Expectations over Fractal Sets

Michael Rose

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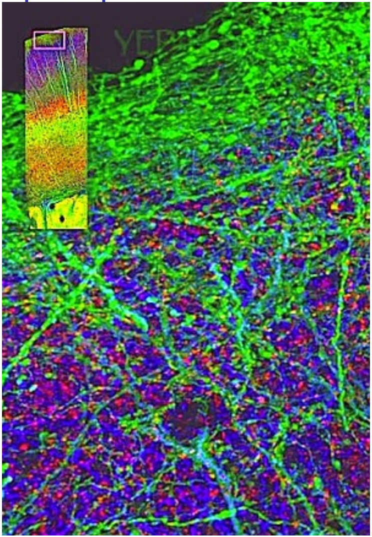
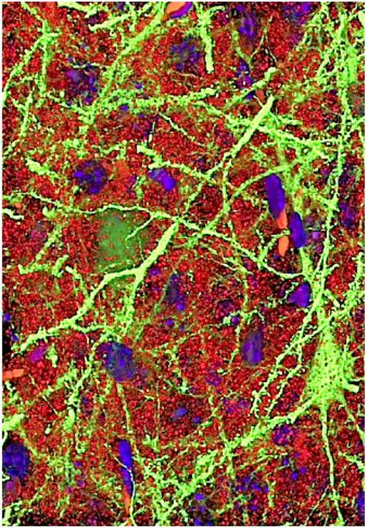
21st June 2015

Synapse spatial distributions





R.E. Crandall, *On the fractal distribution of brain synapses*. In Computational and Analytical Mathematics (2013).

Classical box integrals

$B_n(s)$ is the order-s moment of separation between a random point and a **vertex** of the n-cube:

$$B_n(s) \;:=\; \langle |x|^s \rangle_{x \in [0,1]^n} = \int_{x \in [0,1]^n} |x|^s \; \mathcal{D}x$$

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$\Delta_n(s)$ is the order-s moment of separation between **two random points** in the n-cube:

$$\Delta_n(s) \;:=\; \langle |x-y|^s \rangle_{x,y \in [0,1]^n} = \int_{x,y \in [0,1]^n} |x-y|^s \; \mathcal{D}x\mathcal{D}y$$

R.S. Anderssen, R.P. Brent, D.J. Daley and P.A.P. Moran, *Concerning* $\int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_k^2)^{1/2} \mathrm{d}x_1 \cdots \mathrm{d}x_k$ *and a Taylor Series Method*. SIAM Journal on Applied Mathematics 30 (1976).

D.H. Bailey, J.M. Borwein and R.E. Crandall, *Box integrals*. Journal of Computational and Applied Mathematics 206 (2007).

Outline:

1. Expectations over String-generated Cantor Sets (SCSs).

2. Expectations over Iterated Function System (IFS) attractors.

3. Ongoing Mathematics and Computation.

String-generated Cantor Set (SCS) expectations

String-generated Cantor Sets

Ternary expansion for coordinates of $x = (x_1, \ldots, x_n) \in [0,1]^n$ (with $x_{jk} \in \{0,1,2\}$):

$$\begin{array}{rcl} x_1 & = & 0\,.\,x_{11}\ x_{12}\ x_{13}\ldots \\ x_2 & = & 0\,.\,x_{21}\ x_{22}\ x_{23}\ldots \\ & \vdots & \\ x_n & = & 0\,.\,x_{n1}\ x_{n2}\ x_{n3}\ldots \\ & & \quad\ \uparrow\quad\ \uparrow\quad\ \uparrow \\ & & \quad\ c_1\quad c_2\quad c_3\quad \ldots \end{array}$$

$U(c) := \#\{1\text{'s in ternary vector } c\}$

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Definition (String-generated Cantor set)

Given an embedding space $[0,1]^n$ and an entirely-periodic string $P = P_1 P_2 \ldots P_p$ of non-negative integers with $P_i \le n$ for all $i = 1, 2, \ldots, p$,

