BRAILEY AND SPACES OF NON-POSITIVE CURVATURE

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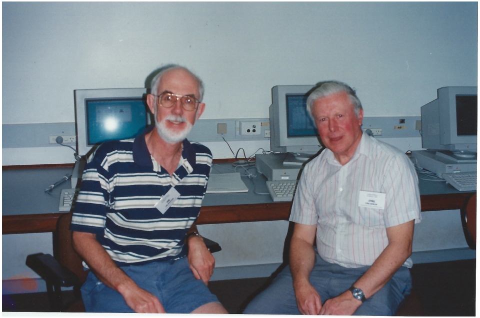
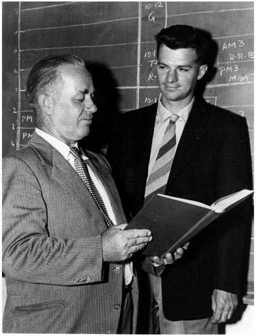
A Story in Three Parts

Prologue

In the beginning...

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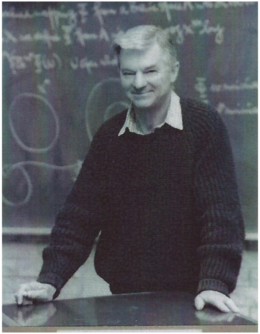
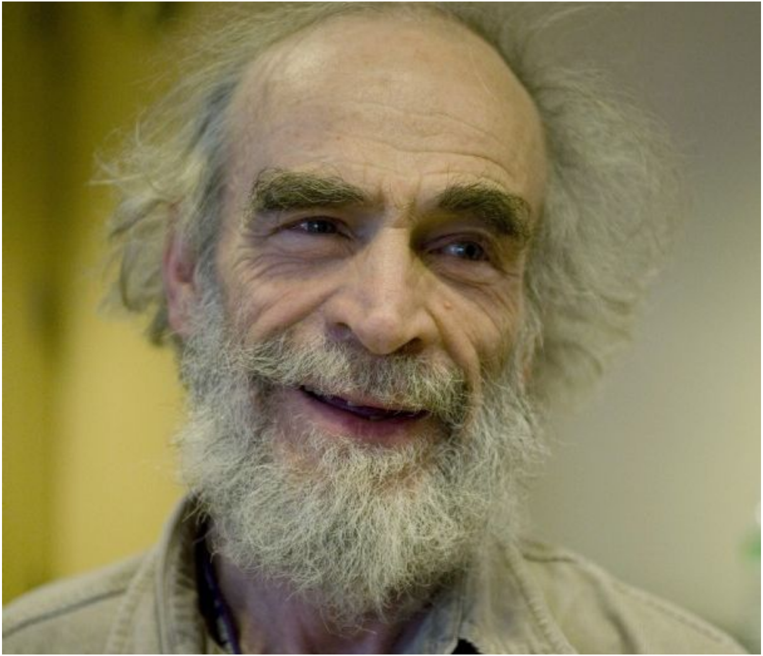
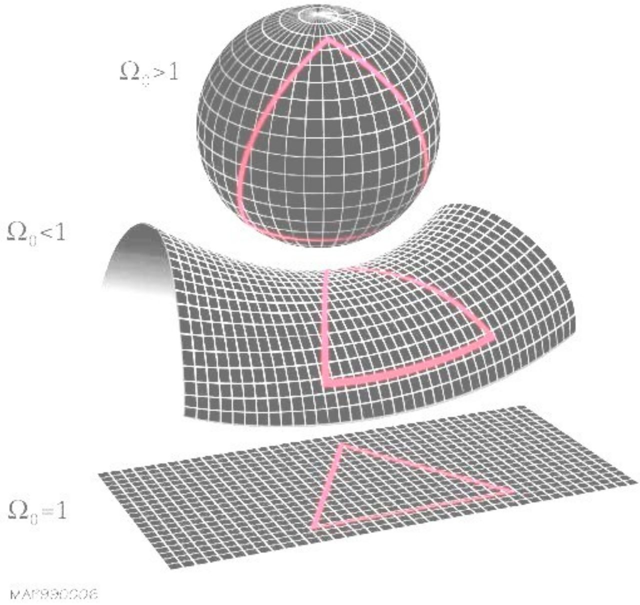


