\{\lambda^k\}$ such that

$$\frac{E\left[\left\|\hat{x}^{(s),k+1}-\hat{x}^{(s),k}\right\|_2^2\right]-E\left[\left\|x^{(s),k+1}-\hat{x}^{(s),k+1}\right\|_2^2\right]}{\max\left\{1,E\left[\left\|x^{(s),k+1}-\hat{x}^{(s),k+1}\right\|_2^2\right]\right\}}>\gamma_2,$$

then decrease the penalty parameter by setting $\rho^{k+1}=\alpha\rho^k$,

b) else if the change in $\{\lambda^k\}$ is dominating the change in $\{\hat{x}^k\}$ such that

$$\frac{E\left[\left\|x^{(s),k+1}-\hat{x}^{(s),k+1}\right\|_2^2\right]-E\left[\left\|\hat{x}^{(s),k+1}-\hat{x}^{(s),k}\right\|_2^2\right]}{\max\left\{1,E\left[\left\|\hat{x}^{(s),k+1}-\hat{x}^{(s),k}\right\|_2^2\right]\right\}}>\gamma_3,$$

then increase the penalty parameter by setting $\rho^{k+1}=\theta\rho^k$,

c) otherwise, keep the penalty parameter fixed by setting $\rho^{k+1}=\rho^k$,

otherwise go to Step 2.

Step 2. If there is no significant change in $\{\hat{x}^k\}$ but the change in the dual sequence $\{\lambda^k\}$ is getting larger, i.e. the nonanticipative violation increase over the iterations:

$$E\left[\left\|x^{(s),k+1}-\hat{x}^{(s),k+1}\right\|_2^2\right]>E\left[\left\|x^{(s),k}-\hat{x}^{(s),k}\right\|_2^2\right],$$

then

a) if the increase is large, i.e.

$$\frac{E\left[\left\|x^{(s),k+1}-\hat{x}^{(s),k+1}\right\|_2^2\right]-E\left[\left\|x^{(s),k}-\hat{x}^{(s),k}\right\|_2^2\right]}{E\left[\left\|x^{(s),k}-\hat{x}^{(s),k}\right\|_2^2\right]}>\nu,$$

then increase the penalty parameter by setting $\rho^{k+1}=\beta\rho^k$,

b) else keep the penalty parameter fixed by setting $\rho^{k+1}=\rho^k$.

Otherwise, go to Step 3.

Step 3. Increase the penalty parameter by setting $\rho^{k+1}=\eta\rho^k$.

A last point to discuss is the choice of the initial penalty parameter ρ^0. As the Lagrange multipliers vector λ^0 is set to zero, the initial augmented Lagrangian is

$$E\left[f\left(x^{(s),0},\xi^{(s)}\right)\right]+\frac{\rho^0}{2}E\left[\left\|x^{(s),0}-\hat{x}^{(s),0}\right\|_2^2\right].$$

This suggests to balance the two terms, leading to

$$\rho^0=\frac{\max\left\{1,2\zeta\left|E[c'^s x^0]\right|\right\}}{\max\left\{1,E\left[\|x^0-\hat{x}^0\|_2^2\right]\right\}}, \tag{9}$$

with $\zeta>0$.

4 Computational study

In order to numerically validate our approach, we compare it with some of the propositions identified in the literature. We first consider fixed values, dynamic update with/without dropping from Mulvey and Vladimirou (1991b), a simplified version of convergence-proportional update by Hvattum and Løkketangen (2009), excluding problem-dependent aspects, and optimality- and NA-proportional update by Zéphyr et al. (2014). We do not examine cost-proportional penalties as some of the test problems present many variables associated to null costs.

We also limit ourselves to linear problems of the form

$$
\begin{aligned}
\min_x \ & \sum_{s\in\mathcal{S}} p_s \left(\sum_{t=1}^{T} {c_t^{(s)}}' x_t^{(s)} \right) \\
\text{s.t. } & H_1^{(s)} x_1^{(s)} = b_1^{(s)}, \\
& \sum_{j<t} G_j^{(s)} x_j^{(s)} + H_t^{(s)} x_t^{(s)} = b_t^{(s)},\ t = 2,\dots,T,\ s \in \mathcal{S}, \\
& x_t^{(s)} \geq 0,\ t = 1,\dots,T,\ s \in \mathcal{S}, \\
& x_t^{(s)} \text{ is nonanticipative},\ t = 1,\dots,T,
\end{aligned}
$$

The problems were collected from SMI (`http://www.coin-or.org/projects/Smi.xml`) and the problems collection SPLIB proposed by V. Zverovich (`https://github.com/vitaut/splib`). We also created a modified version of KW3R by adding stochastic in the constraints matrix and we denote this version by KW3Rb Their characteristics are summarized in Table 2. We used SMI to parse the SMPS files describing them and we solved their deterministic equivalent formulations using CPLEX 12.5 in order to have reference optimal values. We implemented the PHA in C++ and the scenario subproblems are solved with CPLEX. The numerical tests were performed on a cluster of computers with 2.40 GHZ Intel(R) Xeon(R) E5620 CPU (quad-core) with 2 threads each and 98 GB of RAM.