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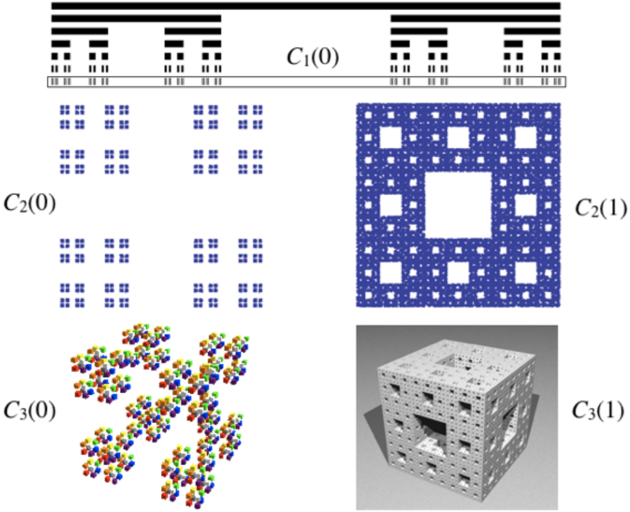
Given an embedding space $[0,1]^n$ and an entirely-periodic string $P = P_1 P_2 \ldots P_p$ of non-negative integers with $P_i \le n$ for all $i = 1, 2, \ldots, p$, the **String-Generated Cantor Set (SCS)**, *denoted $C_n(P)$, is the set of all admissible $x \in [0,1]^n$, where*

$$x \textit{ admissible} \iff U(c_k) \le P_k \quad \forall\, k \in \mathbb{N}$$

String-generated Cantor Sets



D.H. Bailey, J.M. Borwein, R.E. Crandall and M.G. Rose, *Expectations on fractal sets*. Applied Mathematics and Computation 220 (2013).

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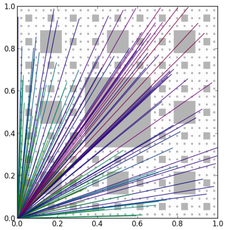
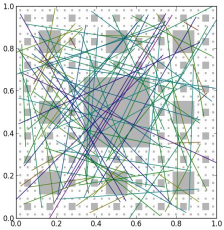
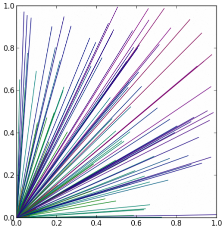
Fractal Box Integrals



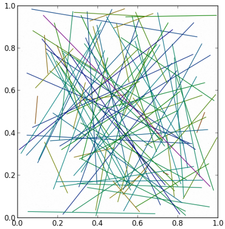
$B(2, C_2(1)) = \frac{11}{16} = 0.6875$



$\Delta(2, C_2(1)) = \frac{3}{8} = 0.375$



$B(2, C_2(2)) = \frac{2}{3} = 0.66\ldots$



$\Delta(2, C_2(2)) = \frac{1}{3} = 0.33\ldots$

Definition of expectation

Definition (Expectation over an SCS)

The **expectation of** $F : \mathbb{R}^n \to \mathbb{C}$ **on an SCS** $C_n(P)$ *is defined by:*

$$\langle\, F(x)\,\rangle_{x\in C_n(P)} \;:=\; \lim_{j\to\infty} \frac{1}{N_1\cdots N_j} \sum_{U(c_i)\le P_i} F\left(\frac{c_1}{3}+\frac{c_2}{3^2}+\cdots+\frac{c_j}{3^j}\right)$$

$$\langle\, F(x-y)\,\rangle_{x,y\in C_n(P)} \;:=\; \lim_{j\to\infty} \frac{1}{N_1^2\cdots N_j^2} \sum_{\substack{U(c_i)\le P_i\\ U(d_i)\le P_i}} F\left(\frac{c_1-d_1}{3}+\cdots+\frac{c_j-d_j}{3^j}\right)$$

when the respective limits exist.

