Semi-, Sub- and Uniform Regularity of Collections of Sets

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Optimization, Nonlinear Analysis, Randomness & Risk

CARMA, The University of Newcastle, 12th July, 2014

Collections of sets

Regularity:

- Constraint qualifications
- Qualification conditions in subdifferential calculus
- Qualification conditions in convergence analysis

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Absence of regularity

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Absence of regularity $\Longleftrightarrow$ Stationarity

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Absence of regularity $\iff$ Stationarity

Optimality $\implies$ Extremality $\implies$ (Approximate) Stationarity

Outline

1 Regularity

2 Examples

3 Metric characterizations

4 Dual characterizations

5 Set-valued mappings

Semiregularity

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\} \subset X$ $(m > 1)$ $\quad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

Semiregularity

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Definition

$\mathbf{\Omega}$ is semiregular at $\bar{x}$ if $\exists \alpha, \delta > 0$ such that

$$\bigcap_{i=1}^{m} (\Omega_i - x_i) \bigcap B_\rho(\bar{x}) \neq \emptyset \qquad \forall \rho \in (0, \delta)$$

$\forall x_i \in X$ $(i = 1, \ldots, m)$ with $\max\limits_{1 \le i \le m} \|x_i\| < \alpha\rho$

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(*regularity* — Kruger, 2006; *property* $(R)_S$ — Kruger, 2009)

Subregularity

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\} \subset X$ $(m > 1)$ $\quad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

Definition

$\mathbf{\Omega}$ is subregular at $\bar{x}$ if $\exists \alpha, \delta > 0$ such that

$$\bigcap_{i=1}^{m} (\Omega_i + (\alpha\rho)\mathbb{B}) \bigcap B_\delta(\bar{x}) \subseteq \bigcap_{i=1}^{m} \Omega_i + \rho\mathbb{B} \qquad \forall \rho \in (0, \delta)$$

Uniform regularity

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\} \subset X \ (m > 1) \qquad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

Definition

$\mathbf{\Omega}$ is uniformly regular at $\bar{x}$ if $\exists \alpha, \delta > 0$ such that

$$\bigcap_{i=1}^{m} (\Omega_i - \omega_i - x_i) \bigcap (\rho\mathbb{B}) \neq \emptyset \qquad \forall \rho \in (0, \delta)$$

$\forall \omega_i \in \Omega_i \cap B_\delta(\bar{x})$ and $x_i \in X \ (i = 1, \ldots, m)$ with $\max\limits_{1 \le i \le m} \|x_i\| < \alpha\rho$

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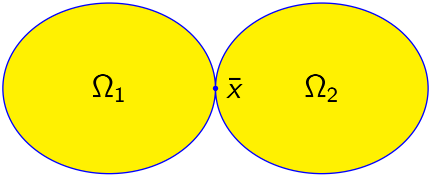
Semiregularity $\Longleftarrow$ Uniform regularity $\Longrightarrow$ Subregularity

Examples: extremality (Kruger, Mordukhovich, 1980)



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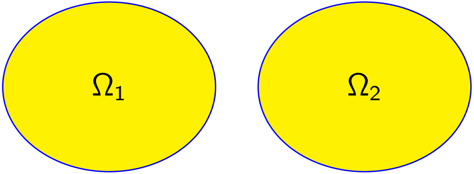

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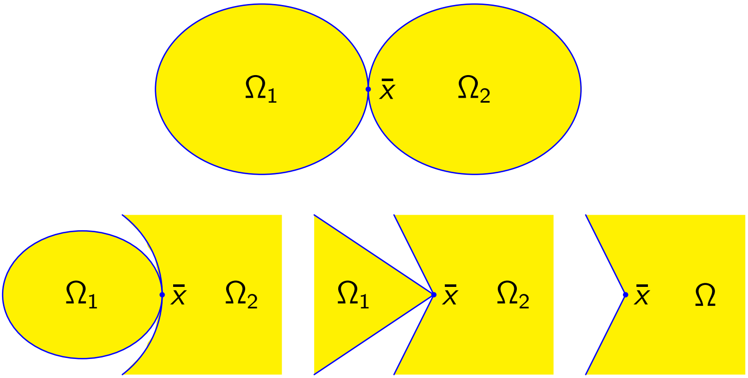

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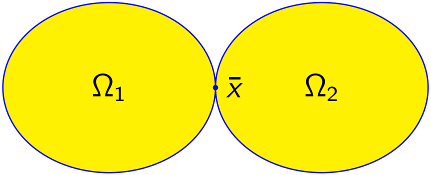