Problem	#stages	#scenarios	Optimal value
KW3R	3	9	2613
KW3Rb	3	9	3204
app0110R	3	9	41.96
SGPF3Y3	3	25	-2967.91
Asset Mgt	4	8	-164.74
SGPF5Y4	4	125	-4031.3
wat10I16	10	16	-2158.75
wat10C32	10	32	-2611.92

Table 2: Problems

We compare the use of a fixed penalty parameter (referred as Fixed) with several of the identified strategies, namely the dynamic update proposed by Mulvey and Vladimirou (1991a) with reduction (referred as M&VR) and without reduction (referred as M&V), the controlled dynamic update designed by Hvattum and Løkketangen (2009) (referred as H&L), the learning update developed by Zéphyr et al. (2014) (referred as Z&L&L) and the adaptive update (referred as Adaptive). The initial penalty parameter ρ^0 is set as in (9), or by using the recommended values in the original papers, and we compute $x_0^{(s)}$, $s = 1, \ldots, S$, by solving the scenario subproblems without the NA constraints. We tried three different settings corresponding to different values of ζ, as the smaller the value of ζ, the less we enforce the initial NA solution. We test the method with $\zeta = 0.01$ (small), $\zeta = 0.10$ (medium), and $\zeta = 0.50$ (large). Moreover, the parameters associated to each approach are chosen as described below. For the M&V update, we set $\rho^{k+1} = (\tau_\rho \rho^k)^\mu$, and consider two settings recommended by Mulvey and Vladimirou (1991a) referring to them by a and b, respectively. In setting a, we define $\tau_\rho = 1.1$, $\mu = 0.80$, $\rho^0 = 0.02$, and $\rho_{min} = 0.05$, while in setting b, we have $\tau_\rho = 1.25$, $\mu = 0.95$, $\rho^0 = 0.05$, and $\rho_{min} = 0.05$. To drop the penalty parameter as they suggested, we check if $\sum_{s \in \mathcal{S}} p_s \|x^{s,k+1} - \hat{x}^{s,k+1}\|_2^2 \leq \varepsilon_d$, with $\varepsilon_d = 10^{-5}$, is satisfied. For H&L, we set $\rho^{k+1} = \delta \rho^k$, if the progress toward dual convergence is negative, and $\rho^{k+1} = \frac{1}{\delta}\rho^k$, if the progress toward primal convergence is negative, with $\delta = 1.8$ and $\rho^0 = 0.3$, We follow Zéphyr et al. (2014) recommendations for the implementation of their strategy. Finally, the adaptive strategy parameters are $\gamma_1 = 10^{-5}$, $\gamma_2 = 0.01$, $\gamma_3 = 0.25$, $\sigma = 10^{-5}$, $\alpha = 0.95$, $\theta = 1.09$, $\nu = 0.1$, $\beta = 1.1$, and $\eta = 1.25$.

The stopping criteria is a normalized version of 7. We stop if

$$\sqrt{\frac{\sum_{s \in \mathcal{S}} p_s \|x^{s,k+1} - \hat{x}^{s,k}\|_2^2}{\max\left\{1, \sum_{s \in \mathcal{S}} p_s \|\hat{x}^{s,k}\|_2^2\right\}}} \leq \varepsilon, \tag{10}$$

with $\varepsilon = 10^{-5}$, and set the iteration limit to 500 and time limit to 36000 seconds (or 10 hours). We declare convergence if the difference with the optimal value is less than 0.1% within the time and iteration limits. A summary of our main results is given in Tables 3–5, where the methods are compared when the same initial penalty parameter is used, while we present detailed results in Appendix B, problem by problem. When we reach the time or iterations limit, we indicate "Limit" if the final solution is within a 0.1% optimality gap, otherwise we mention "Wrong" if the final solution has an optimality gap greater than 0.1%, and "Infeasible" if the NA constraints are not satisfied. If (10) is satisfied within the defined limits, we report the number of PHA iterations and the optimality gap in brackets in case we have converged to a suboptimal solution, i.e. the optimality gap is greater than 0.1%.

We also graphically compare the methods by means of performance profiles Dolan and Moré (2002). In the figures, P designates the percentage of problems which are solved within a factor τ of the best solver, using the number of iterations as our performance metric. We first compare the methods for a given ζ, and then compare the choice of ζ for fixed and adaptive strategies. Figures 1, 2, and 3 exhibit that existing strategies have difficulties to converge towards

Method	KW3R	KW3Rb	app0110R	SGPF3Y3
Fixed	30	Infs.	Infs.	9
M&VRa	43	Limit	Wrong	12(1.1%)
M&VRb	48	Limit	Limit	13(0.3%)
M&Va	27	346	105	12(1.1%)
M&Vb	23	244	92	13(0.3%)
H&L	115	Infs.	Infs.	9
Z&L&L	27	261	109	18(1.9%)
Adaptive	25	139	108	10
Method	Asset-Mgt	SGPF5Y4	wat10I16	wat10C32
Fixed	6	Limit	342	Limit
M&VRa	8	20(7.9%)	43	Infs.
M&VRb	8	18(4.4%)	275	44
M&Va	Wrong	20(7.9%)	Wrong	Wrong
M&Vb	55(0.11%)	18(4.4%)	44(2.6%)	46(2%)
H&L	8	Infs.	Infs.	Infs.
Z&L&L	Wrong	32(11.2%)	Wrong	Wrong
Adaptive	6	46	48	73