Functional equations for expectations

Proposition (**Functional equations for expectations**)

For x, y in R^n and appropriate F the expectations pertaining to the box integrals B and Δ satisfy the **functional equations***:*

$$\langle F(x)\rangle_{x\in C_n(P)} = \frac{1}{\prod_{j=1}^{p} N_j} \sum_{U(c_k)\le P_k} \left\langle F\left(\frac{x}{3^p} + \sum_{j=1}^{p} \frac{c_j}{3^j}\right)\right\rangle$$

$$\langle F(x-y)\rangle_{x,y\in C_n(P)} = \frac{1}{\prod_{j=1}^{p} N_j^2} \sum_{\substack{U(b_k)\le P_k \\ U(a_k)\le P_k}} \left\langle F\left(\frac{x-y}{3^p} + \sum_{j=1}^{p} \frac{(b_j - a_j)}{3^j}\right)\right\rangle$$

Special case - second moments

The functional expectation relations lead directly to:

Theorem (Closed forms for $B(2, C_n(P))$ and $\Delta(2, C_n(P))$)

For any embedding dimension n and SCS $C_n(P)$ the box integral $B(2, C_n(P))$ is **rational**, *given by the closed form:*

$$B(2, C_n(P)) \;=\; \frac{n}{4} + \frac{1}{1-9^{-p}} \sum_{k=1}^{p} \frac{1}{9^k} \frac{\sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}(n-j)}{\sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}}$$

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and the corresponding box integral $\Delta(2, C_n(P))$ is also **rational**, *given by:*

$$\Delta(2, C_n(P)) \; = \; 2B(2, C_n(P)) - \frac{n}{2}$$

Special case - second moments

The classical box integrals over the unit n-cube are:

$$B_n(2) = \frac{n}{3} \quad \text{and} \quad \Delta_n(2) = \frac{n}{6}$$

which matches the output of our closed forms when $P = n$.

Iterated Function System (IFS)
attractor expectations

IFS Attractors

Let (X, d) be a metric space and let $(\mathcal{H}(X), h(d))$ be the associated space of non-empty compact subsets of X equipped with the Hausdorff metric $h(d)$.

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Definition

For each $i \in \{1, 2, \ldots, m\}$ (where $m \geq 2$), let $f_i : X \to X$ be a contraction mapping with contractivity factor $0 < c_i < 1$ (so $d(f_i(x), f_i(y)) \leq c_i \cdot d(x, y)$) and associated probability $0 < p_i < 1$ (where $\sum_{i=1}^{m} p_i = 1$). A **hyperbolic iterated function system (IFS)** *is the collection*

$$\{X; f_1, \ldots, f_m\}$$

.

IFS Attractors

Theorem

Let $\{X; f_1, \ldots, f_m\}$ *be a hyperbolic IFS. Then the transformation* $\mathcal{F} : \mathcal{H}(X) \to \mathcal{H}(X)$ *defined by* $\mathcal{F}(S) = \bigcup_{n=1}^{m} f_n(S)$ *for all* $S \in \mathcal{H}(X)$ *is a contraction mapping on* $\mathcal{H}(X)$ *with contractivity factor* $C = \max\{c_1, \ldots, c_m\}$.

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Theorem (The Contraction Mapping Theorem)

The mapping $\mathcal{F}$ possesses a unique fixed point $A \in H(X)$, which satisfies:

$$A = \mathcal{F}(A) = \bigcup_{n=1}^{m} f_n(A)$$

and which is referred to as the **attractor** *of the IFS.*

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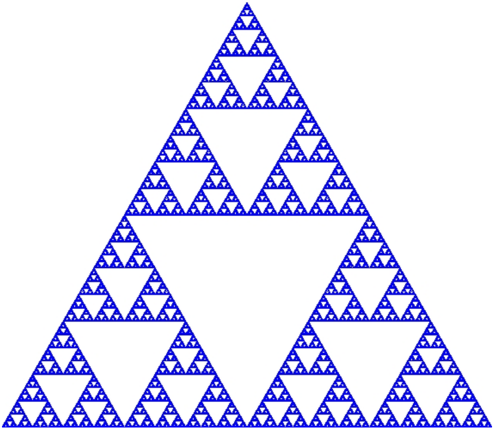
We will take as our 'deterministic fractals' those sets that can be so expressed as an IFS attractor.

