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NEAREST POINTS AND DELTA CONVEX FUNCTIONS IN BANACH SPACES

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Abstract

Given a closed set C in a Banach space $(X, \|\cdot\|)$, a point $x \in X$ is said to have a nearest point in C if there exists $z \in C$ such that $d_C(x) = \|x - z\|$, where d_C is the distance of x from C. We survey the problem of studying the size of the set of points in X which have nearest points in C. We then turn to the topic of delta convex functions and indicate how it is related to finding nearest points.

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1. Nearest points in Banach spaces

1.1. Background. Let $(X, \|\cdot\|)$ be a real Banach space and let $C \subseteq X$ be a nonempty closed set. Given $x \in X$, its distance from C is given by

$$d_C(x) = \inf_{y \in C} \|x - y\|.$$

If there exists $z \in C$ with $d_C(x) = \|x - z\|$, we say that x has a *nearest point* in C. Let

$$N(C) = \{x \in X : x \text{ has a nearest point in } C\}.$$

One can then ask questions about the structure of the set $N(C)$. Such questions have been studied in [4, 11, 13, 20, 22, 27, 28, 30, 35] to name just a few articles. More specifically, the following questions are at the heart of this note:

Given a nonempty closed set $C \subseteq X$, how large is the set $N(C)$?
When is $N(C)$ nonempty?

One way to answer such questions is to consider sets which are large in the set-theoretic topological sense, such as dense G_δ sets. We begin with some definitions.

Definition 1.1. If $N(C) = X$, that is, if every point in X has a nearest point in C, then C is said to be proximinal. If $N(C)$ contains a dense G_δ set, then C is said to be almost proximinal.

In passing, we recall that if every point in X is uniquely proximinal, then C is said to be a *Chebyshev set*. It has been conjectured for over half a century that, in Hilbert space, Chebyshev sets are necessarily convex, but this is only proven for weakly closed sets [7]. See also [17] for a recent survey of this topic that, in particular, gives a clear construction of a nonconvex Chebyshev set in an incomplete inner product space.

Closed convex sets in reflexive spaces are proximinal, as are all closed sets in finite-dimensional spaces (see [4]). One can also consider stronger notions of 'large' sets, as in Section 1.4.

Definition 1.2. A Banach space is said to be a (sequentially) Kadec space if for each sequence $\{x_n\}$ that converges weakly to x with $\lim \|x_n\| = \|x\|$, $\{x_n\}$ converges to x in norm, that is,

$$\lim_{n\to\infty} \|x - x_n\| = 0.$$

All locally uniformly convex Banach spaces are Kadec spaces, as are all finite-dimensional spaces. With the above definitions in hand, the following lovely result holds.

Theorem 1.3 [4, 22]. *If X is a reflexive Kadec space and $C \subseteq X$ is closed, then C is almost proximinal.*

The assumptions on X are in fact necessary.

Theorem 1.4 [20]. *If X is not both Kadec and reflexive, then there exist $C \subseteq X$ closed and $U \subseteq X\backslash C$ open such that no $x \in U$ has a nearest point in C.*

It is known that under stronger assumptions on X one can obtain stronger results on the set $N(C)$ (see Section 1.4).

1.2. Fréchet sub-differentiability and nearest points. We begin with a definition.

Definition 1.5. Assume that $f : X \to \mathbb{R}$ is a real-valued function with $f(x)$ finite. Then f is said to be Fréchet sub-differentiable at $x \in X$ if there exists $x^* \in X^*$ such that

$$\liminf_{y\to 0} \frac{f(x+y) - f(x) - x^*(y)}{\|y\|} \geq 0. \tag{1.1}$$

The set of points in X^* that satisfy (1.1) is denoted by $\partial f(x)$.

Sub-derivatives have been found to have many applications in approximation theory (see, for example, [4, 5, 7, 8, 25]).

One of the connections between sub-differentiability and the nearest-point problem was studied in [4]. Given $C \subseteq X$ closed, the following modification of a construction of [22] was introduced. Consider

$$L_n(C) = \Big\{x \in X\backslash C : \exists x^* \in \mathbb{S}_{X^*} \text{ with } \sup_{\delta>0} \inf_{z\in C\cap B(x,d_C(x)+\delta)} x^*(x-z) > (1-2^{-n})d_C(x)\Big\},$$

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where $\mathbb{S}_{X^*}$ denotes the unit sphere of X^*. Also, let

$$L(C) = \bigcap_{n=1}^{\infty} L_n(C).$$

The following is known.