The Main Game

Spaces of non-positive curvature

"Equations are just the boring parts of mathematics. I attempt to see things in terms of geometry." - Stephen Hawking




$\Omega_0 > 1$
$\Omega_0 < 1$
$\Omega_0 = 1$
MAP990006

The study of spaces of non-positive curvature originated with the discovery of hyperbolic spaces, the work of Jacques Hadamard at the beginning of the last century, and the work of Élie Cartan in the 1920's.

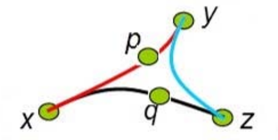
The idea of what it means for a geodesic metric space to have non-positive curvature (or, more generally, curvature bounded above by a real number κ) goes back to the independent, but very similar, discoveries of the American mathematician Herbert Busemann and the Russian mathematician A.D. Alexandrov in the 1950's.

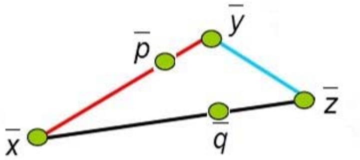
Of particular importance to the revival of interest in this topic are the lectures which Mikhael Gromov gave in 1981 at the Collège de France in Paris. In these lectures Gromov explained the main features of global Riemannian geometry especially by basing his account wholly on the metric structure of the so-called CAT(0) spaces.

The acronym CAT is derived from the names Cartan, Alexandrov and Toponogov in recognition of their pioneering work in the area.

About ten years ago mathematicians realized that many of the standard ideas and methods of nonlinear analysis and Banach space theory, with particular emphasis on fixed point theory, carry over nicely to the CAT(0) spaces. Generally speaking, while there are many many similarities between Hilbert spaces and CAT(0) spaces, there are fundamental differences.

CAT(0) spaces are characterised by the fact that angles in geodesic triangles are no larger than corresponding angles in the euclidean plane and therefore in many ways have a similar geometric structure. Thus Hilbert spaces are CAT(0) spaces.


y
p
x
q
z


$\bar{y}$
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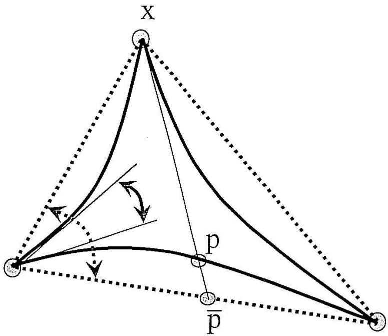
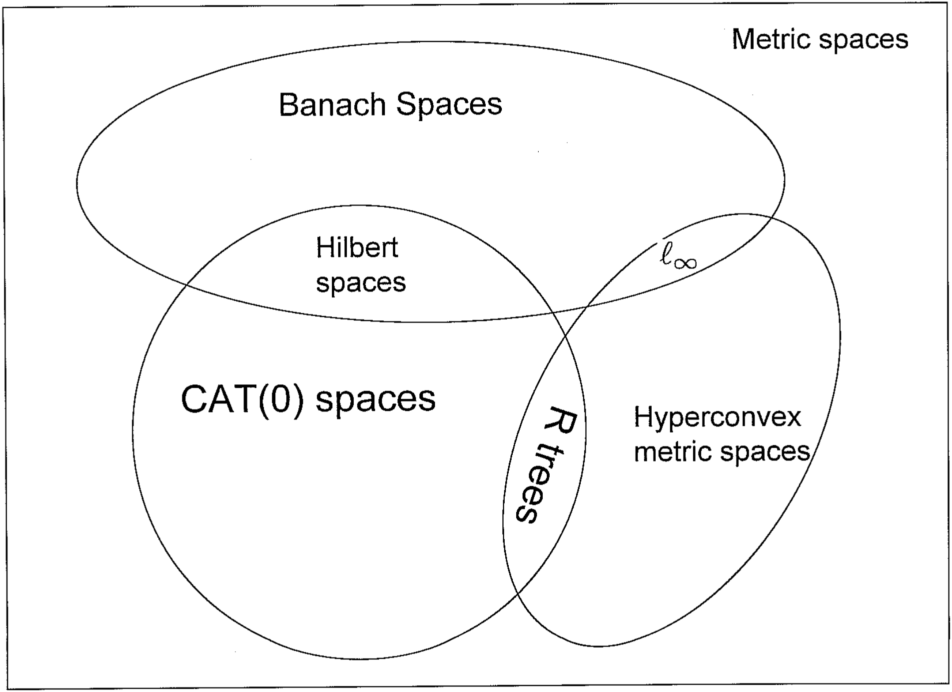