IFS Attractors



$$f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$$

$$f_2(x, y) = \left(\frac{x+1}{2}, \frac{y+\sqrt{3}}{2}\right)$$

$$f_3(x, y) = \left(\frac{x+2}{2}, \frac{y}{2}\right)$$

SCS in IFS framework

Any given SCS can be expressed as the attractor of an IFS in the following manner:

Proposition

The IFS corresponding to the SCS $C_n(P)$ is:

$$\{[0,1]^n \subset \mathbb{R}^n; f_1, f_2, \ldots, f_i, \ldots, f_m\}$$

where $f_i(x) = \left(\frac{1}{3}\right)^p x + \left(\frac{1}{3}\right) c_{1_i} + \left(\frac{1}{3}\right)^2 c_{2_i} + \ldots + \left(\frac{1}{3}\right)^p c_{p_i}$ *for* $i \in \{1, 2, \ldots, m\}$ *ranging over all admissible columns* c_k*, where* $m = \prod_{k=1}^{p} N_k$ *and* $N_k = \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}$*.*

Code Space

Definition

Given an IFS $\{X; f_1, \ldots, f_m\}$, the associated code space Σ_m is defined as:

$$\Sigma_m := \{\sigma = \sigma_1\sigma_2 \ldots \mid \sigma_i \in \{1, 2, \ldots, m\} \quad \forall i \in \mathbb{N}\}$$

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The address function $\phi : \Sigma_m \to X$ is defined by:

$$\phi(\sigma) := \lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \ldots \circ f_{\sigma_k}(x)$$

for any $x \in A$.

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for any $x \in A$. The attractor of the IFS can also be represented by:

$$A = \{\phi(\sigma) \mid \sigma \in \Sigma_m\}$$

Expectations on IFS Attractors

Definition (Fundamental definition of expectation)

Let $\{X; f_1, \ldots, f_m\}$ *be an IFS with attractor* $A \in H(X)$*. Let* $F : X \to \mathbb{C}$ *be a complex-valued function over* X*. The expectation of* F *over* A*,* $\langle F(x) \rangle_{x \in A}$*, is defined as:*

$$\langle F(x) \rangle_{x \in A} \; := \; \lim_{j \to \infty} \frac{1}{m^j} \sum_{\sigma_{|j|} \in \Sigma_m} F\left(\phi(\sigma_{|j|})\right)$$

when the limit exists.

Expectations on IFS Attractors

Corollary

(Fundamental definition of separation (using code-space)) Let $\{X; f_1, \ldots, f_m\}$ *be an IFS with attractor* $A \in H(X)$*. Let* $F : X \to \mathbb{C}$ *be a complex-valued function over* X*. The separation expectation of* F *over* A*,* $\langle F(x-y) \rangle_{x,y \in A}$*, is defined as:*

$$\langle F(x-y) \rangle_{x,y \in A} := \lim_{j \to \infty} \frac{1}{m^{2j}} \sum_{\sigma_{|j|} \in \Sigma_m} \sum_{\tau_{|j|} \in \Sigma_m} F\left(\phi(\sigma_{|j|}) - \phi(\tau_{|j|})\right)$$

when the limit exists.

The invariant IFS measure

Definition

Let B be a Borel subset of a metric space (X, d). The residence measure is defined as:

$$\mu(B) := \lim_{n\to\infty} \frac{1}{n} \left(\# \left\{ k \ : \ f^k(x) \in B,\ 1 \leq k \leq n \right\} \right)$$

The residence measure is a normalised, invariant measure over the attractor of any IFS.

The invariant IFS measure

Theorem (Elton's Theorem - special case)

Let (X, d) *be a compact metric space and let* $\{X; f_1, \ldots, f_m\}$ *be a hyperbolic IFS.*

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Let (X, d) *be a compact metric space and let* $\{X; f_1, \ldots, f_m\}$ *be a hyperbolic IFS. Let* $\{x_n\}_{n=0}^{\infty}$ *denote a chaos game orbit of the IFS starting at* $x_0 \in X$*,*

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$$x_n = f_{\sigma_n} \circ \ldots \circ f_{\sigma_1}(x_0)$$

where the maps are chosen independently according to the probabilities $p_1, \ldots, p_m$ *for* $n \in \mathbb{N}$*.*

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where the maps are chosen independently according to the probabilities $p_1, \ldots, p_m$ for $n \in \mathbb{N}$. Let μ be the unique invariant measure for the IFS. Then, with probability 1,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} F(x_k) = \int_X F(x) \mathrm{d}\mu(x)$$