Proposition 1.6 [4]. *For every $n \in \mathbb{N}$, $L_n(C)$ is open. In particular, $L(C)$ is G_δ.*

Finally, let

$$\Omega(C) = \Big\{x \in X\backslash C : \exists x^* \in \mathbb{S}_{X^*}, \text{ such that } \forall \epsilon > 0, \exists \delta > 0,$$
$$\inf_{z \in C \cap B(x, d_C(x)+\delta)} x^*(x - z) > (1 - \epsilon) d_C(x)\Big\}.$$

While $L(C)$ is G_δ by Proposition 1.6, under the assumption that X is reflexive, the following is known.

Proposition 1.7 [4]. *If X is reflexive then $\Omega(C) = L(C)$. In particular, $\Omega(C)$ is G_δ.*

The connection to sub-differentiability is given in the following proposition.

Proposition 1.8 [4]. *If $x \in X\backslash C$ and $\partial d_C(x) \neq \emptyset$, then $x \in \Omega(C)$.*

The following fundamental result is available.

Theorem 1.9 [6]. *If f is lower semicontinuous on a reflexive Banach space, then f is Fréchet sub-differentiable on a dense set.*

In fact, Theorem 1.9 holds under a weaker assumption (see [4, 6]). Since the distance function is lower semicontinuous, it follows that it is sub-differentiable on a dense subset, and therefore, by the above propositions, $\Omega(C)$ is a dense G_δ set. Thus, in order to prove Theorem 1.3, it is only left to show that every $x \in \Omega(C)$ has a nearest point in C. Indeed, if $\{z_n\} \subseteq C$ is a minimising sequence, then by extracting a subsequence, assume that $\{z_n\}$ has a weak limit $z \in C$. By the definition of $\Omega(C)$, there exists $x^* \in \mathbb{S}_{X^*}$ such that

$$\|x - z\| \geq x^*(x - z) = \lim_{n\to\infty} x^*(x - z_n) \geq d_C(x) = \lim_{n\to\infty} \|x - z_n\|.$$

On the other hand, by weak lower semicontinuity of the norm,

$$\lim_{n\to\infty} \|x - z_n\| \geq \|x - z\|,$$

and so $\|x - z\| = \lim \|x - z_n\|$. Since it is known that $\{z_n\}$ converges weakly to z, the Kadec property implies that in fact $\{z_n\}$ converges in norm to z. Thus z is a nearest point. This completes the proof of Theorem 1.3.

This scheme of proof, taken from [4], shows that differentiation arguments can be fruitfully used to prove that $N(C)$ is large.

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1.3. Nearest points in non-Kadec spaces. It was previously mentioned that closed convex sets in reflexive spaces are proximinal. It also known that nonempty 'Swiss cheese' sets (sets whose complement is a mutually disjoint union of open convex sets) in reflexive spaces are almost proximinal [4]. These two examples show that for some classes of closed sets, the Kadec property can be removed. Moreover, one can consider another, weaker, way to 'measure' whether a set $C \subseteq X$ has 'many' nearest points: ask whether the set of nearest points in C to points in $X \backslash C$ is dense in the boundary of C. Note that if C is almost proximinal, then nearest points are dense in the boundary. The converse, however, is not true. In [4] an example of a non-Kadec reflexive space was constructed where for every closed set, the set of nearest points is dense in its boundary. The following general question is still open, even in renormings of Hilbert space.

Question 1.10. Let $(X, \|\cdot\|)$ be a reflexive Banach space and suppose $C \subseteq X$ is closed. Is the set of nearest points in C to points in $X \backslash C$ dense in its boundary?

Relatedly, if the set C is norm closed and bounded in a space with the *Radon–Nikodym property*, as is the case for a reflexive space, then $N(C)$ is nonempty and is large enough so that $\overline{\mathrm{conv}} C = \overline{\mathrm{conv}} N(C)$ [4].

1.4. Porosity and nearest points. As was mentioned in Section 1.2, one can consider stronger notions of 'large' sets. One is the following notion.

Definition 1.11. A set $S \subseteq X$ is said to be porous if there exists $c \in (0, 1)$ such that for every $x \in X$ and every $\epsilon > 0$, there is a $y \in B(0, \epsilon) \backslash \{0\}$ such that

$$B(x + y, c\|y\|) \cap S = \emptyset.$$

A set is said to be σ-porous if it a countable union of porous sets. Here and in what follows, $B(x, r)$ denotes the closed ball around x with radius r.

