The hidden face of the $3x + 1$ problem
Part I : Intrinsic algorithm

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Abstract

In 1937 Lothar Collatz stated a conjecture and of 2018, the problem is still open. Jeffrey C. Lagarias published two detailed bibliographies [1, 2] on the $3x + 1$ problem, that are periodically annotated. In this paper, we define the intrinsic algorithm derivated from the original one, and build by induction the tree which allows to go up in trajectories of naturals by this algorithm. We define the two-dimensional map M, use it to compute the first iterations of the intrinsic algorithm starting from any odd natural, and prove the existency of nodes between trajectories.

Keywords: 3x+1 problem, Collatz conjecture

1 Intrinsic algorithm of the $3x + 1$ problem

In this section we recall the definitions of the Collatz and the reduced Collatz functions, then we define the step by step algorithm and finally the intrinsic algorithm. We use the notations: $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ for the set of nonzero naturals, and $\mathcal{O} = \{2k + 1 \,|\, k \in \mathbb{N}\}$ for the set of odd naturals.

1.1 Recall: Collatz algorithm

The Collatz function is the map $f_c : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ such that:

$$\forall x \in \mathbb{N}^* \quad \begin{cases} x \equiv 0 \pmod 2 \implies & f_c(x) = x/2 \\ x \equiv 1 \pmod 2 \implies & f_c(x) = 3x + 1 \end{cases}$$

The Collatz conjecture states: $\forall x \in \mathbb{N}^* \quad \exists n \in \mathbb{N} \quad f_c^n(x) = 1$

We observe the trajectories end in the cycle $4 \mapsto 2 \mapsto 1 \mapsto 4$
The Collatz tree looks irregular.

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1.2 Recall: Reduced Collatz algorithm

Let the function $\rho : \mathbb{N}^* \longrightarrow \mathcal{O}$ such that:
$\quad \forall (n,k) \in \mathbb{N}^2 \quad \rho\,(2^n\,(2k+1)) = 2k+1$
The reduced Collatz function is the map $f_{cr} : \mathcal{O} \longrightarrow \mathcal{O}$ such that:
$\quad \forall x \in \mathcal{O} \quad f_{cr}(x) = \rho(3x+1)$
The Collatz conjecture is equivalent to: $\forall x \in \mathcal{O} \quad \exists n \in \mathbb{N} \quad f_{cr}^n(x) = 1$

1.3 Definition: Step by step algorithm

The map f_{cr} defined in 1.2 verifies the properties:

$$\forall k \in \mathbb{N} \quad \begin{cases} f_{cr}(3+4k) = 5+6k \\ f_{cr}(1+8k) = 1+6k \\ f_{cr}(5+8k) = f_{cr}(1+2k) \end{cases}$$

The classes $\{3+4k \,|\, k \in \mathbb{N}\}$, $\{1+8k \,|\, k \in \mathbb{N}\}$, $\{5+8k \,|\, k \in \mathbb{N}\}$ achieving a partition of odd naturals, these properties are sufficient to define f_{cr}

Let the map $f_s : \mathcal{O} \longrightarrow \mathcal{O}$ such that:

$$\forall k \in \mathbb{N} \quad \begin{cases} f_s(3+4k) = 5+6k \\ f_s(1+8k) = 1+6k \\ f_s(5+8k) = 1+2k \end{cases}$$

The behavior of the algorithm obtained by iteration of f_s is same as the one obtained by iteration of f_{cr} , the only difference is that f_s inserts steps:

$$5+8k \xrightarrow{f_{cr}} f_{cr}(1+2k)$$
$$5+8k \xrightarrow{f_s} 1+2k \xrightarrow{f_s} f_s(1+2k)$$

The Collatz conjecture is equivalent to: $\forall x \in \mathcal{O} \quad \exists n \in \mathbb{N} \quad f_s^n(x) = 1$

1.4 Definition: Intrinsic algorithm

Let f_s the map defined in 1.3

Let φ the bijection $\mathbb{N} \longrightarrow \mathcal{O} \qquad x \longmapsto \varphi(x) = 2x+1$

Let the map $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by: $f = \varphi^{-1} \circ f_s \circ \varphi$

$$\forall k \in \mathbb{N} \quad \begin{cases} f(1+2k) = 2+3k \\ f(4k) = 3k \\ f(2+4k) = k \end{cases}$$

$$\forall x \in \mathbb{N} \quad \begin{cases} x \equiv 1 \pmod 2 \implies f(x) = f_1(x) = (3x+1)/2 \\ x \equiv 0 \pmod 4 \implies f(x) = f_2(x) = 3x/4 \\ x \equiv 2 \pmod 4 \implies f(x) = f_3(x) = (x-2)/4 \end{cases}$$

The Collatz conjecture is equivalent to: $\forall x \in \mathbb{N} \quad \exists n \in \mathbb{N} \quad f^n(x) = 0$