Expectations over IFS attractors

Corollary

Let $\{X; f_1, f_2, \ldots, f_m\}$ *be a contractive IFS with attractor* $A \in \mathcal{H}(X)$*. Given a complex-valued function* $F : X \to \mathbb{C}$*, the expectation of* F *over* A *is given by the integral:*

$$\langle F(x) \rangle_{x \in A} = \int_X F(x) \mathrm{d}\mu(x)$$

Functional equations

Proposition (Functional equations for expectations)

For points x, y in the attractor A of a non-overlapping IFS, the expectations for a complex-valued function F satisfy the functional equations:

$$\langle F(x) \rangle = \frac{1}{m} \sum_{j=1}^{m} \langle F\left(f_j(x)\right) \rangle$$

$$\langle F(x-y) \rangle = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{k=1}^{m} \langle F\left(f_j(x) - f_k(y)\right) \rangle$$

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and more generally

$$\langle F(x_1, x_2, \ldots, x_n) \rangle = \frac{1}{m^n} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_n=1}^{m} \langle F\left(f_{j_1}(x_1), f_{j_2}(x_2), \ldots, f_{j_n}(x_n)\right) \rangle$$

Ongoing mathematics and computation

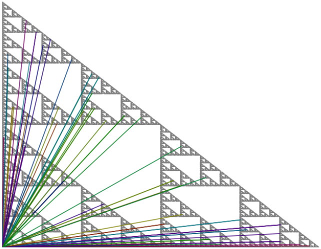
Exact evaluation of even moments

- Substitute a given IFS and function F into the functional equation:

$$\langle F(x-y)\rangle = \frac{1}{m^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\langle F\left(f_j(x)-f_k(y)\right)\rangle$$

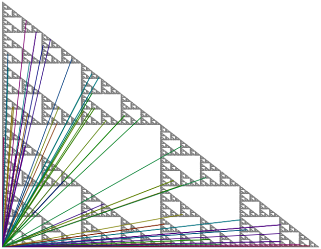
- Simplify the resulting expression into a (linear) combination of n simpler expectations.
- Feed these expectations back into the functional equation to generate a system of n (linear) equations in the n unknown expectations.
- Solve the system of equations and hence determine the expectation.

Orthogonal Sierpinski Triangle



$$B(2, A) = \frac{10}{27}$$

Orthogonal Sierpinski Triangle



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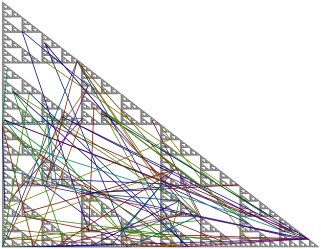
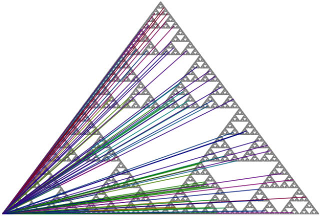


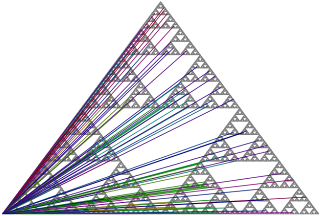
$$\Delta(2, A) = \frac{8}{27}$$

Equilateral Sierpinski Triangle



$$B(2, A) = \frac{4}{9}$$

Equilateral Sierpinski Triangle



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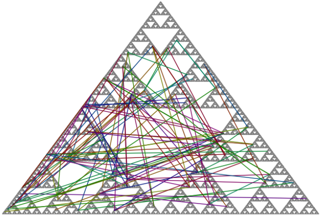
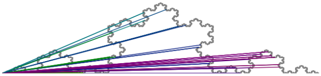


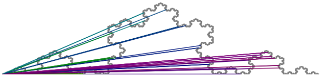
$$\Delta(2, A) = \frac{2}{9}$$

Koch Curve



$$B(2, A) = \frac{1}{3}$$

Koch Curve



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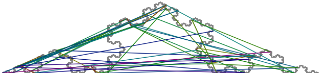