Table 3: Number of PHA iterations with $\zeta = 0.01$

Method	KW3R	KW3Rb	app0110R	SGPF3Y3
Fixed	28	373	215	95
M&VRa	28	Limit	Limit	11(1.1%)
M&VRb	36	Limit	Limit	11(0.4%)
M&Va	28	299	102	11(1.08%)
M&Vb	25	234	86	11(0.4%)
H&L	83	Infs.	Infs.	49
Z&L&L	59	265	118	17(1.4%)
Adaptive	24	155	83	62
Method	Asset-Mgt	SGPF5Y4	wat10I16	wat10C32
Fixed	19	109	Limit	144
M&VRa	23	20(7.9%)	Limit	185
M&VRb	21	17(5%)	Limit	203
M&Va	Wrong	20(7.9%)	Wrong	Wrong
M&Vb	60(1.2%)	17(5%)	46(4.5%)	45(3.3%)
H&L	Wrong	Limit	286	95
Z&L&L	179	31(9.7%)	Wrong	Wrong
Adaptive	16	32	41	62

Table 4: Number of PHA iterations with $\zeta = 0.1$

Method	KW3R	KW3Rb	app0110R	SGPF3Y3
Fixed	44	324	109	467
M&VRa	29	234	126	11(1.2%)
M&VRb	60	Limit	Limit	11(0.8%)
M&Va	35	290	99	11(1.2%)
M&Vb	49	267	82	11(0.8%)
H&L	105	Limit	Infs.	45(0.4%)
Z&L&L	82	421	137	17(1.34%)
Adaptive	39	189	67	88
Method	Asset-Mgt	SGPF5Y4	wat10I16	wat10C32
Fixed	90	38	Limit	Limit
M&VRa	92	19(7.7%)	Limit	Limit
M&VRb	92	16(6.6%)	Limit	Limit
M&Va	Wrong	19(7.7%)	Wrong	Wrong
M&Vb	58(1.5%)	16(6.6%)	50(8.1%)	51(6.8%)
H&L	39(0.39%)	Limit	38(0.6%)	359(0.83%)
Z&L&L	Wrong	31(9.2%)	Wrong	Wrong
Adaptive	38	24	56	95

Table 5: Number of PHA iterations with $\zeta = 0.5$

the optimal solution, a fixed parameter strategy being surprisingly more efficient when the initial penalty parameter is not chosen small enough. For a small initial penalty parameter ($\zeta = 0.01$), the approach designed by Mulvey and Vladimirou (1991a) has a slight advantage over the other methods, that disappears when increasing the initial penalty parameter value. Overall, none of the existing approaches has been found to significantly outperforms the other ones. The adaptive strategy proved to be quite efficient, as it is the fastest approach for most of the problems, whatever the choice of ρ^0, and it is the only one to always deliver the correct solution. It is therefore interesting to explore the influence of the initial penalty parameter. We draw the performance profile of the fixed and adaptive strategy in Figure 4. We again see that the adaptive strategy is more efficient than keeping the penalty parameter fixed, and there is a slight advantage to start with a small initial value for the penalty parameter. The fixed approach is more sensitive to the choice of the initial penalty, a medium penalty being the best compromise in our experiments. The problems set being limited, we have to remain careful before we can derive strong conclusions, but the numerical results are nevertheless encouraging.

5 Conclusion

The PHA remains a popular scenarios decomposition method, but practical issues are still often observed. In particular, the choice of the penalty parameter

