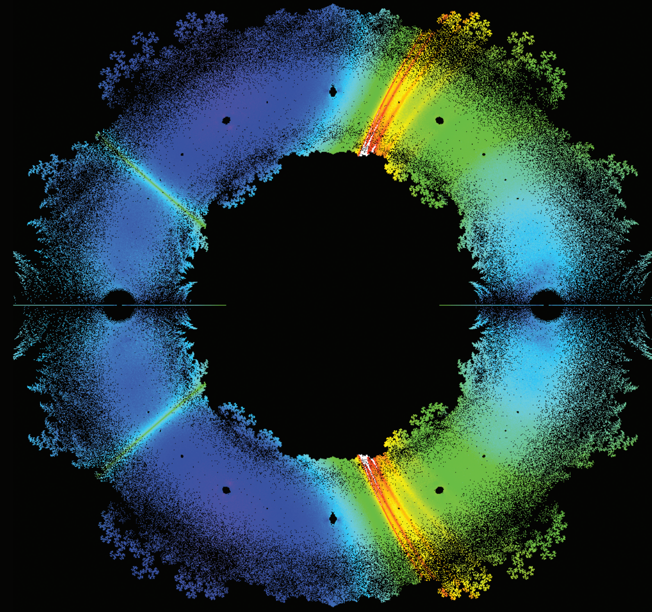
Experimental mathematics leads to **new insights**



PROFESSOR JON BORWEIN

In this discussion, **Professor Jonathan Borwein** of the University of Newcastle, Australia explains how modern computer technology has greatly broadened the ability to discover new mathematical results

To start, what is your research speciality?

I am a bit of a jack of all trades: I have a 1974 Oxford DPhil in optimisation theory and have held professorships as a pure mathematician, an applied mathematician, an operations researcher and a computer scientist. My current research interests span pure (analysis), applied (optimisation), computational (numerical and computational analysis) mathematics, and high performance computing.

I have authored well over a dozen books, including a 2010 publication on convex functions, which was a Choice 2011 Outstanding Academic Book, and over 350 refereed articles. My most recent co-authored book, *Lattice Sums Then and Now*, sits on the boundary between mathematical physics and number theory.

Which area of mathematics did your grant proposal focus on?

Over the past decade I have (co-)authored half a dozen books – both research monographs and textbooks – on experimental mathematics and related mathematical computation. The given grant proposal focused on this area and is entitled Computer Assisted Research Mathematics and its Applications (CARMA), which is also the name of the centre I direct at Newcastle University. You might say CARMA brought me to Australia!

How would you describe the field of experimental mathematics?

Experimental applied mathematics comprises the use of modern computer technology as an active agent of research for the purposes of gaining insight and intuition, discovering new patterns and relationships, testing and conjectures, and confirming analytically derived results, much in the same spirit that laboratory experimentation is employed in the physical sciences. It is closely related to what is known as 'experimental mathematics' in pure mathematics, as has been described elsewhere, including by the late Herb Wilf in the *Princeton Companion to Mathematics*.

Depending on the context, the role of rigorous proof in experimental applied mathematics may be much reduced or may be unchanged from that of its pure sister. There are many complex applied problems where there is little point to proving the validity of a minor component, rather than finding strong evidence for the appropriateness of the general method.

In what way does your study appeal to both applied and pure mathematics? How will it change the culture of mathematics?

Mathematics has traditionally been seen as a deductive science, as opposed to the inductive methods of the physical and biological sciences. My colleagues and I are breaking down this somewhat false barrier, which is being rendered obsolete by the power and versatility of modern computing hardware and mathematical software (Moore's law and on). We do mathematics in a more laboratory-like mode.

The mathematical research community is facing a great challenge to re-evaluate the role of proof in light of the growing power of current computer systems, modern mathematical computing packages, and the growing capacity to data-mine on the Internet. Add to that the enormous complexity of many modern capstone results such as the Poincaré conjecture, Fermat's last theorem, and the classification of finite simple groups. As the need and prospects for inductive mathematics blossom, the requirement to ensure the role of proof is properly founded remains undiminished both in research and teaching.

Similarly, what have you found with analysis into Giuga's Conjecture? Could you explain some of your methodology?

Guiga's is a 63-year-old conjecture about prime numbers that no one has any idea how to prove. By clever computation based on smart mathematics, we have shown that the conjecture can only fail for counting numbers with more than 20,000 digits – as my collaborator David Bailey says, 'computo ego sum'. Such kinds of mathematical computations have a long history of uncovering hardware and software bugs that more standard scientific computational tests do not. A famous example is the so-called Pentium bug.