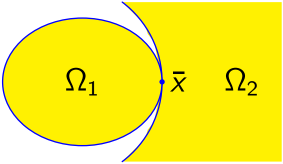

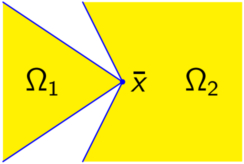

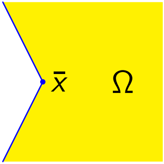

No semiregularity

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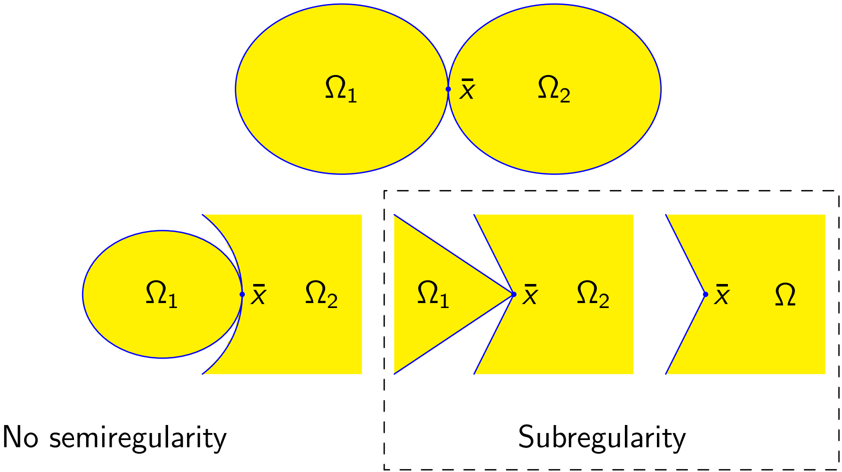
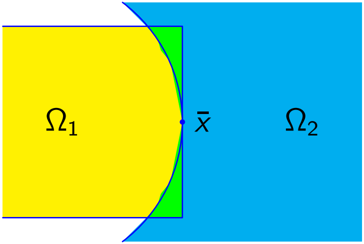


Examples: stationarity



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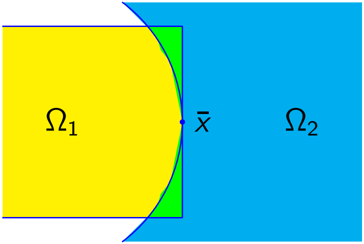
Examples: stationarity



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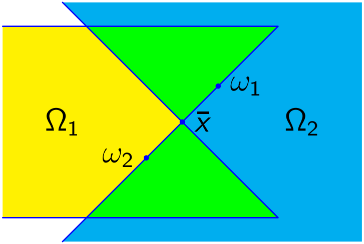
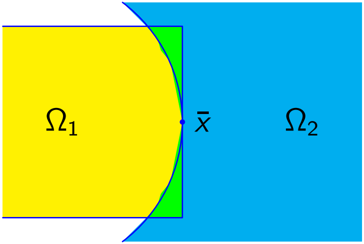


No uniform regularity



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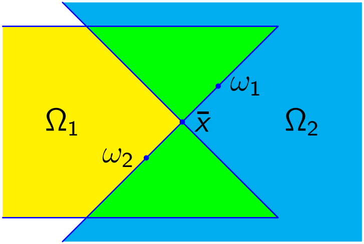
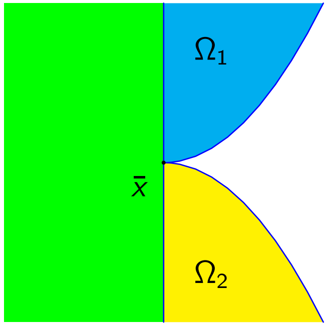


No uniform regularity

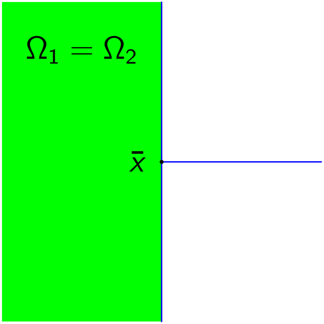
Subregularity

Examples: subregularity vs semiregularity



Semiregularity   No subregularity

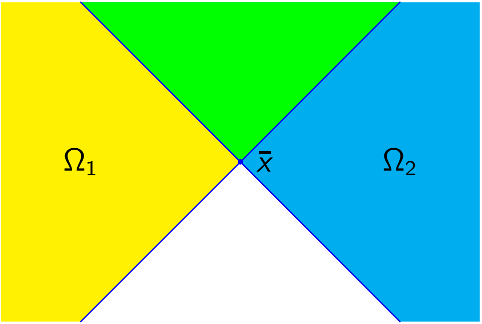
Examples:
sub-/semi-regularity vs uniform regularity



