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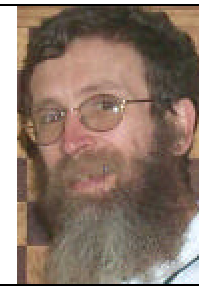
How Euler Did It

by Ed Sandifer



Derangements

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Euler worked for a king, Frederick the Great of Prussia. When the King asks you to do something, he's not really "asking." In the late 1740's and early 1750's, the King "asked" Euler to work on a number of practical problems. For example, the King had a party palace named Sans Souci. Euler was asked to design the hydraulics to run the fountains at Sans Souci. He also asked Euler to do the engineering on a canal. Another time, when the King was running out of money, he asked Euler to calculate the probabilities so the King could try to pay his debts by running a lottery.

At about the same time, Euler was turning his talents to analyzing ordinary and frivolous things. He solved the Königsburg Bridge Problem, and the Knight's Tour problem, as well as analyzing some lotteries other than the one the King asked about. Among these other problems was a card game, called "le jeu de rencontre," or "the game of coincidence." He reported his results in a paper, E-201, published in the *Mémoires* of the Berlin Academy under the title "Calcul de la probabilité dans le jeu de rencontre." Richard Pulskamp's translation of this article is available on line [P], and the original, besides appearing in Series I Volume 7 of the *Opera Omnia*, is on line through the Euler Archive [EA] and the Berlin Academy [B].

CALCUL DE LA PROBABILITE'
DANS LE JEU DE RENCONTRE,
PAR M. EULER.

Rencontre takes two players, whom Euler names A and B. Their descendents still populate mathematics problems worldwide. The players have identical decks of cards. They both turn over cards, one at a time and at the same time. If they turn over the same card at the same time, there is a coincidence, and A wins. If they go all the way through the deck without a coincidence, then B wins.

The problem is to calculate the probability that A will win. The probability will, of course, depend on the number of cards in the decks. Euler takes this number to be m. The problem still appears in many modern texts on probability, and the solutions given usually resemble Euler's. Since Jakob Bernoulli's *Ars Conjectandi* had appeared in 1713, Euler has at his command many of the standard tools of probability. In particular, we assume that all of the m! possible permutations of the deck of cards are equiprobable (Euler uses neither the notation $m!$, nor the term “equiprobable”) and that the probability that A wins is the number of successful arrangements divided by the number of possible arrangements.

First, we get a couple of simplifying assumptions that do not cost us any generality. First, we assume that the cards have numbers, 1, 2, 3, … m, rather than designs. Second, we assume that A turns over cards in the order 1, 2, 3, … m, so that the outcome of the game depends only on the order of B’s cards.

Euler proceeds in his classical expository style. He starts with the easiest examples, m = 1, 2, 3 and 4. He does the case m = 4 two different ways, with the second method providing the idea that leads to a general solution.

The case m = 1 is trivial; A wins. If m = 2, there are two arrangements for B’s cards, 1,2 and 2,1. A wins in the first case and loses in the second, so the probability that A wins is ½.

We get some hints that Euler has some interesting ideas when he shows us the case m = 3. He gives us the table below, enumerating the six possible orders for B’s cards, and with some entries crossed out. Before we explain what Euler says about this table, the reader should try to figure out what the table is about on his/her own:

	B					
A	1	2	3	4	5	6
1	1	1	2	2	3	3
2	~~2~~	~~3~~	1	3	2	1
3	~~3~~	~~2~~	3	1	~~1~~	2

One thing to do would be to cross out all the 1’s in row 1, all the 2’s in row 2 and the 3’s in row 3. Then, any column that still has all its numbers represents an arrangement of B’s cards that results in a win for B. This would show that B wins twice, columns 4 and 6, and A wins four times.

That’s not what the table does. Columns 1 and 2 describe games in which A wins on the first move, so Euler has crossed out all the outcomes after row 1; they don’t matter. Column 5 represents the game in which A wins on the second move, so Euler has crossed out the outcome after row 2. Column 3 describes the game in which A wins on the last move. There’s nothing left to cross out, so Euler explains in the text that Column 3 represents the game in which A wins on the last move.