x
p
$\bar{p}$

However, in addition to an euclidean geometric structure, Hilbert spaces have an underlying algebraic framework which the CAT(0) spaces lack. As a result certain corresponding properties that Hilbert spaces enjoy are not available in this more general framework and the Hilbert space theory does not automatically carry over.



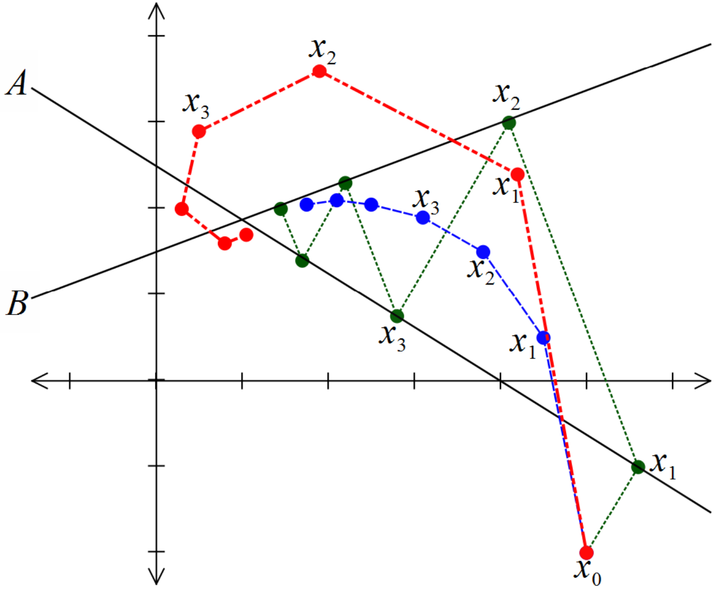
Metric spaces
Banach Spaces
Hilbert
spaces
ℓ_∞
CAT(0) spaces
R trees
Hyperconvex
metric spaces

There is an increasing interest in analysis in the absence of linear structure (such as in the analysis of a state space) either from a pure research or applications perspective.
Interest in both of these aspects has been expanding for some time and continues to grow.

Areas of application currently include, phylogenomic trees, configuration spaces for robotic movements, cognitive models employing prototype theory and the application of the projection methods in inverse problems.

The feasibility problem associated with nonempty closed convex sets A and B is to find some $x \in A \cap B$. Projection algorithms in general aim to compute such a point. These algorithms play key roles in optimization and have many applications outside mathematics - for example in medical imaging.

Until recently convergence results were only available in the setting of linear spaces (more particularly, Hilbert spaces) and where the two sets are closed and convex. The extension into geodesic metric spaces allows their use in spaces where there is no natural linear structure.



A
B
x_2
x_3
x_2
x_1
x_3
x_2
x_3
x_1
x_1
x_0

The alternating projection method

Let X be a complete CAT(0) space with $A, B \subset X$ closed convex sets. Then the alternating projection method produces the sequence

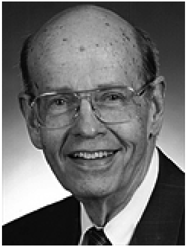
$$x_{2n-1} = P_A(x_{2n-2}), \quad x_{2n} = P_B(x_{2n-1}) \quad n \in N$$