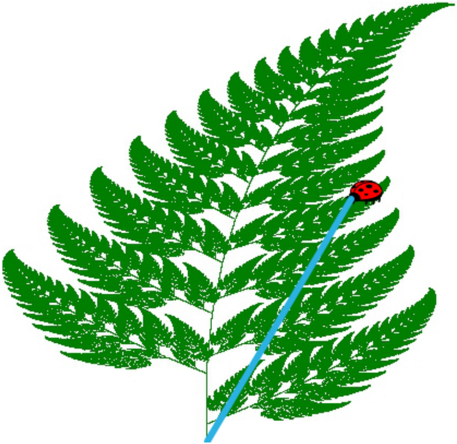
$$\Delta(2, A) = \frac{4}{27}$$

Barnsley Fern

Barnsley Fern

Barnsley Fern

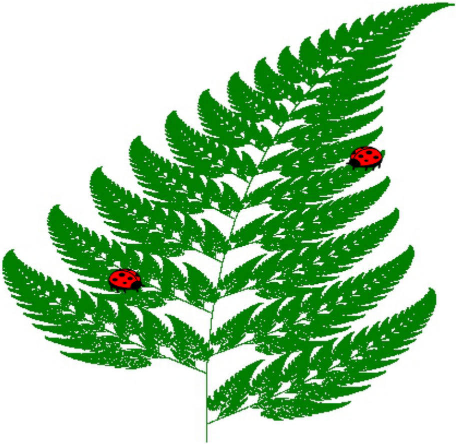
Barnsley Fern



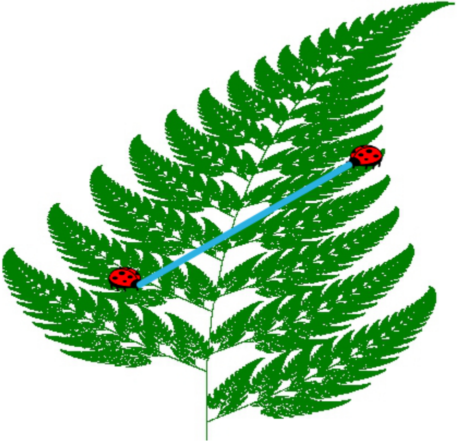
$$B(2, A) = \frac{2049440803137681904}{580160660775546421} \approx 3.5$$

Barnsley Fern

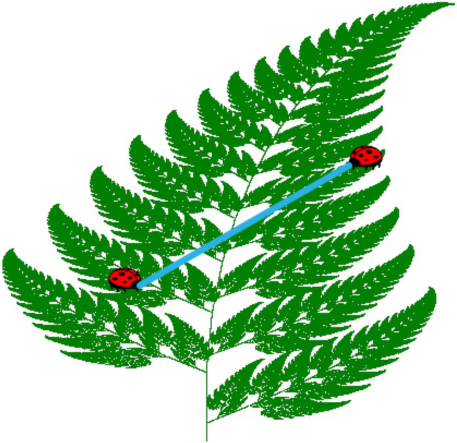
Barnsley Fern



Barnsley Fern



Barnsley Fern



$$\Delta(2, A) = \frac{1561818604387599983932186}{5411303523218715355277225} \approx 2.9$$

Current research: Poles for box integrals

For classical box integrals over unit hypercubes, we have the following:

Theorem (Absolutely-convergent analytic series for $B_n(s)$)

For all $s \in \mathbb{C}$,

$$B_n(s) = \frac{n^{1+s/2}}{s+n} \sum_{k=0}^{\infty} \gamma_{n-1,k} \left(\frac{2}{n}\right)^k$$

where the $\gamma_{m,k}$ *are fixed real coefficients defined by the two-variable recursion:*

$$(1 + 2k/m)\gamma_{m,k} = (k - 1 - s/2)\gamma_{m,k-1} + \gamma_{m-1,k}$$

for $m, k \geq 1$, *with initial conditions* $\gamma_{0,k} := \delta_{0,k}$, $\gamma_{m,0} := 1$.

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Note the single pole at $s = -n$, the negated dimension of the embedding space.