This is our first clue to what Euler intends to count. He will calculate the number of ways that A can win on move i if there are m cards. So far, his table would look kind of like this:

		Number of cards		
		I	II	III
Number of ways that A can win on move number --	I	1	1	2
	II		0	1
	III			1

That is skipping ahead a bit, though. Euler sticks to form and considers the case $m = 4$. He gives the following table:

A	B 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4
2	2	2	3	3	4	4	3	3	4	4	1	1	4	4	1	1	2	2	1	1	2	2	3	3
3	3	4	4	2	2	3	4	1	1	3	4	3	1	2	2	4	1	4	2	3	3	1	1	2
4	4	3	2	4	3	2	1	4	3	1	3	4	2	3	4	2	4	1	3	2	1	3	2	1

It takes some squinting, but we see that there are 6 ways that A wins on the first move, 4 ways to win on the second, 3 ways to win on the third, and 2 ways to win on the last move. This would add another column of data to the table we made a little earlier.

Now, Euler sets out to figure out how the table works. He studies the case $m = 4$ and asks about the games in which A wins on the third card. He extracts from the table above all the games for which there is a 3 in row 3, and gets the following sub-table:

A	B					
1	1	1	2	2	4	4
2	2	4	4	1	1	2
4	4	2	1	4	2	1

We notice that this table is almost exactly like the first table, the table of outcomes for the 3-card game, but all the 3's have been changed to 4's (though, for no apparent reason, the columns have been rearranged a little bit.) From these, he takes away those games in which A wins on the first card (two games) or the second card (one game), and the three games that remain must be the ones in which A wins the 4-card game on the third card. Euler has discovered the seeds of a recurrence relation, by which the number of ways to win a 4-card game on a particular moved depends on the number of ways of winning various 3-card games. It will take some notation to untangle it. Unfortunately, subscripts had not yet been invented, so Euler has to make do without them.

Suppose there are m cards in the deck, and that the total number of possible games is M. We know that $M = m!$, but the factorial notation hadn't been invented yet, either. Now, let

a be the number of cases for which A wins on the first move,
b be the number for which A wins on the second move,
c be the number for the third move,
etc.

Easy analysis shows that $a = \frac{M}{m}$.

Now, consider the game with m + 1 cards. Euler denotes by *M'*, *a'*, *b'*, *c'*, etc. the corresponding numbers for the larger game, and asks how the numbers for the (m + 1)-card game are related to those for the m-card game.

Some parts are easy; *M'* = M (m + 1) and $a' = \frac{M'}{m+1} = M$.

Now, there are *M* cases in which A turns over a 2 on the second card, but some of these must be excluded, since they are cases in which A has already turned over a 1 on the first card. The analysis Euler did on the reduced table tells us to look at the m – card games to find that there are a such arrangements, so that

$$b' = M - a.$$

Likewise, there are *M* cases in which A turns over a 3 on the third card, but from these we must subtract those cases in which A has already won on the first or the second card. That is

$$c' = M - a - b.$$

The pattern continues. We can write these relations in a simpler form:

$$a' = M$$
$$b' = a' - a$$
$$c' = b' - b$$
etc.

Euler uses these results to calculate the following table for up to 10 cards. This is the same table we derived ourselves for up to 3 cards.

NOMBRE DES CARTES

	I	II	III	IV	V	VI	VII	VIII	IX	X
a	1	1	2	6	24	120	720	5040	40320	362880
b	-	0	1	4	18	96	600	4320	35280	322560
c	-	-	1	3	14	78	504	3720	30960	287280
d	-	-	-	2	11	64	426	3216	27240	256320
e	-	-	-	-	9	53	362	2790	24024	229080
f	-	-	-	-	-	44	309	2428	21234	205056
g	-	-	-	-	-	-	265	2119	18806	183822
h	-	-	-	-	-	-	-	1854	16687	165016
i	-	-	-	-	-	-	-	- -	14833	148329
k	-	-	-	-	-	-	-	- -	- -	133496

The hard work is over, but Euler promised to calculate the probabilities, too. Let A be the probability that A wins on the first move of an n – 1-card game, B the probability he wins on the second, C the probability he wins on the third, and so forth, and let N be the number of possible n – 1 – card games. That is, $N = (n-1)!$. Similarly, let A', B', C' and N' be the corresponding probabilities for an n – card game.