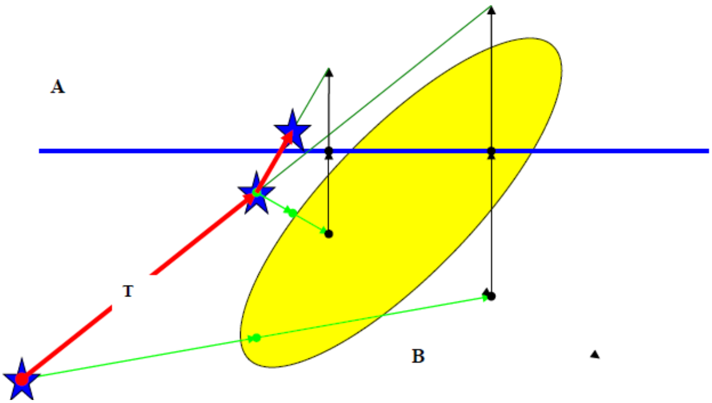
where $x_0 \in X$ is a given starting point.

In CAT(0) space

Let X be a complete CAT(0) space and $A, B \subset X$ convex, closed subsets such that $A \cap B \neq \varnothing$. Let $x_0 \in X$ be a starting point and $(x_n) \subset X$ be the sequence generated by the algorithm above. Then

(x_n) weakly converges to a point $x \in A \cap B$.





A
T
B

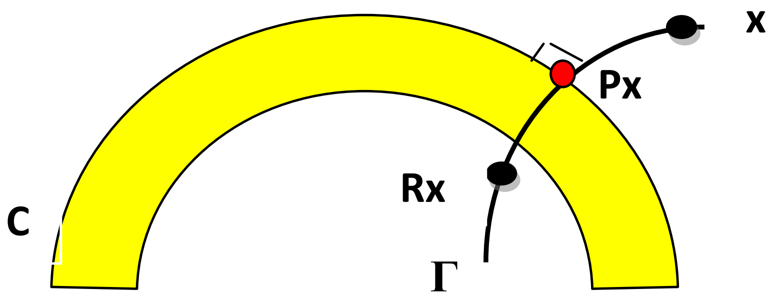
Reflections in CAT(0) spaces

To discuss reflections in CAT(0) spaces we require the geodesics to be extendable and unique (eg spaces with curvature bounded below have non-bifurcating geodesics).

With the above conditions we can define the reflection of a point x in a closed convex set C of a CAT(0) space as the point $R_C(x)$ such that

$$d(R_C(x), P_C(x)) = d(x, P_C(x)),$$

where $P_C x$ is the projection of x onto the set C.



x
Px
Rx
C
Γ

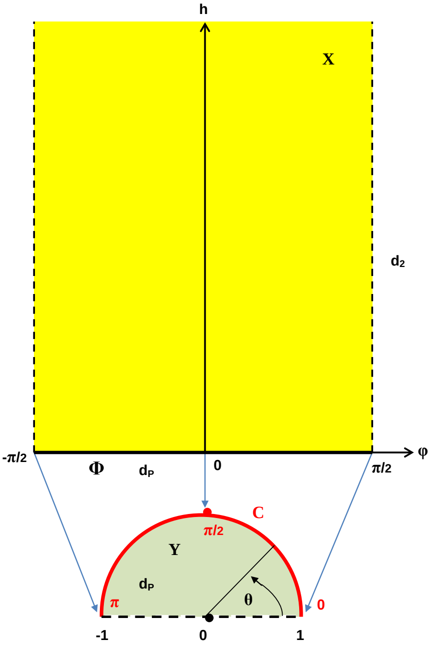
It is well known that reflections in Hilbert space are non-expansive. This follows since the closest point projection is firmly nonexpansive. While projections are also firmly nonexpansive in an appropriate sense in CAT(0) spaces, it no longer follows that reflections are nonexpansive.

Aurora Fernández-Leon and Adriana Nicolae proved reflections in the space M_κ^n of constant curvature to be nonexpansive.