It is known that every σ-porous set is of the first category, that is, a union of nowhere dense sets. Moreover, it is also known that the class of σ-porous sets is a proper sub-class of the class of first category sets. When $X = \mathbb{R}^n$, one can show that every σ-porous set has Lebesgue measure zero. This is not the case for every first category set: $\mathbb{R}$ can be written as a disjoint union of a set of the first category and a set of Lebesgue measure zero. Hence, the notion of porosity automatically gives a stronger notion of large sets: every set whose complement is σ-porous is also a dense G_δ set. We recommend [23, 36] for a more detailed discussion on porous sets.

We recall that a Banach space $(X, \|\cdot\|)$ is said to be *uniformly convex* if the function

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x, y \in \mathbb{S}_X, \|x - y\| \geq \epsilon\right\} \tag{1.2}$$

is strictly positive whenever $\epsilon > 0$. Here $\mathbb{S}_X$ denotes the unit sphere of X. In [11] the following was shown.

Theorem 1.12 [11]. *If X is uniformly convex, then $N(C)$ has a σ-porous complement.*

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In fact, [11] contains a stronger result, namely that for every x outside a σ-porous set, the minimisation problem is *well posed*, that is, there is unique minimiser to which every minimising sequence converges. See also [16, 27, 28] for closely related results in this direction.

The proof of Theorem 1.12 builds on ideas developed in [30]. It would, however, be interesting to know whether one may use differentiation arguments as in Section 1.2. This raises the following question.

QUESTION 1.13. Can differentiation arguments be used to give an alternative proof of Theorem 1.12?

More specifically, if one can show that $\partial d_C \neq \emptyset$ outside a σ-porous set, then by the arguments presented in Section 1.2, it would follow that $N(C)$ has a σ-porous complement. Next, we mention two important results regarding differentiation in Banach spaces. See also [23, Section 3.3].

THEOREM 1.14 [26]. *If X has a separable dual and $f : X \to \mathbb{R}$ is continuous and convex, then X is Fréchet differentiable outside a σ-porous set.*

Theorem 1.14 implies that if, for example, d_C is a linear combination of convex functions (see more on this in Section 2), then $N(C)$ has a σ-porous complement. Also, we have the following theorem.

THEOREM 1.15 [10]. *If X has a separable dual and $f : X \to \mathbb{R}$ is Lipschitz, then the set of points where f is Fréchet sub-differentiable but not differentiable is σ-porous.*

Since d_C is 1-Lipschitz (nonexpansive), the issues from a porosity perspective of seeking points of sub-differentiability or points of differentiability are similar. We also observe that Theorems 1.14 and 1.15 remain true if we consider $f : A \to \mathbb{R}$ where $A \subseteq X$ is open and convex.

We now turn from results depending on the geometry of the space to those exploiting the finer structure of d_C or C.

2. DC functions and DC sets

2.1. Background.

DEFINITION 2.1. A function $f : X \to \mathbb{R}$ is said to be delta convex (DC) if it can be written as a difference of two convex functions on X.

This notion was introduced in [18] and was later studied by many authors, see, for example, [2, 8, 9, 12, 14, 21, 24, 34]. In particular, [2] gives a concise introduction to this topic. We will discuss here only the parts that are closely related to the nearest-point problem.

The following is an important and attractive proposition. See, for example, [19, 33] for a proof.

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Proposition 2.2. *Assume that $f_1, \dots, f_k$ are DC functions and $f : X \to \mathbb{R}$ is continuous and $f(x) \in \{f_1(x), \dots, f_n(x)\}$ for every $x \in X$. Then f is also DC.*

The result remains true if we replace the domain X by any convex subset.

2.2. DC functions and nearest points. Showing that a given function is in fact DC is a powerful tool, as it allows us to use many known results about convex and DC functions. For example, if a function is DC on a Banach space with a separable dual, then by Theorem 1.14, it is differentiable outside a σ-porous set. In the context of the nearest-point problem, if we know that the distance function is DC, then using the scheme presented in Section 1.2, it follows that $N(C)$ has a σ-porous complement. The same holds if we have a difference of a convex function and, say, a smooth function.