For that reason, some of my algorithms for Pi were run on every Cray Inc. supercomputer before it left the manufacturer, from around 1986 until the company was sold. In 1986, Bailey had replicable hardware and software errors that Cray was unaware of during a then-record computation of 29 million decimal digits of Pi. The record is now 10 trillion digits!

Number crunch!

A study being conducted at the **University of Newcastle** in Australia is building on previous research that shows how modern computers can discover completely unexpected relationships and formulas

USING CONTEMPORARY COMPUTER technology for active research is commonly referred to as experimental applied mathematics. This method of study is utilised for insight and intuition: to discover correlations and relationships, test conjectures and confirm results derived by analytics (similar to laboratory experiments in physical science). The practice is not far removed from what is known in pure mathematics as experimental mathematics.

Over the years, advances in computing have led the way for progression in the field. This is particularly true for Professor Jonathan Borwein of the University of Newcastle, Australia. Over the past 25 years, Borwein has developed and cultivated several series of results that were not possible via traditional methods. This was done across three research centres – in Vancouver, Canada and now Newcastle – all of which were built by Borwein.

Among the plethora of highly technical findings gathered by Borwein and his group, perhaps the most famous are the computer-generated reverse-engineered results, as found by Borwein's brother, Peter, along with two other researchers, David Bailey and Simon Plouffe. They discovered a formula – the eponymous Bailey-Borwein-Plouffe (BBP) formula – that allows the binary digits of Pi and other constants to be determined without knowing the previous digits, as Borwein elaborates: "Last year, 25 hexadecimal digits (100 bits) of Pi starting at the quadrillionth (10 to the power 15) position were computed by Ed Karrel at Nvidia, the graphics processing unit company. Until 1996 it was thought to be impossible to ever compute things like this. The discoverers of the BBP formula were the only mathematical finalists for the first Edge of Computation Prize, and sat alongside the founders of Google, Netscape, Celera, among others". The prize was ultimately won by the founder of quantum computing, David Deutsch.

EXPERIMENTAL MATH-ODOLOGY

There are several main areas of research in the field of experimental mathematics that are of notable importance at present. For years, the majority of applied mathematicians and researchers have actively integrated computer technology into their studies. Characteristics of such computationally assisted, applied mathematical research includes: computation and simulation for exploration and discovery, symbolic computing, high-precision arithmetic, integer relation algorithms, graphics and visualisation, and connections with non-traditional mathematics.

The key findings of Borwein's study have thus far been nothing short of spectacular – particularly the results on the structure of short, random walks and flights; randomness of the distribution of digits of numbers; and other technical areas, including the creation of fast algorithms used for hard image reconstruction issues. However, it is the evolution of the study's methodological underpinnings that Borwein finds most exciting. He calls it 'experimental mathodology', a name derived from a fortuitous misspelling of 'methodology' (Borwein liked it and decided to keep it). These underpinnings are: gaining insight and intuition, discovering new relationships, visualising math principles, testing (especially falsifying conjectures), exploring a possible result to see if it merits formal proof, suggesting approaches for formal proof, computing and thereby replacing lengthy hand derivations, and confirming analytically derived results.

The study has garnered much academic insight, the lessons of which will develop experimental mathematics in the classroom. Borwein explains: "My postgraduate student Matt Skerritt and I have co-authored two Springer-Verlag books: Modern Mathematical Computation with Maple (2011) and Modern Mathematical Computation with Mathematica (2012) that introduce these tools into the undergraduate classroom. We have even taught the course using modern collaboration tools to a class with 15 students in Newcastle and 15 at James Cook University in Northern Queensland. This experimental approach to mathematics makes the subject lively and accessible to students who are either proficient or challenged in the field.

EVIDENCING GIUGA'S CONJECTURE

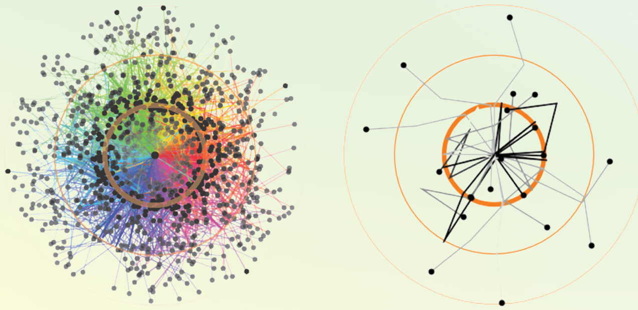
Computing the digits of Pi is just one example of the work carried out by Borwein in experimental mathematics. Another area that has benefited from the use of modern computer technology and exploration is a mathematical conundrum called Giuga's conjecture: a number theory conjecture which postulates that, with any positive integer n, we can confirm if n is a prime number by checking Guiga's condition. This is achieved through calculating a sum, in which n is contained in the exponent of the summands.