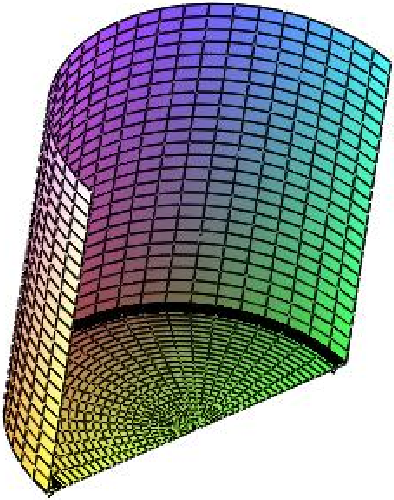
They also established convergence of the Douglas-Rachford algorithm in M_κ^n. Strong convergence is obtained since the model spaces are proper metric spaces. A space is proper if every closed ball is compact.

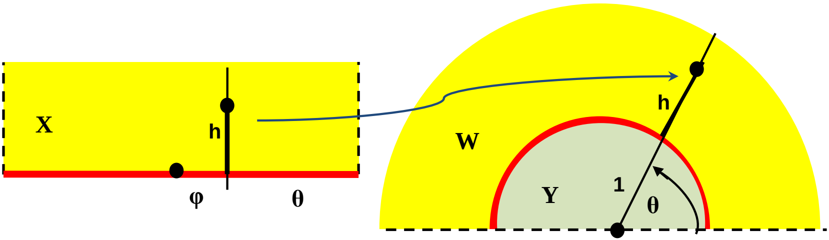
Let $\kappa \leq 0$ and $n \in N$. Suppose A and B are two nonempty closed and convex subsets of M_κ^n with $A \cap B \neq \varnothing$ and $T : X \to X$ defined by $T = \dfrac{I + R_A R_B}{2}$. Let $x_0 \in M_\kappa^n$ and (x_n) be the sequence starting at x_0 generated by the Douglas-Rachford algorithm. Then (x_n) converges to some fixed point x of the mapping T and $P_B(x) \in A \cap B$.

However reflections in CAT(0) spaces need not in general be nonexpansive. To see this we created a prototype CAT(0) space of non-constant curvature.

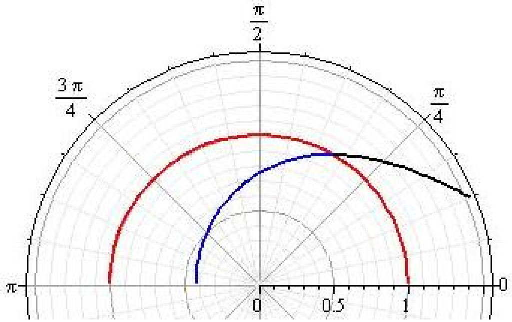


h
X
d_2
$-\pi/2$
Φ
d_P
0
$\pi/2$
φ
C
$\pi/2$
Y
d_P
π
θ
0
-1
0
1

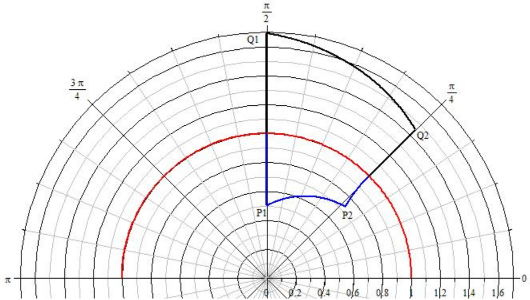




X
h
φ
θ
W
h
Y
1
θ



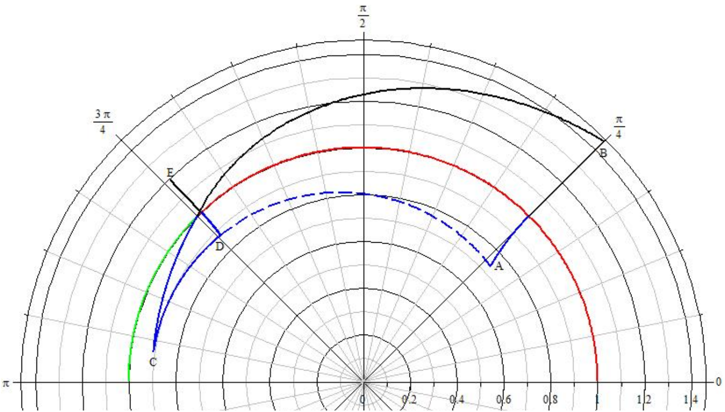
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
$\frac{\pi}{4}$
π
0
0
0.5
1