The simplest and best-known example, originally due to Asplund, is that when $(X, \|\cdot\|)$ is a Hilbert space, we have

$$\begin{aligned} d_C^2(x) &= \inf_{y\in C} \|x-y\|^2 \\ &= \inf_{y\in C} [\|x\|^2 - 2\langle x, y\rangle + \|y\|^2] \\ &= \|x\|^2 - 2\sup_{y\in C}[\langle x, y\rangle - \|y\|^2/2], \end{aligned}$$

and the function $x \mapsto \sup_{y\in C}[\langle x, y\rangle - \|y\|^2/2]$ is convex as a supremum of affine functions. Hence d_C^2 is DC on X. Moreover, in a Hilbert space we have the following result (see [8, Section 5.3]).

Theorem 2.3. *If $(X, \|\cdot\|)$ is a Hilbert space, d_C is locally DC on $X\backslash C$.*

Proof. Fix $y \in C$ and $x_0 \in X\backslash C$. It can be shown that if we let $f_y(x) = \|x - y\|$, then f_y satisfies

$$\|f_y'(x_1) - f_y'(x_2)\|_{X^*} \le L_{x_0}\|x_1 - x_2\|, \quad x_1, x_2 \in B_{x_0},$$

where $L_{x_0} = 4(d_S(x_0))^{-1}$ and $B_{x_0} = B(x_0, \frac{1}{2}d_C(x_0))$. In particular,

$$(f_y'(x + tv_1) - f_y'(x + t_2v))(v) \le L_{x_0}(t_2 - t_1), \quad v \in \mathbb{S}_X,\ t_2 > t_1 \ge 0, \tag{2.1}$$

whenever $x + t_1v, x + t_2v \in B_{x_0}$. Next, the convex function $F(x) = \frac{1}{2}L_{x_0}\|x\|^2$ satisfies

$$(F'(x_1) - F'(x_2))(x_1 - x_2) \ge L_{x_0}\|x_1 - x_2\|^2 \quad \forall x_1, x_2 \in X. \tag{2.2}$$

In particular,

$$(F'(x + t_2v) - F'(x + t_1v))(v) \ge L_{x_0}(t_2 - t_1), \quad v \in \mathbb{S}_X,\ t_2 > t_1 \ge 0. \tag{2.3}$$

Altogether, if $g_y(x) = F(x) - f_y(x)$, then

$$(g_y'(x + t_2v) - g_y'(x + t_1v))(v) \overset{(2.1)\wedge(2.3)}{\ge} 0, \quad v \in \mathbb{S}_X,\ t_2 > t_1 \ge 0,$$

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whenever $x + t_1 v, x + t_2 v \in B_{x_0}$. This monotonicity implies that g_y is convex on B_{x_0} [7]. It then follows that

$$d_C(x) = \tfrac{1}{2}L_{x_0}\|x\|^2 - \sup_{y\in C}[\tfrac{1}{2}L_{x_0}\|x\|^2 - \|x-y\|] = h(x) - \sup_{y\in C} g_y(x)$$

is DC on B_{x_0}. □

REMARK 2.4. Even in $\mathbb{R}^2$ there are sets for which d_C is not DC everywhere (not even locally DC), as was shown in [2]. Thus, the most one could hope for in Theorem 2.3 is a locally DC function on $X\backslash C$.

Given $q \in (0, 1]$, a norm $\|\cdot\|$ is said to be *q-Hölder smooth* at a point $x \in X$ if there exists a constant $K_x \in (0, \infty)$ such that for every $y \in \mathbb{S}_X$ and every $\tau > 0$,

$$\frac{\|x+\tau y\|}{2} + \frac{\|x-\tau y\|}{2} \le 1 + K_x \tau^{1+q}.$$

If $q = 1$ then $(X, \|\cdot\|)$ is said to be *Lipschitz smooth* at x. The spaces L_p, $p \ge 2$, are known to be Lipschitz smooth. In general, L_p, $p > 1$, is s-Hölder smooth with $s = \min\{1, p-1\}$.

A Banach space is said to be *p-uniformly convex* if for every $x, y \in \mathbb{S}_X$,

$$1 - \left\|\frac{x+y}{2}\right\| \ge L\|x-y\|^p.$$

Note that this is similar to assuming that $\delta(\epsilon) = L\epsilon^p$ in (1.2). The spaces L_p, $p > 1$, are r-uniformly convex with $r = \max\{2, p\}$.

One might well ask whether the scheme of proof of Theorem 2.3 can be used in a more general setting. The results which follow indicate that this is not possible.