Semiregularity Subregularity

No uniform regularity

Examples: uniform regularity



Metric characterizations

- $\mathbf{\Omega}$ is semiregular at $\bar{x}$ $\quad\Longleftrightarrow\quad$ $\exists\gamma,\delta>0$ such that

$$\gamma d\left(\bar{x},\bigcap_{i=1}^{m}(\Omega_i-x_i)\right)\le\max_{1\le i\le m}\|x_i\|\quad\forall x_i\in\delta\mathbb{B}\ (i=1,\ldots,m)$$

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- $\boldsymbol{\Omega}$ is subregular at $\bar{x}$ $\iff$ $\exists\gamma,\delta>0$ such that

$$\gamma d\left(x,\bigcap_{i=1}^{m}\Omega_i\right)\le\max_{1\le i\le m}d(x,\Omega_i)\quad\forall x\in B_\delta(\bar{x})$$

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  - (Bounded, local) linear regularity (Bauschke, Borwein, 1993)
  - Linear estimate, linear coherence (Penot, 1998, 2013)
  - Metric inequality (Ngai, Théra, 2001)
  - (Dolecki, 1982; Ioffe, 1989; Jourani, 1995; . . .)

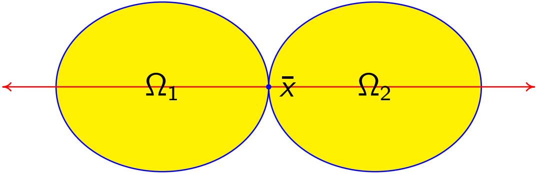
Metric characterizations

- $\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ $\exists\gamma,\delta>0$ such that

$$\gamma d\left(x,\bigcap_{i=1}^{m}(\Omega_i-x_i)\right)\le\max_{1\le i\le m}d(x+x_i,\Omega_i)$$

for any $x\in B_\delta(\bar{x})$, $x_i\in\delta\mathbb{B}$ $(i=1,\ldots,m)$

Dual characterizations: extremality



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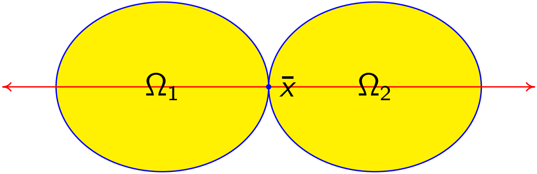

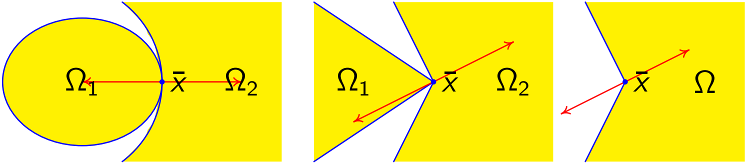

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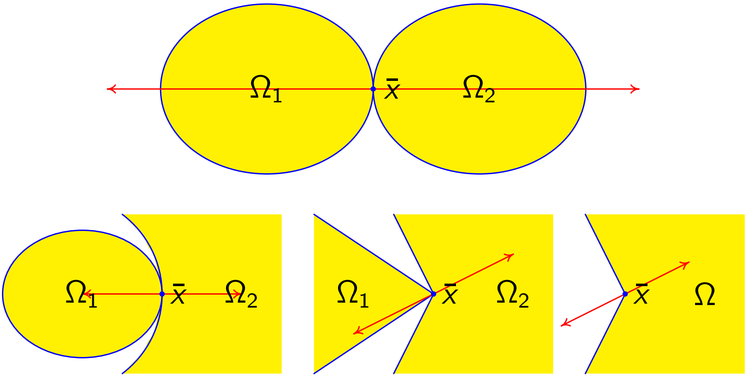

Extremal principle – separabilty
(Kruger, Mordukhovich, 1980; Mordukhovich, Shao, 1996)

Dual characterizations: Fréchet normals

$x \in \Omega$

Fréchet normal cone to Ω at x:

$$N_\Omega(x) := \left\{ x^* \in X^* \middle| \limsup_{u \to x,\, u \in \Omega \setminus \{x\}} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \le 0 \right\}$$