The sum would have a specific value – s for example – only if n is a prime number. In other words, the sum would not have the value of s if n is composite.

Giua's conjecture (1950)
An integer $n > 1$ is prime if and only if

$$s := \sum_{k=1}^{n-1} k^{n-1} \equiv -1 (\mathrm{mod}\ n).$$

Despite dating back to the 1950s, the theory has never been proven. It is often considered too daunting for traditional mathematical methods. Borwein, however, has made significant findings through the use of computers. With colleagues, he was recently able to show



VISUALISING THREE-STEP RANDOM WALKS. THE THICK CIRCLE SHOWS THE EXPECTED DISTANCE TRAVELLED

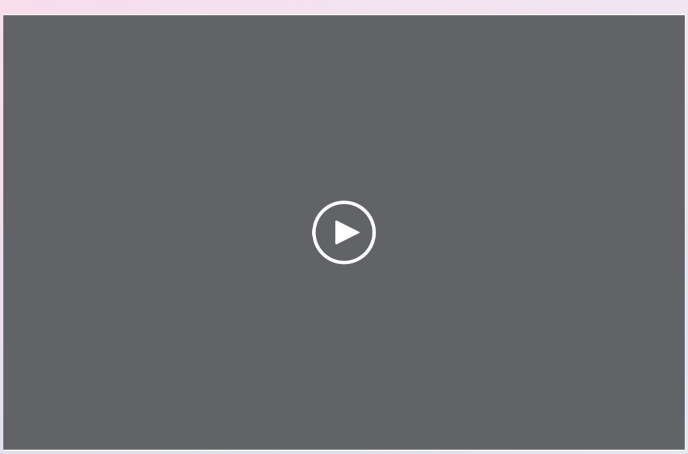
that any number proving an exception to the theory must have more than 4,771 distinct prime factors and be greater than 19,907 decimal digits long. This means that any shorter composite number cannot yield the value of s. Although this doesn't confirm the theory, it provides fascinating evidence in its favour. Indeed, it is only upon realising small breakthroughs such as these that mathematicians are compelled and inspired to seek full proof of a theory.

THE FUTURE OF EXPERIMENTAL MATHEMATICS

In terms of the future direction of his studies, Borwein is clear: "I expect much more emphasis on advanced visualisation in two and three dimensions as illustrated in the description of my recent work on Pi and other fundamental constants. I am heavily involved in attempts to make computational science research more reproducible and reliable," he reveals. "This is an enormous task as it requires much more robust code than is currently available, which no one wishes to pay for, and a series of culture changes."

First, researchers need to be convinced that ensuring reproducibility is worth their while, which will lead to increased productivity, less time spent recovering data or code, and a more efficient conversion from data to published papers. Second, the education system needs to provide more rewards at every institutional level, including departmental decisions, grant funding and the publication of papers. Currently, in the educational and industrial spheres, the main focus is on publication and project funding, rather than reproducibility. However, this does not favour those who spend much time following or developing community standards. Finally, the standards set for peer review need to be more robust: professionals, such as editors and reviewers, need to emphasise the importance of verification, validity testing and the full disclosure of computational details. Indeed, some details could be demoted to an online database or website, providing the information remains accessible.

Despite these challenges, the future remains bright for the computational experimental approach in applied mathematics – an approach that has undeniably accelerated in the 21st Century, and will continue to do so.



INTELLIGENCE

COMPUTER ASSISTED RESEARCH MATHEMATICS AND ITS APPLICATIONS

OBJECTIVES

This proposal has a pure and an applied part. The applied research is on a class of algorithms used successfully by many, but without real justification, for core signal reconstruction problems in astronomy, physics, bioscience, genomics, geoscience and medicine. Success will provide much better methods for such reconstructions and, equally importantly, an understanding of why and when they work. Success with the purer research, which still has immediate applications in other sciences, is also aimed at helping change the culture of mathematics at both a research and a pedagogical level.

FUNDING

Over the four years of the ARC funded project roughly AUS $550,000 came from ARC, with another $1 million from University of Newcastle funding for CARMA (the research centre)

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THE UNIVERSITY OF NEWCASTLE AUSTRALIA

CARMA