PROPOSITION 2.5. *Let $(X, \|\cdot\|)$ be a Banach space and $C \subseteq X$ a closed set, and fix $x_0 \in X\backslash C$ and $y \in C$. Assume that there exists r_0 such that $f_y(x) = \|x-y\|$ has a Lipschitz derivative on $B(x_0, r_0)$:*

$$\|f_y'(x_1) - f_y'(x_2)\| \le L_{x_0}\|x_1 - x_2\|. \tag{2.4}$$

Then the norm is Lipschitz smooth on $-y + B_{x_0} = B(x_0 - y, r_0)$. If, in addition, there exists a function $F : X \to \mathbb{R}$ satisfying

$$(F'(x_1) - F'(x_2))(x_1 - x_2) \ge L_{x_0}\|x_1 - x_2\|^2 \quad \forall x_1,\ x_2 \in B(x_0, r_0), \tag{2.5}$$

then $(X, \|\cdot\|)$ admits an equivalent norm which is 2-uniformly convex. In particular, if $X = L_p$ then $p = 2$.

PROOF. To prove the first assertion, note that (2.4) is equivalent to

$$\|x-y+h\| + \|x-y-h\| - 2\|x-y\| \le L_{x_0}\|h\|^2, \quad x \in B_{x_0}$$

(see, for example, [15, Proposition 2.1]).

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To prove the second assertion, note that a function that satisfies (2.5) is also known as *strongly convex*: one can show that (2.5) is in fact equivalent to the condition

$$f\Big(\frac{x_1+x_2}{2}\Big) \le \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) - C\|x_1 - x_2\|^2,$$

for some constant C (see, for example, [29, Appendix A]). This implies that there exists an equivalent norm which is 2-uniformly convex [7, Theorem 5.4.3]. □

REMARK 2.6. From [1] it is known that if $F : X \to \mathbb{R}$ satisfies

$$(F'(x_1) - F'(x_2))(v) \ge L\|x_1 - x_2\|^2,$$

for *all* $x_1, x_2 \in X$, and F is also twice (Fréchet) differentiable at one point, then $(X, \|\cdot\|)$ is isomorphic to a Hilbert space.

REMARK 2.7. If we replace the Lipschitz condition by a Hölder condition

$$\|f'_y(x_1) - f'_y(x_2)\| \le \|x_1 - x_2\|^\beta, \quad \beta < 1,$$

then in order to follow the same scheme as in the proof of Theorem 2.3, instead of (2.2) we would need a function F satisfying

$$(F'(x_1) - F'(x_2))(x_1 - x_2) \ge \|x_1 - x_2\|^{1+\beta}, \quad x_1, x_2 \in B_{x_0},$$

which implies

$$\|F'(x_1) - F'(x_2)\| \ge \|x_1 - x_2\|^\beta, \quad x_1, x_2 \in B_{x_0}. \tag{2.6}$$

If $G = (F')^{-1}$, then we get

$$\|Gx_1 - Gx_2\| \le \|x_1 - x_2\|^{1/\beta}, \quad x_1, x_2 \in F'(B_{x_0}),$$

which can occur only if G is a constant. Hence (2.6) cannot hold and the scheme of proof cannot be generalised if we replace the Lipschitz condition by a Hölder condition.

2.3. DC sets and DC representable sets. The next definition is quite natural.

DEFINITION 2.8. A set C is is said to be a DC set if $C = A\backslash B$ where A, B are convex.

We can also consider the following class of sets.

DEFINITION 2.9. A set $C \subseteq X$ is said to be DC representable if there exists a DC function $f : X \to R$ such that $C = \{x \in X : f(x) \le 0\}$.

Note that if $C = A\backslash B$ is a DC set, then we can write $C = \{\mathbb{1}_B - \mathbb{1}_A + 1/2 \le 0\}$, where $\mathbb{1}_A$, $\mathbb{1}_B$ are the convex indicator functions of A, B, respectively. Therefore, C is DC representable. Moreover, we have the following theorem.

THEOREM 2.10 [31]. *Assume that X and Y are two Banach spaces and $T : Y \to X$ is a surjective bounded linear map which is not an isomorphism, that is,* $\ker(T) \ne \{0\}$. *Then for any set $M \subseteq X$ there exists a DC representable set $D \subseteq Y$, such that $M = T(D)$.*

Also, the following 'converse' is known (see [19]).