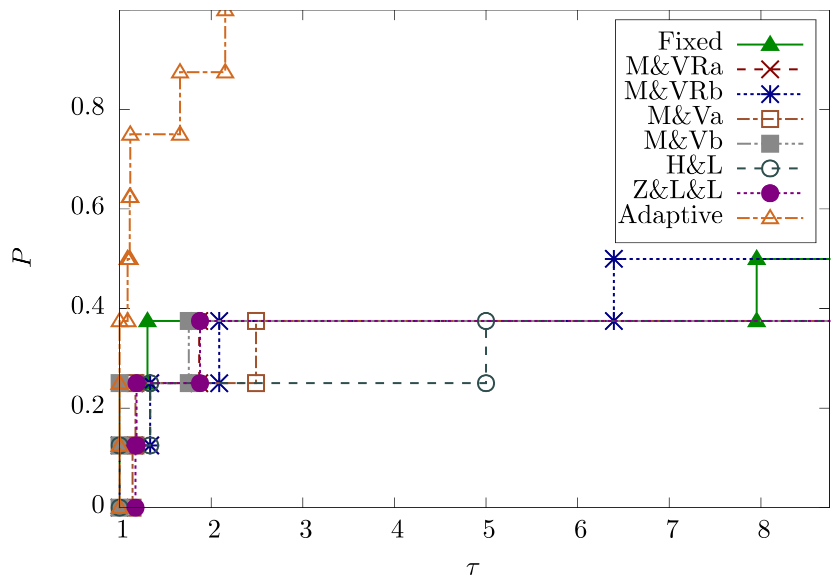

Figure 1: Performance profile with $\zeta = 0.01$

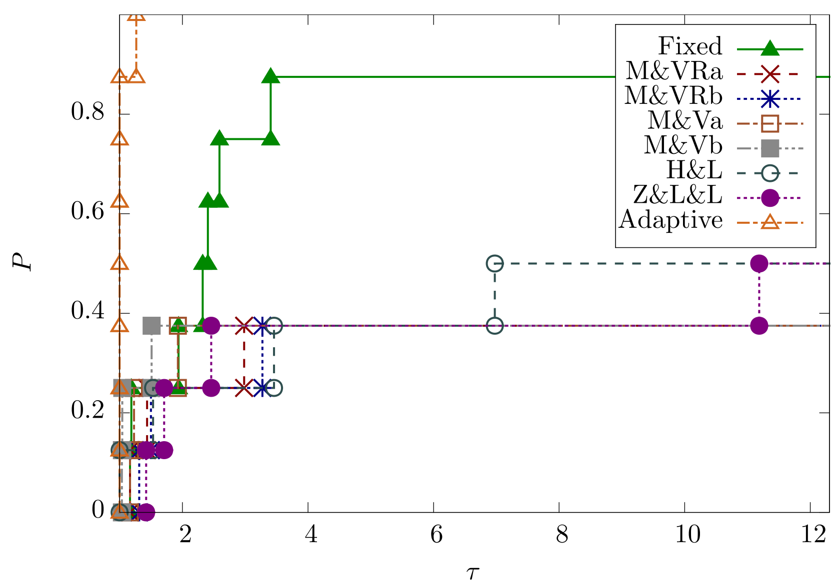

Figure 2: Performance profile with $\zeta = 0.1$

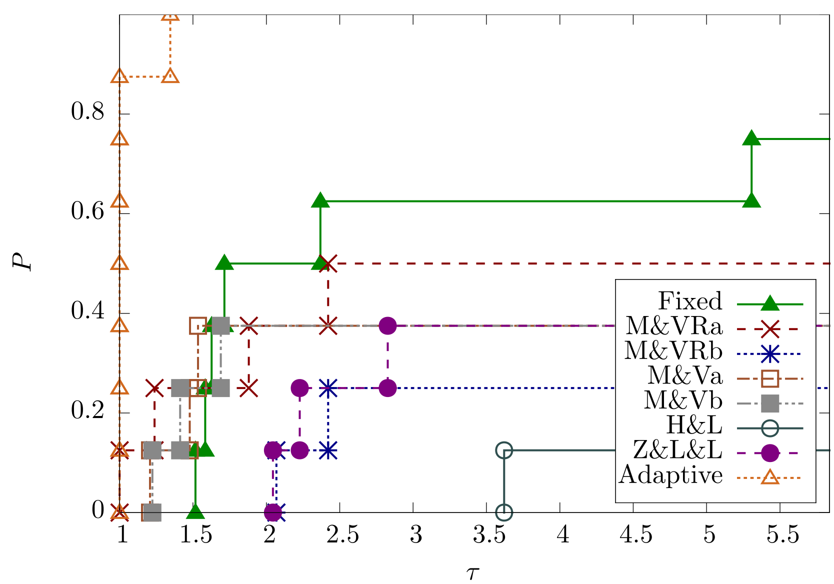

Figure 3: Performance profile with $\zeta = 0.5$

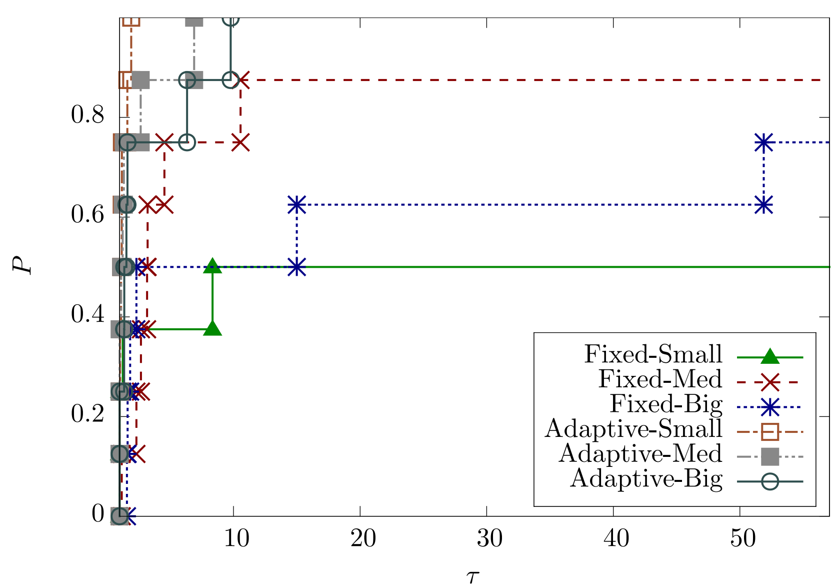

Figure 4: Performance profile with fixed and adaptive strategies

value significantly influences the speed of convergence. A low value can produce a very slow convergence, while a large value will allow faster convergence, but the returned solution can be suboptimal. In order to circumvent these problems, many researchers have proposed heuristics to update the penalty parameters, but the study of their efficiency and robustness is often limited, and valid for the application under consideration only. In this paper, we have reviewed several approaches proposed in the literature, and observed that even for simple problems, we can face convergence issues. We then propose a dynamic update that allows to increase or reduce the penalty parameter value, aiming to enforce the NA constraints only when they are correctly identified. While the proposed approach is still heuristic, we have observed a large improvement over the other strategies for the test problems, the method being fast and robust. More research would however be needed to offer better theoretical guarantees, and to assess its performance on larger classes of problems, especially with respect to proximal-penalization techniques.