Dual characterizations: uniform regularity

X – Asplund space, $\Omega_1, \ldots, \Omega_m$ – closed

Theorem

$\boldsymbol{\Omega}$ *is uniformly regular at* $\bar{x}$ $\quad\Longleftrightarrow\quad$ $\exists \alpha, \delta > 0$ *such that*

$$\left\| \sum_{i=1}^{m} x_i^* \right\| \geq \alpha$$

$\forall \omega_i \in \Omega_i \cap B_\delta(\bar{x})$, $x_i^* \in N_{\Omega_i}(\omega_i)$ $(i = 1, \ldots, m)$ *satisfying* $\sum_{i=1}^m \|x_i^*\| = 1$

Dual characterizations: uniform regularity

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(*property* $(\mathrm{URD})_S$ — Kruger, 2009)

Dual characterizations: subregularity

X – Asplund space, $\Omega_1, \ldots, \Omega_m$ – closed

Theorem

$\boldsymbol{\Omega}$ *is subregular at* $\bar{x}$ *if* $\exists \alpha, \delta, \varepsilon > 0$ *such that*

$$\left\| \sum_{i=1}^{m} x_i^* \right\| > \alpha$$

$\forall x \in B_\delta(\bar{x})$, $\omega_i \in \Omega_i \cap B_\delta(x)$, $x_i^* \in N_{\Omega_i}(\omega_i) + \delta \mathbb{B}^*$ $(i = 1, \ldots, m)$ *satisfying*

- $\omega_j \neq x$ *for some* $j \in \{1, \ldots, m\}$
- $\sum_{i=1}^{m} \|x_i^*\| = 1$
- $x_i^* = 0$ *if* $\|x - \omega_i\| < \max_{1 \le j \le m} \|x - \omega_j\|$
- $\langle x_i^*, x - \omega_i \rangle \ge \|x_i^*\| (\|x - \omega_i\| - \varepsilon)$ $(i = 1, \ldots, m)$

Collections of sets vs set-valued mappings

X – Banach space

$\mathbf{\Omega} := \{\Omega_1, \ldots, \Omega_m\} \subset X$ $(m > 1)$ $\quad \bar{x} \in \bigcap_{i=1}^m \Omega_i$

$F : X \rightrightarrows X^m$: $\quad F(x) := (\Omega_1 - x) \times \ldots \times (\Omega_m - x)$ $\quad$ (Ioffe, 2000)

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Proposition

$\mathbf{\Omega}$ is semiregular at $\bar{x}$ $\iff$ F is metrically semiregular at $(\bar{x}, 0)$, i.e., $\exists \gamma, \delta > 0$ such that

$$\gamma d\left(\bar{x}, F^{-1}(y)\right) \leq \|y\| \quad \forall y \in \delta \mathbb{B}^m$$

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Proposition

$\mathbf{\Omega}$ *is subregular at* $\bar{x}$ $\iff$ F *is metrically subregular at* $(\bar{x}, 0)$, *i.e.,* $\exists \gamma, \delta > 0$ *such that*

$$\gamma d\left(x, F^{-1}(0)\right) \leq d(0, F(x)) \quad \forall x \in B_\delta(\bar{x})$$

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Proposition

$\boldsymbol{\Omega}$ is uniformly regular at $\bar{x}$ $\iff$ F is metrically regular at $(\bar{x}, 0)$, i.e., $\exists \gamma, \delta > 0$ such that

$$\gamma d\left(x, F^{-1}(y)\right) \leq d\left(y, F(x)\right) \quad \forall x \in B_\delta(\bar{x}),\ y \in \delta \mathbb{B}^m$$

Collections of sets vs set-valued mappings

X, Y – Banach spaces
$F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$

$\Omega_1 = \operatorname{gph} F$, $\Omega_2 = X \times \{\bar{y}\} \in X \times Y$, $\mathbf{\Omega} := \{\Omega_1, \Omega_2\}$

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Theorem

1. *F is metrically semiregular at $(\bar{x}, \bar{y})$ $\iff$ $\boldsymbol{\Omega}$ is semiregular at $(\bar{x}, \bar{y})$*
2. *F is metrically subregular at $(\bar{x}, \bar{y})$ $\iff$ $\boldsymbol{\Omega}$ is subregular at $(\bar{x}, \bar{y})$*
3. *F is metrically regular at $(\bar{x}, \bar{y})$ $\iff$ $\boldsymbol{\Omega}$ is uniformly regular at $(\bar{x}, \bar{y})$*

Concluding remarks

- Quantitative characterizations

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- Hölder-like properties

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- Quantitative characterizations
- Hölder-like properties
- Infinite collections

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Thank you