Figure 1: Tree of the step by step algorithm f_s

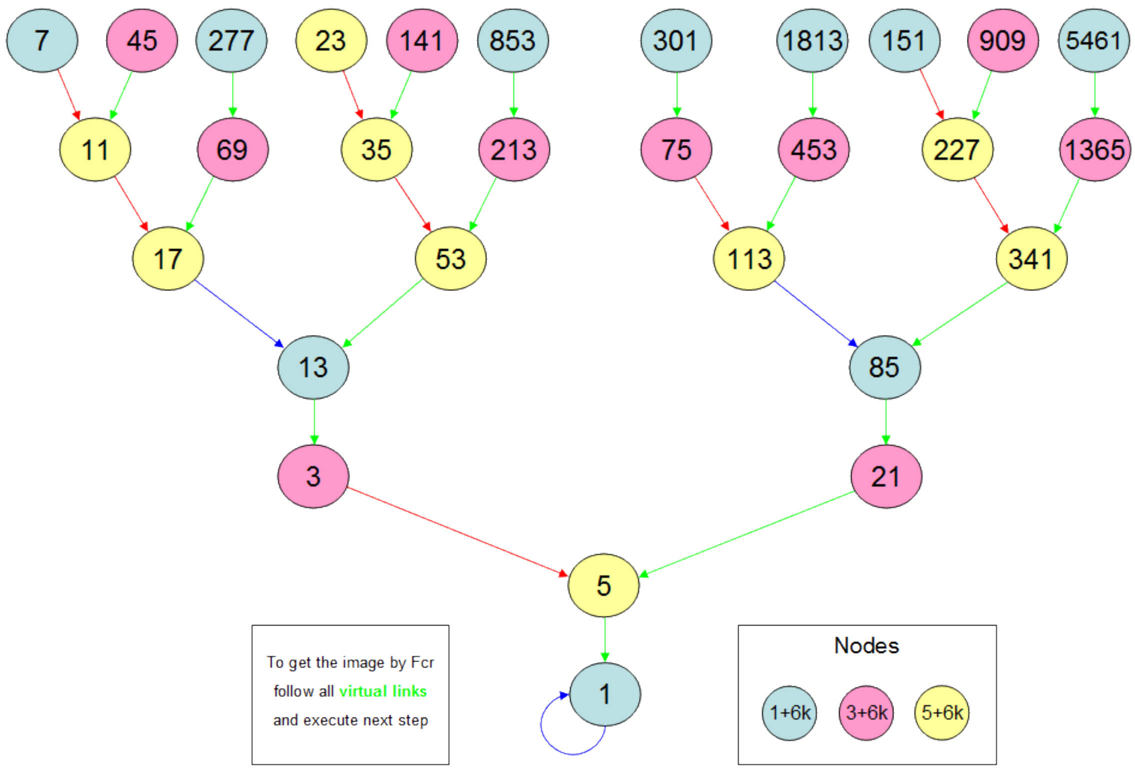
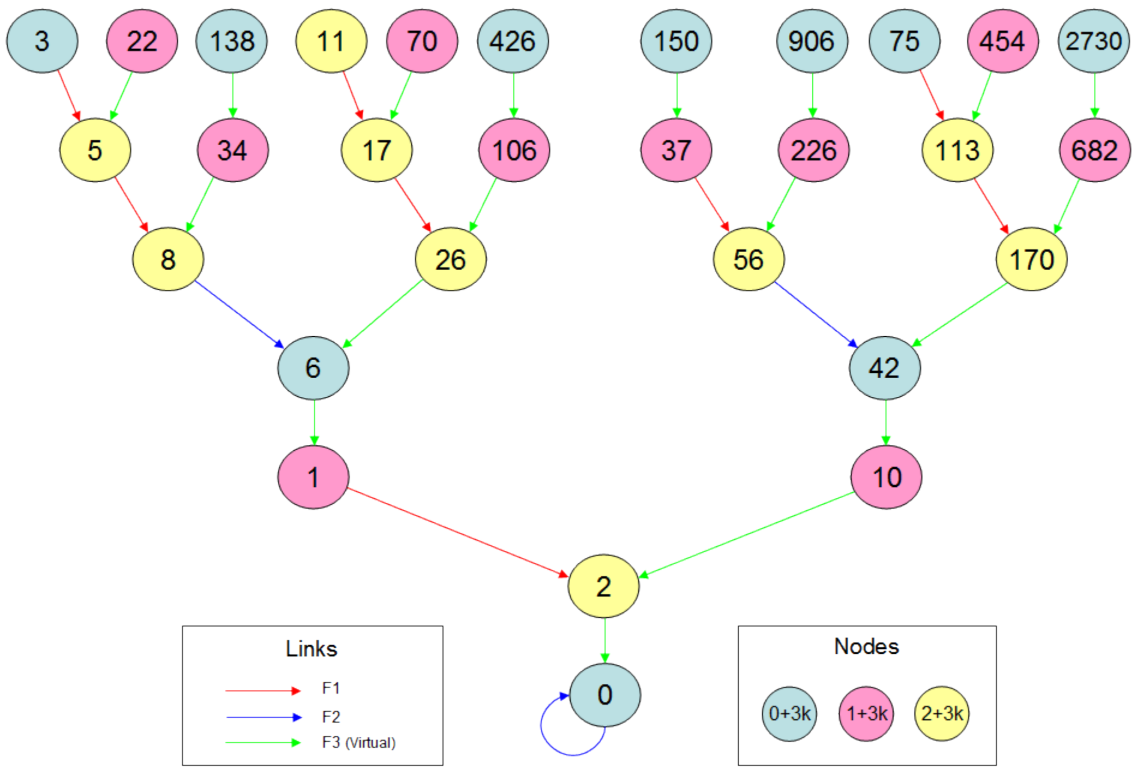