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A Proof of equality (8)

In this appendix, we develop the proof of the equality (8)

$$\sum_{s\in\mathcal{S}} p_s \|x^{s,k+1} - \hat{x}^{s,k}\|_2^2 = \sum_{s\in\mathcal{S}} p_s \|x^{s,k+1} - \hat{x}^{s,k+1}\|_2^2 + \sum_{s\in\mathcal{S}} p_s \|\hat{x}^{s,k+1} - \hat{x}^{s,k}\|_2^2.$$

Consider the space $\mathcal{D} = \{X : \mathcal{S} \to \mathcal{R}^n \mid X(s) = (X_1(s), \ldots, X_T(s)),\ X_t : \mathcal{S} \to \mathcal{R}^{n_t}\}$, with $\sum_{t=1}^{T} n_t = n$, and $\mathcal{N} = \{X \in \mathcal{D} \mid X_t \text{ is nonanticipative}\}$. We equip $\mathcal{D}$ with the inner product $\langle X, Y\rangle_{\mathcal{D}} = E[X(s) \cdot Y(s)]$, where $\cdot$ is the componentwise product, and the associated norm $\|X\|_{\mathcal{D}} = \sqrt{E[\|X(s)\|_2^2]}$. Let J be the orthogonal projection operator from $\mathcal{D}$ on the $\mathcal{N}$, and $K = I - J$ the orthogonal projection operator on the subspace of $\mathcal{D}$ complementary to $\mathcal{N}$, denoted by $\mathcal{M} = \mathcal{N}^{\perp} = \{X \in \mathcal{D} \mid JX = 0\}$. Then, we have $\mathcal{N} = \{X \in \mathcal{D} \mid KX = 0\}$, To prove the equality (8), we work on each term separately. We first have

$$\sum_{s\in\mathcal{S}} p_s \|x^{s,k+1} - \hat{x}^{s,k}\|_2^2 = \|x^{k+1} - \hat{x}^k\|_{\mathcal{D}}^2 = \|x^{k+1}\|_{\mathcal{D}}^2 - 2\langle x^{k+1}, \hat{x}^k\rangle + \|\hat{x}^k\|_{\mathcal{D}}^2. \quad (11)$$

Similarly,

$$\sum_{s\in\mathcal{S}} p_s \|\hat{x}^{s,k+1} - \hat{x}^{s,k}\|_2^2 = \|\hat{x}^{k+1} - \hat{x}^k\|_{\mathcal{D}}^2 = \|\hat{x}^{k+1}\|_{\mathcal{D}}^2 - 2\langle \hat{x}^{k+1}, \hat{x}^k\rangle + \|\hat{x}^k\|_{\mathcal{D}}^2, \quad (12)$$

Moreover, we have

$$\begin{aligned}\sum_{s\in\mathcal{S}} p_s \|x^{s,k+1} - \hat{x}^{s,k+1}\|_2^2 &= \sum_{t} \sum_{\mathcal{B}_t^s} \sum_{s'\in\mathcal{B}_t^s} p^{s'} \|x_t^{s',k+1} - \hat{x}_t^{s',k+1}\|_2^2 \\ &= \sum_{t} \sum_{\mathcal{B}_t^s} \sum_{s'\in\mathcal{B}_t^s} p^{s'} \left(\|x_t^{s',k+1}\|_2^2 - \|\hat{x}_t^{s',k+1}\|_2^2\right) \quad (13) \\ &= \|x^{k+1}\|_{\mathcal{D}}^2 - \|\hat{x}^{k+1}\|_{\mathcal{D}}^2,\end{aligned}$$

Combining (12) and (13), we have

$$\sum_{s\in\mathcal{S}} p_s \|x^{s,k+1} - \hat{x}^{s,k+1}\|_2^2 + \sum_{s\in\mathcal{S}} p_s \|\hat{x}^{s,k+1} - \hat{x}^{s,k}\|_2^2 = \|x^{k+1}\|_{\mathcal{D}}^2 - 2\langle \hat{x}^{k+1}, \hat{x}^k\rangle + \|\hat{x}^k\|_{\mathcal{D}}^2,$$

that corresponds to (11) as

$$\langle x^{k+1} - \hat{x}^{k+1}, \hat{x}^k\rangle = \langle Kx^{k+1}, Jx^k\rangle = \langle x^{k+1}, KJx^k\rangle = 0,$$

since K is an orthogonal projection operator. This concludes the proof.

B Detailed numerical results

We provide in Tables 6–13 the detailed results of our numerical experimentations over the eight test problems. For each problem, we compare thirty penalty parameter update strategies, and provide the final objective value along with the gap to optimal value in brackets. We also report the number of iterations and the computation time. In case we reach the iteration or time limit before declaring convergence, we mention "Limit" in the corresponding cell.