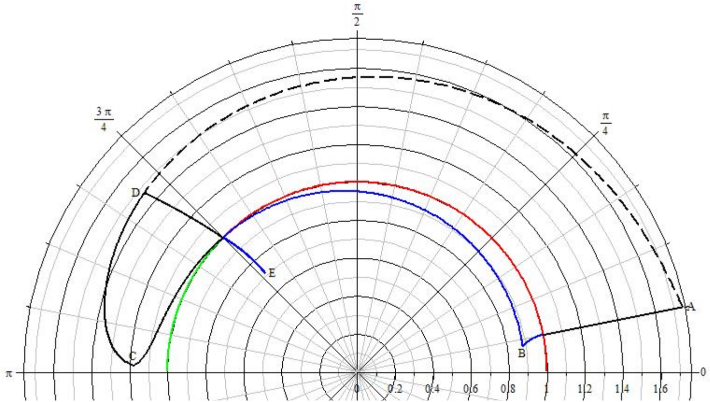
If $P_1 := \frac{i}{2}$ and $P_2 := 0.5439 + 0.4925i$ we have $Q_1 := R_C(P_1) = 1.6931i$ and $Q_2 := R_C(P_2) = 1.453e^{\Pi/4}$. So $d(Q_1, Q_2) = 0.9135 < d(P_1, P_2) = 1.0476.$

The Douglas-Rachford algorithm in
a CAT(0) of non-constant curvature

	$n=1$	$n=2$	$n=3$
x_n	$(0.733799, 0.735867)$	$(0.872208, 2.341632)$	$(0.874992, 2.365024)$
$P_{C_2}x_n$	$(1, 0.785398)$	$(1, 2.332662)$	$(1, 2.356056)$
$R_{C_2}x_n$	$(1.452881, 0.785398)$	$(1.190054, 2.332662)$	$(1.189967, 2.356056)$
$P_{C_1}R_{C_2}x_n$	$(1, \frac{3\pi}{4})$	$(1, \frac{3\pi}{4})$	$(1, \frac{3\pi}{4})$
$R_{C_1}R_{C_2}x_n$	$(0.903444, 3.005137)$	$(0.877656, 2.387882)$	$(0.875025, 2.365298)$
x_{n+1}	$(0.872208, 2.341632)$	$(0.874992, 2.365024)$	$(0.875008, 2.365161)$

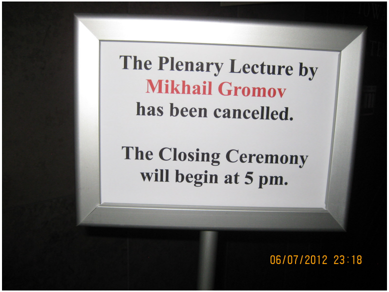


	$n=1$	$n=2$
x_n	$(1.75, \frac{\pi}{16})$	$(1.464634, 2.442068)$
$P_{C_2}x_n$	$(1, \frac{\pi}{16})$	$(1, \frac{3\pi}{4})$
$R_{C_2}x_n$	$(0.882893, 0.152446)$	$(0.707278, 2.320714)$
$P_{C_1}R_{C_2}x_n$	$(1, \frac{3\pi}{4})$	$(1, \frac{3\pi}{4})$
$R_{C_1}R_{C_2}x_n$	$(1.179269, 3.115386)$	$(1.464634, 2.442068)$
x_{n+1}	$(1.464634, 2.442068)$	$(1.464634, 2.442068)$



Section 3

Epilogue


The Plenary Lecture by
Mikhail Gromov
has been cancelled.
The Closing Ceremony
will begin at 5 pm.
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The remaining story ....

THANK YOU