Current research: Poles for box integrals

Theorem (Fractal Dimension with the Open Set Condition)

Suppose that the open set condition holds for the IFS $\mathcal{F} = \{\mathbb{R}^n;\ f_1, f_2, \ldots, f_m\}$ *(with associated contraction factors* $\{c_1, c_2, \ldots, c_m\}$*). That is, the attractor contains a non-empty set* $O \subset A$ *which is open in the metric space* A *such that*

1. $f_i(O) \cap f_j(O) = \emptyset$ *for all* $i, j \in \{1, 2, \ldots, m\}$ *with* $i \neq j$
2. $\bigcup_{i=1}^{m} f_i(O) \subset O$

Then the Hausdorff dimension and Minkowski box-counting dimension of the attractor of the IFS are equal and take the value δ*, where:*

$$\sum_{i=1}^{m} (c_i)^{\delta} = 1.$$

Current research: Poles for box integrals

Using the functional expectation relations, we can prove:

Proposition (SCS: Pole of $B(s, C_n(P))$)

For any **SCS** $C_n(P)$*, the box integral* $B(s, C_n(P))$ *has a pole at*

$$s = -\delta(C_n(P)).$$

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Proposition (IFS: Pole of $B(s)$ over Uniform Affine IFSs)

Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ *be a* **contractive affine IFS** *satisfying the* **open set condition** *with* **uniform contraction factors**; *that is,* $c_1 = c_2 = \ldots = c_m$. *Then the box integral* $B(s, A)$ *over the attractor* $A \in \mathbb{H}(X)$ *has a pole at*

$$s = -\delta(A)$$

Current research: Poles for box integrals

Proposition (IFS: Bounds on pole of $\Delta(s)$ over Similarity IFSs)

Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a **contractive similarity IFS** *satisfying the* **open set condition***; that is, $|f_i(x) - f_i(y)| = c_i|x - y|$ for all i. Then, if the box integral $\Delta_A(s)$ over the attractor $A \in \mathbb{H}(X)$ has a pole on the real axis, the pole is bounded by:*

$$\frac{\log(m)}{\log(c_{\max})} \leq s \leq \frac{\log(m)}{\log(c_{\min})}$$

where $c_{\max} = c = \max\{c_1, \ldots, c_m\}$ and $c_{\min} = \min\{c_1, \ldots, c_m\}$.

Current research: Odd-order moments

n	s	$B(s, C_2(2))$
2	-4	$-\frac{1}{4} - \frac{\pi}{8}$
2	-3	$-\sqrt{2}$
2	-2	∞
2	-1	$2\log(1+\sqrt{2})$
2	1	$\frac{1}{3}\sqrt{2} + \frac{1}{3}\log(1+\sqrt{2})$
2	3	$\frac{7}{20}\sqrt{2} + \frac{3}{20}\log(1+\sqrt{2})$

Table : Closed-form results for B box integrals of various order s over the unit square and unit cube. For the unit n-cube all integer values for $1 \leq n \leq 5$ have closed forms. Ti_2 is a generalized tangent (polylog) value and G is Catalan's constant.

Current research: Odd-order moments

n	s	$B(s, C_3(3))$
3	-5	$-\frac{1}{6}\sqrt{3} - \frac{1}{12}\pi$
3	-4	$-\frac{3}{2}\sqrt{2}\arctan\frac{1}{\sqrt{2}}$
3	-3	∞
3	-2	$-3G + \frac{3}{2}\pi\log(1+\sqrt{2}) + 3\ \mathrm{Ti}_2(3-2\sqrt{2})$
3	-1	$-\frac{1}{4}\pi + \frac{3}{2}\log\left(2+\sqrt{3}\right)$
3	1	$\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2+\sqrt{3}\right)$
3	3	$\frac{2}{5}\sqrt{3} - \frac{1}{60}\pi + \frac{7}{20}\log\left(2+\sqrt{3}\right)$

Current research: Odd-order moments

n	s	$\Delta(s, C_2(2))$
2	-5	$\frac{4}{3} + \frac{8}{9}\sqrt{2}$
2	-2,-3,-4	∞
2	-1	$\frac{4}{3} - \frac{4}{3}\sqrt{2} + 4\log(1+\sqrt{2})$
2	1	$\frac{2}{15} + \frac{1}{15}\sqrt{2} + \frac{1}{3}\log(1+\sqrt{2})$