Update	Objective value	Iterations	Time (s)
Fixed-Sml	2613 (0)	30	3.53
Fixed-Med	2613 (0)	28	3.14
Fixed-Big	2613 (0)	44	4.77
M&VRa	2613 (0)	Limit	122.22
M&VRa-Small	2613 (0)	43	5.11
M&VRa-Medium	2613 (0)	28	3.18
M&VRa-Large	2613 (0)	29	3.32
M&VRb	2613 (0)	64	8.1
M&VRb-Small	2613 (0)	48	5.7
M&VRb-Medium	2613 (0)	36	4.07
M&VRb-Large	2613 (0)	60	6.94
M&Va	2613 (0)	29	3.56
M&Va-Small	2613 (0)	27	3.1
M&Va-Medium	2613 (0)	28	3.24
M&Va-Large	2613 (0)	35	3.97
M&Vb	2613 (0)	25	2.88
M&Vb-Small	2613 (0)	23	2.6
M&Vb-Medium	2613 (0)	25	2.97
M&Vb-Large	2613 (0)	49	5.77
H&L	2613 (0)	118	17.6
H&L-Small	2613 (0)	115	16.53
H&L-Medium	2613 (0)	83	11.85
H&L-Large	2613 (0)	105	15.3
Z&L&L	2613 (0)	28	2.53
Z&L&L-Small	2613 (0)	27	3.15
Z&L&L-Medium	2613 (0)	59	7.14
Z&L&L-Large	2613 (0)	82	10.99
Adaptive-Small	2613 (0)	25	2.87
Adaptive-Medium	2613 (0)	24	2.65
Adaptive-Large	2613 (0)	39	4.36

Table 6: KW3R problem

Update	Objective value	Iterations	Time (s)
Fixed-Sml	3197.34 (-0.21)	Limit	123.92
Fixed-Med	3204 (0)	373	74.8
Fixed-Big	3204 (0)	324	60.67
M&VRa	3204.25 (0.008)	Limit	124.33
M&VRa-Small	3204.93 (0.03)	Limit	124.31
M&VRa-Medium	3204.05 (0.001)	Limit	123.91
M&VRa-Large	3204 (0)	234	39.58
M&VRb	3204 (0)	Limit	122.23
M&VRb-Small	3204.16 (0.005)	Limit	121.3
M&VRb-Medium	3204 (0)	Limit	120.62
M&VRb-Large	3204.26 (0.008)	Limit	118.19
M&Va	3204 (0)	348	69.29
M&Va-Small	3204 (0)	346	69.81
M&Va-Medium	3204 (0)	299	56.12
M&Va-Large	3204 (0)	290	52.48
M&Vb	3204 (0)	244	41.04
M&Vb-Small	3204 (0)	244	40.6
M&Vb-Medium	3204 (0)	234	39
M&Vb-Large	3204 (0)	267	46.39
H&L	3193 (-0.3)	Limit	123.75
H&L-Small	3151.08 (-2)	Limit	125.7
H&L-Medium	3154.16 (-1)	Limit	124.58
H&L-Large	3204 (0)	Limit	122.29
Z&L&L	3204 (0)	266	41.92
Z&L&L-Small	3204 (0)	261	44.38
Z&L&L-Medium	3204 (0)	265	44.85
Z&L&L-Large	3204 (0)	421	91.72
Adaptive-Small	3204 (0)	139	20.98
Adaptive-Medium	3204 (0)	155	23.55
Adaptive-Large	3204 (0)	189	29.91

Table 7: Modified KW3R problem

Update	Objective value	Iterations	Time (s)
Fixed-Sml	39.4 (-6)	Limit	424.77
Fixed-Med	41.96 (0)	215	91.35
Fixed-Big	41.96 (0)	109	30.75
M&VRa	41.96 (0)	Limit	421.14
M&VRa-Small	42.01 (0.12)	Limit	423.34
M&VRa-Medium	41.96 (0)	Limit	422.51
M&VRa-Large	41.96 (0)	126	38.72
M&VRb	41.963 (0.007)	Limit	407.18
M&VRb-Small	41.96 (0)	Limit	430.5
M&VRb-Medium	42 (0.09)	Limit	422.19
M&VRb-Large	41.96 (0)	Limit	422.97
M&Va	41.96 (0)	105	29.34
M&Va-Small	41.96 (0)	105	29.13
M&Va-Medium	41.96 (0)	102	28.42
M&Va-Large	41.96 (0)	99	25.79
M&Vb	41.96 (0)	88	21.89
M&Vb-Small	41.96 (0)	92	23.03
M&Vb-Medium	41.96 (0)	86	21.41
M&Vb-Large	41.96 (0)	82	20.1
H&L	34.83 (-17)	Limit	423.63
H&L-Small	19.25 (-54)	Limit	424.79
H&L-Medium	31.47 (-25)	Limit	421.84
H&L-Large	37.81 (-9.9)	Limit	422.05
Z&L&L	41.96 (0)	125	35.75
Z&L&L-Small	41.96 (0)	109	29.15
Z&L&L-Medium	41.96 (0)	118	34.58
Z&L&L-Large	41.96 (0)	137	43.91
Adaptive-Small	41.96 (0)	108	29.04
Adaptive-Medium	41.96 (0)	83	20.53
Adaptive-Large	41.96 (0)	67	14.48