Proposition 2.11. *If C is a DC representable set, then there exist $A, B \subseteq X \oplus \mathbb{R}$ convex, such that $x \in C \iff (x, x') \in A \backslash B$.*

Proof. Suppose that $C = \{x \in X : f(x) \leq 0\}$ where $f = f_1 - f_2$ and f_1, f_2 are convex. Define $g_1(x, x') = f_1(x) - x'$, $g_2(x, x') = f_2(x) - x'$. Let $A = \{(x, x') : g_1(x, x') \leq 0\}$, $B = \{(x, x') : g_2(x, x') \leq 0\}$. Then $x \in C \iff (x, x') \in A \backslash B$. □

In particular, every DC representable set in X is a projection of a DC set in $X \oplus \mathbb{R}$. The next theorem was proved in [32].

Theorem 2.12 [32]. *If X is a reflexive Banach space and $C \subseteq X$ is closed, then C is DC representable.*

This makes relevant the following question.

Question 2.13. Are there any classes of spaces X, say uniformly convex spaces, such that there exists $\alpha > 0$ such that d_C^α is locally DC on $X \backslash C$ whenever C is a DC representable set?

If the answer to Question 2.13 is positive, then by the discussion in Section 1.2 we can conclude that $N(C)$ has a σ-porous complement, thus giving an alternative proof of Theorem 1.12. One may also ask Question 2.13 for DC sets instead of DC representable sets.

To end this note, we discuss some simple cases where DC and DC representable sets can be used to study the nearest-point problem.

Proposition 2.14. *Assume that $C = X \backslash \bigcup_{a \in \Lambda} U_a$, where each U_a is an open convex set. Then d_C is locally DC (in fact, locally concave) on $X \backslash C$.*

Proof. First, it is shown in [4, Section 3] that if $a \in \Lambda$, then $d_{X \backslash U_a}$ is concave on U_a. Next, it is also shown in [4] that if $x \in U_a$ then $d_{X \backslash U_a}(x) = d_C(x)$. In particular, d_C is concave on U_a. □

Proposition 2.15. *Assume that $C = A \backslash B$ is a closed DC set and that A is closed and B is open. Then d_C is convex whenever $d_C(x) \leq d_{A \cap B}$.*

Proof. Since $A = (A \backslash B) \cup B$, we have

$$d_A(x) = \min\{d_{A \backslash B}(x), d_{A \cap B}(x)\} = \min\{d_C(x), d_{A \cap B}(x)\}.$$

Hence, if $d_C(x) \leq d_{A \cap B}(x)$ then $d_C(x) = d_A(x)$ is convex. □

Proposition 2.16. *Assume that C is a DC representable set, with the representation $C = \{x \in X : f_1(x) - f_2(x) \leq 0\}$, and that $f_2(x) = \max_{1 \leq i \leq m} \varphi_i(x)$, where φ_i is affine. Then d_C is DC on X.*

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Proof. Write

$$\begin{aligned} C &= \{x : f_1(x) - f_2(x) \le 0\} \\ &= \Big\{x : f_1(x) - \max_{1\le i\le m} \varphi_i(x) \le 0\Big\} \\ &= \Big\{x : \min_{1\le i\le m} (f_1(x) - \varphi_i(x)) \le 0\Big\} \\ &= \bigcup_{i=1}^{n} \{x : f_1(x) - \varphi_i(x) \le 0\}, \end{aligned}$$

where the sets $\{x : f_1(x) - \varphi_i(x) \le 0\}$ are convex sets. Hence,

$$d_C(x) = \min_{1\le i\le m} d_{C_i}(x)$$

is a minimum of convex sets and therefore, by Proposition 2.2, a DC function. □

In [9] it was shown that if X is super-reflexive, then any Lipschitz map is a uniform limit of DC functions. See also [7, Section 5.1]. We have the following simple partner result.

Proposition 2.17. *Suppose that X is separable. Then d_C is a limit (not necessarily uniform) of DC functions.*

Proof. If X is separable, that is, there exists a countable set $Q = \{q_1, q_2, \ldots\} \subseteq X$ with $\overline{Q} = X$, we have

$$d_C(x) = \inf_{z\in C} \|x - z\| = \inf_{z\in C\cap Q} \|x - z\| = \lim_{n\to\infty} \Big[\min_{z\in C\cap Q_n} \|x - z\|\Big],$$

where $Q_n = \{q_1, q_2, \ldots, q_n\}$. Again by Proposition 2.2, we have that $\min_{z\in C\cap Q_n} \|x - z\|$ is a DC function as a minimum of convex functions. □

3. Conclusion

Despite many decades of study, the core questions addressed in this note are still far from settled. We hope that our analysis will encourage others to take up the quest and also to reconsider the related Chebshev problem [3, 7, 17].

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