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Current research: Odd-order moments

n	s	$\Delta(s, C_3(3))$
3	–2	$2\pi - 12\ G + 12\,\mathrm{Ti}_2\left(3 - 2\sqrt{2}\right) + 6\pi \log\left(1 + \sqrt{2}\right)$ $+2\log 2 - \frac{5}{2}\log 3 - 8\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)$
3	-1	$\frac{2}{5} - \frac{2}{3}\pi + \frac{2}{5}\sqrt{2} - \frac{4}{5}\sqrt{3} + 2\log\left(1 + \sqrt{2}\right)$ $+12\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) - 4\log\left(2 + \sqrt{3}\right)$
3	1	$-\frac{118}{21} - \frac{2}{3}\pi + \frac{34}{21}\sqrt{2} - \frac{4}{7}\sqrt{3}$ $+2\ \log\left(1 + \sqrt{2}\right) + 8\ \log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$
3	3	$-\frac{1}{105} - \frac{2}{105}\pi + \frac{73}{840}\sqrt{2} + \frac{1}{35}\sqrt{3}$ $+\ \frac{3}{56}\ \log\left(1 + \sqrt{2}\right) + \frac{13}{35}\ \log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$

Current research: What are the odd moments?

Insight into the evaluation of odd moments can be gleaned from *computer-assisted mathematics*. In particular, we are considering a modified Richardson Extrapolation Technique combined with the PSLQ Integer Relation Algorithm to hunt for closed forms (joint work with Nathan Clisby).

Current research: High-precision numerics

A vector $x = (x_1, x_2 \cdots x_n)$ of real numbers has an **integer relation** if there exists integers a_i, not all zero, such that:

$$\sum_{i=1}^{n} a_i x_i = 0.$$

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- Given a vector $x = (x_1, x_2 \cdots x_n)$, PSLQ iteratively constructs a sequence of integer-matrices B_n that reduce the vector xB_n.
- The process continues until either:
    1. The smallest entry of the latest B_n abruptly decreases to within drops to within ϵ of 0. This signals the detection of an integer relation, which PSLQ will produce as one of the columns of the last B_n.
    2. The available precision is exhausted. In this case, PSLQ will establish a bound on the size of any possible integer relation.

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  2. The available precision is exhausted. In this case, PSLQ will establish a bound on the size of any possible integer relation.

In order to find an integer relation among n terms, PSLQ requires at least nd accurate digits in all terms, where d is the number of digits of the largest integer a_i.

Current research: High-precision numerics

The Richardson Extrapolation Technique combines multiple lower-accuracy evaluations to eliminate the highest-order error terms and thereby obtain a higher-accuracy evaluation.

- Start with an approximation formula $A_1(h)$ for quantity of interest x, accurate to $O(h^1)$. That is,

$$x = A_1(h) + O(h^1) = A_1(h) + c_1 h + c_2 h^2 + \dots .$$

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- Thus, halving the step-size h in the power series,

$$x = A_1\left(\frac{h}{2}\right) + c_1 \frac{h}{2} + c_2 \frac{h^2}{4} + \ldots.$$

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- Combine these equations to eliminate $c_1 h$, yielding:

$$x = A_2(h) - \frac{1}{2} c_2 h^2 - \frac{3}{4} c_3 h^3 + \ldots = A_2(h) + O(h^2)$$

where $A_2(h) = A_1\left(\frac{h}{2}\right) + \frac{1}{2^1 - 1}\left(A_1\left(\frac{h}{2}\right) - A_1(h)\right)$.

Current research: High-precision numerics

Successive iterations over smaller step-sizes yield $x = A_k(h) + O(h^k)$, where

$$A_k(h) = A_{k-1}\left(\frac{h}{2}\right) + \frac{1}{2^{k-1}-1}\left(A_{k-1}\left(\frac{h}{2}\right) - A_{k-1}(h)\right).$$

Adapting this process to IFS attractors, Nathan Clisby has computed 112 digits of $B(1, \text{Sierpiński Triangle})$:

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0.6180082171582247074177418624555167834492481641438960879796572765
2894992781724125962846495857367069910610756180 7 . . .

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$$0.61800821715822470741774186245551678344924816414389608797965727652894992781724125962846495857367069910610756180 7\ldots$$

We are aiming to improve this method by constructing an analogue of Bulirsch-Stöer extrapolation modified for IFS attractors, wherein the sequence of estimates is fitted to a rational function of h and evaluated at $h = 0$.

Thank you!