Table 8: app0110R problem

Update	Objective value	Iterations	Time (s)
Fixed-Sml	-2967.90 (0.0003)	9	6.93
Fixed-Med	-2967.90 (0.0003)	95	385.7
Fixed-Big	-2967.90 (0.0003)	467	8475.53
M&VRa	-2927.3 (1.4)	14	13.64
M&VRa-Small	-2935.81 (1.1)	12	11.09
M&VRa-Medium	-2935.86 (1.1)	11	9.66
M&VRa-Large	-2931.62 (1.22)	11	9.5
M&VRb	-2879.93 (3)	15	15.21
M&VRb-Small	-2958.85 (0.3)	13	12.3
M&VRb-Medium	-2954.63 (0.4)	11	9.93
M&VRb-Large	-2943.71 (0.8)	11	9.61
M&Va	-2927.30 (1.37)	14	13.52
M&Va-Small	-2935.81 (1.08)	12	10.82
M&Va-Medium	-2935.86 (1.08)	11	9.83
M&Va-Large	-2931.62 (1.22)	11	9.82
M&Vb	-2879.93 (3)	15	15.34
M&Vb-Small	-2958.85 (0.3)	13	12.38
M&Vb-Medium	-2954.63 (0.4)	11	9.64
M&Vb-Large	-2943.71 (0.8)	11	9.55
H&L	-2895.27 (2.4)	19	21.98
H&L-Small	-2967.90 (0.0003)	9	6.81
H&L-Medium	-2967.83 (0.003)	49	113.37
H&L-Large	-2957.07 (0.4)	45	98.7
Z&L&L	-2917.65 (1.7)	17	17.66
Z&L&L-Small	-2912.12 (1.9)	18	20.34
Z&L&L-Medium	-2926.65 (1.4)	17	18.79
Z&L&L-Large	-2928.06 (1.34)	17	18.95
Adaptive-Small	-2967.90 (0.0003)	10	8.07
Adaptive-Medium	-2967.84 (0.002)	62	176.83
Adaptive-Large	-2967.83 (0.003)	88	324.23

Table 9: SGPF3Y3 problem

Update	Objective value	Iterations	Time (s)
Fixed-Sml	-164.74 (0)	6	0.52
Fixed-Med	-164.74 (0)	19	1.63
Fixed-Big	-164.74 (0)	90	9.93
M&VRa	-164.74 (0)	173	25.05
M&VRa-Small	-164.74 (0)	8	0.7
M&VRa-Medium	-164.74 (0)	23	2.76
M&VRa-Large	-164.74 (0)	92	11.35
M&VRb	-164.74 (0)	175	20.86
M&VRb-Small	-164.74 (0)	8	0.68
M&VRb-Medium	-164.74 (0)	21	1.86
M&VRb-Large	-164.74 (0)	92	10.12
M&Va	-162.57 (1.3)	Limit	92.28
M&Va-Small	-163.18 (1.0)	Limit	94.27
M&Va-Medium	-162.84 (1.2)	Limit	94.61
M&Va-Large	-162.55 (1.3)	Limit	93.6
M&Vb	-162.12 (1.6)	57	6.21
M&Vb-Small	-164.55 (0.1)	55	5.42
M&Vb-Medium	-162.72 (1.2)	60	5.67
M&Vb-Large	-162.19 (1.5)	58	5.39
H&L	-162.09 (1.6)	29	2.25
H&L-Small	-164.74 (0)	8	0.68
H&L-Medium	-164.16 (0.4)	Limit	90.85
H&L-Large	-164.17	(0.3)	39
Z&L&L	-164.73 (0.006)	Limit	95.71
Z&L&L-Small	-164.44 (0.2)	Limit	89.82
Z&L&L-Medium	-164.74 (0)	179	24.1
Z&L&L-Large	-163.09 (1)	Limit	84.82
Adaptive-Small	-164.74 (0)	6	0.51
Adaptive-Medium	-164.74 (0)	16	1.36
Adaptive-Large	-164.74 (0)	38	3.43

Table 10: Asset problem

Update	Objective value	Iterations	Time (s)
Fixed-Sml	-4031.30 (0)	134	Limit
Fixed-Med	-4031.30 (0)	109	24542.4
Fixed-Big	-4031.30 (0)	38	3132.49
M&VRa	-3573.08 (11.4)	26	1553.29
M&VRa-Small	-3711.83 (7.9)	20	897
M&VRa-Medium	-3711.38 (7.9)	20	898.47
M&VRa-Large	-3722.37 (7.7)	19	818.35
M&VRb	-3372.32 (16.3)	25	1447.19
M&VRb-Small	-3853.78 (4.4)	18	744.29
M&VRb-Medium	-3829.95 (5)	17	661.26
M&VRb-Large	-3764.81 (6.6)	16	605.09
M&Va	-3573.08 (11.4)	26	1588.28
M&Va-Small	-3711.83 (7.9)	20	897.03
M&Va-Medium	-3711.38 (7.9)	20	895.65
M&Va-Large	-3722.37 (7.7)	19	821.27
M&Vb	-3372.32 (16.3)	25	1440.74
M&Vb-Small	-3853.78 (4.4)	18	735.41
M&Vb-Medium	-3829.95 (5)	17	668.51
M&Vb-Large	-3764.81 (6.6)	16	603.77
H&L	-3575.56 (11.3)	111	25407.78
H&L-Small	-4138.81 (-2.7)	134	Limit
H&L-Medium	-4031.30 (0)	134	Limit
H&L-Large	-4029.73 (0.04)	134	Limit
Z&L&L	-3638.27 (9.8)	32	2318.17
Z&L&L-Small	-3578 (11.2)	32	2310.63
Z&L&L-Medium	-3640.91 (9.7)	31	2191.27
Z&L&L-Large	-3660.43 (9.2)	31	2169.85
Adaptive-Small	-4031.30 (0)	46	4551.63
Adaptive-Medium	-4031.30 (0)	32	2317.18
Adaptive-Large	-4031.30 (0)	24	1318.47