It is easy to see that $A = \frac{a}{N} = \frac{1}{n-1}$ and $A' = \frac{a'}{N'} = \frac{1}{n}$.

Now, $B' = \frac{b'}{N'}$. But, $b' = a' - a$, and $N' = nN$, so

$$B' = \frac{b'}{N'} = \frac{a'-a}{N'} = \frac{a'}{N'} - \frac{a}{N'} = \frac{1}{n} - \frac{a}{nN} = \frac{1}{n} - \frac{1}{n(n-1)} = \frac{n-2}{n(n-1)}$$

Changing n's to $(n-1)$'s gives that

$$B = \frac{n-3}{(n-2)(n-1)}$$

Similar calculations show a clear pattern:

$$A' = \frac{1}{n}$$

$$B' = \frac{1}{n} - \frac{1}{n(n-1)}$$

$$C' = \frac{1}{n} - \frac{2}{n(n-1)} + \frac{1}{n(n-1)(n-2)}$$

$$D' = \frac{1}{n} - \frac{3}{n(n-1)} + \frac{3}{n(n-1)(n-2)} - \frac{1}{n(n-1)(n-2)(n-3)}$$

Numerators are rows from Pascal's triangle. Denominators are permutation numbers.

Euler sums the columns. The probability we seek, that player A wins on *some* move, is the sum of the n probabilities on the left hand side, $A' + B' + C' + \dots$. He sums the right hand sides as columns, since the denominators match. This is easier than it looks, since he knows lots of identities about Pascal's triangle.

For the first column,

$$1 + 1 + \dots + 1 = n,$$

so the first term of the sum will be $n/n = 1$.

For the second column,

$$1 + 2 + 3 + \dots + n = \frac{n(n-1)}{2}$$

so the second term of the sum will be $\frac{-\left(\frac{n(n-1)}{2}\right)}{n(n-1)} = \frac{-1}{1 \cdot 2}$

For the third column,

$$1 + 3 + 6 + \dots + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

so the third term of the sum will be $\frac{1}{1 \cdot 2 \cdot 3}$.

Wow! Numerators alternate 1 and –1, while denominators are factorials! So, the probability that A wins playing with an n – card deck is

$$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ldots\pm\frac{1}{n!}$$

As n grows, this converges rapidly to $1/e$. For n = 10, it is already accurate to six decimal places. It is an astonishing result.

Now, about our title, "derangements." In discrete mathematics, combinatorics and abstract algebra courses, we learn about permutations, one-to-one and onto functions from a set to itself. They have all sorts of wonderful properties; they form a group and they are fun to count.

A *derangement* is a special kind of permutation, s, with no fixed points. That is, it never happens that $s(x) = x$. As a permutation, *everything* gets moved. Derangements correspond to those rearrangements of the deck for which A wins the game of rencontre.

Derangements sometimes appear as "the hat-check problem." One (obviously rather dated) version goes like this:

> Ten men go into a restaurant and check their hats. As they are leaving, the lights go out, and each man gets a hat at random. What is the probability that at least one man gets his own hat?

This is obviously 10-card rencontre in disguise. Now that we know how Euler did it, we know that the answer, to at least six decimal places, is $\frac{1}{e}$.

References:

[B] Digitalisierte Akademieschriften und Schriften zur Geschichte der Königlich Preussischen Akademie der Wissenschaften (Digital library of the Royal Prussian Academy of Sciences, 1700-1900) http://www.bbaw.de/bibliothek/digital/.

[EA] The Euler Archive, www.EulerArchive.org

[E201] Euler, Leonhard, "Calcul de la probabilité dans le jeu de rencontre", M*émoires de l'Academie des Sciences de Berlin*, **7** (1751) 1753, pp. 255-270, reprinted in *Opera Omnia* Series I vol 7 pp. 11-25. Available through The Euler Archive at , www.EulerArchive.org.

[P] Pulskamp, Richard, "Leonhard Euler on Probability and Statistics," http://cerebro.xu.edu/math/Sources/Euler/index.html.

All web links were alive on August 23, 2004.

Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 32 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org)