Table 11: SGPF5Y4 problem

Update	Objective value	Iterations	Time (s)
Fixed-Sml	-2158.75 (0)	342	3800.11
Fixed-Med	-2158.74 (0.0005)	Limit	8003.73
Fixed-Big	-2158.74 (0.0005)	Limit	8061.06
M&VRa	-2134.91 (1.1)	Limit	8108.3
M&VRa-Small	-2158.74 (0.0005)	43	78.45
M&VRa-Medium	-2158.74 (0.0005)	Limit	7921.8
M&VRa-Large	-2158.74 (0.0005)	Limit	7940.46
M&VRb	-2148.12 (0.5)	Limit	7789.87
M&VRb-Small	-2158.75 (0)	275	2476.09
M&VRb-Medium	-2158.74 (0.0005)	Limit	8026.43
M&VRb-Large	-2158.74 (0.0005)	Limit	8070.88
M&Va	-1931.3 (10.5)	Limit	8067.18
M&Va-Small	-2013.88 (6.7)	Limit	8078.32
M&Va-Medium	-2001.70 (7.3)	Limit	8094.26
M&Va-Large	-1975.13 (8.5)	Limit	8086.62
M&Vb	-1790.44 (17.1)	Limit	7962.29
M&Vb-Small	-2102.07 (2.6)	44	80.36
M&Vb-Medium	-2061.52 (4.5)	46	86.18
M&Vb-Large	-1983.14 (8.1)	50	100.94
H&L	-1728.57 (20)	67	172.87
H&L-Small	-2251.52 (-4.3)	Limit	7983.85
H&L-Medium	-2158.75 (0)	286	2687.91
H&L-Large	-2145.81 (0.6)	38	62.14
Z&L&L	-2100.79 (2.68)	Limit	8052.37
Z&L&L-Small	-2076.36 (3.8)	Limit	8092.45
Z&L&L-Medium	-2088 (3.3)	Limit	7864.39
Z&L&L-Large	-2110.99 (2.2)	Limit	8037.98
Adaptive-Small	-2158.74 (0.0005)	48	94.54
Adaptive-Medium	-2158.71 (0.002)	41	65.97
Adaptive-Large	-2158.71 (0.002)	56	121.74

Table 12: wat10I16 problem

Update	Objective value	Iterations	Time (s)
Fixed-Sml	-2611.92 (0)	Limit	33954.18
Fixed-Med	-2611.92 (0)	144	2930.41
Fixed-Big	-2611.87 (0.002)	Limit	33785.49
M&VRa	-2572.14 (1.5)	Limit	34108.57
M&VRa-Small	-2612.96 (-0.04)	Limit	34040.47
M&VRa-Medium	-2611.01 (0.03)	185	4960.15
M&VRa-Large	-2610.72 (0.04)	Limit	34069.22
M&VRb	-2585.06 (1)	Limit	34029.37
M&VRb-Small	-2611.21 (0.03)	44	289.91
M&VRb-Medium	-2611.84 (0.003)	203	5944.1
M&VRb-Large	-2611.77 (0.006)	Limit	34000.12
M&Va	-2377.84 (9)	Limit	34112.42
M&Va-Small	-2464.55 (5.6)	Limit	33858.15
M&Va-Medium	-2457.85 (5.9)	Limit	33813.32
M&Va-Large	-2428.80 (7)	Limit	33953.17
M&Vb	2260.01 (13.5)	72	777.52
M&Vb-Small	-2558.96 b	46	338.36
M&Vb-Medium	-2525.67 (3.3)	45	309.53
M&Vb-Large	-2434.35 (6.8)	51	406.14
H&L	-2216.59 (15.1)	35	203.9
H&L-Small	-2809.08 (-7.5)	Limit	34057
H&L-Medium	-2611.92 (0)	95	1353.35
H&L-Large	-2590.12 (0.83)	359	18151.14
Z&L&L	-2564.12 (1.83)	Limit	33928.35
Z&L&L-Small	-2530.93 (3.1)	Limit	33985.33
Z&L&L-Medium	-2551.90 (2.3)	Limit	34031.9
Z&L&L-Large	-2571.58 (1.5)	Limit	34090.2
Adaptive-Small	-2611.92 (0)	73	795.2
Adaptive-Medium	-2611.92 (0)	62	587.24
Adaptive-Large	-2611.87 (0.002)	95	1380.99

Table 13: wat10C32 problem