Figure 2: Tree $\mathbb{E}$ of the intrinsic algorithm f



2 Tree of antecedents for the intrinsic algorithm

In this section we construct by induction the tree of antecedents for the intrinsic algorithm, starting from zero. Any natural x gets the antecedent $f_3^{-1}(x) = 4x+2$ a natural $x \equiv 2 \pmod 3$ gets the second antecedent $f_1^{-1}(x) = (2x-1)/3$, and a natural $x \equiv 0 \pmod 3$ gets the second antecedent $f_2^{-1}(x) = 4x/3$. We also define the binary relation $\boxtimes$ which express the existency of a common node between two trajectories by the intrinsic algorithm.

2.1 Definition: Set $\mathbb{E}$

Let f the map defined in 1.4

Let $\begin{cases} \mathbb{A}_0 = \{0\} \\ \mathbb{A}_1 = \{2\} \\ \forall n \in \mathbb{N}^* \quad \mathbb{A}_{n+1} = \{x \in \mathbb{N} \,|\, f(x) \in \mathbb{A}_n\} \end{cases}$

Let $\forall n \in \mathbb{N} \quad \mathbb{E}_n = \mathbb{A}_0 \bigcup \ldots \bigcup \mathbb{A}_n$

Let $\mathbb{E} = Lim_{\,n\to\infty}\, \mathbb{E}_n$

The following statements hold:

$$\forall n \in \mathbb{N}^* \quad \mathbb{A}_n = \{x \in \mathbb{N} \,|\, (f^n(x) = 0) \text{ and } (\forall p \in \mathbb{N} \ (p < n) \Longrightarrow (f^p(x) \neq 0))\}$$
$$\forall (p,q) \in \mathbb{N}^2 \quad (p \neq q) \Longrightarrow (\mathbb{A}_p \cap \mathbb{A}_q = \emptyset)$$

The Collatz conjecture is equivalent to: $\mathbb{E} = \mathbb{N}$

2.2 Definition: Binary relation $\boxtimes$

Let f the map defined in 1.4. We define the binary relation $\boxtimes$ as follow:

$$\forall (x,y) \in \mathbb{N}^2 : \quad (x \boxtimes y) \Longleftrightarrow \left(\exists (p,q) \in \mathbb{N}^2 \quad f^p(x) = f^q(y)\right)$$

$\boxtimes$ is reflexive, symmetrical and transitive.
The following fundamental statement holds:

$$\forall\, (x,y) \in \mathbb{N} \times \mathbb{E} : \quad (x \boxtimes y) \Longrightarrow (x \in \mathbb{E})$$

where $\mathbb{E}$ is the set defined in 2.1

3 The two-dimensional map M

In this section we define the bijective two-dimensional map M which is totally suitable to the $3x+1$ problem, and much more natural than the Cantor's bijection proving $\mathbb{N}^2$ equipotent to $\mathbb{N}$. It uses same induction than the Mersenne's numbers, that is why it is called like that. As usual for the binomial coefficients C_n^p we note $\mathrm{M}_n^k = \mathrm{M}(n,k)$ where k is not an exposant.

3.1 Definition of the M map

Let the map $\mathrm{M} : \mathbb{N}^2 \longrightarrow \mathbb{N} \quad (n,k) \longmapsto \mathrm{M}_n^k = \mathrm{M}(n,k)$ such that:

$$\forall (n,k) \in \mathbb{N}^2 \quad \begin{cases} \mathrm{M}_0^k = 2k \\ \mathrm{M}_{n+1}^k = 1 + 2 \times \mathrm{M}_n^k \end{cases}$$

From the theory of arithmetico-geometrical progressions we get immediately:

$$\forall (n,k) \in \mathbb{N}^2 \quad \mathrm{M}_n^k = 2^n(2k+1) - 1$$

The numbers n and k are respectively called class and rank of the natural M_n^k
If the rank k is null we get:

$$\forall n \in \mathbb{N} \quad \mathrm{M}_n^0 = 2^n - 1 = \mathrm{M}_n$$

3.2 The map M is bijective

Proof: Let $x \in \mathbb{N}$ and $y = x + 1 \in \mathbb{N}^*$ According to the fundamental theorem of arithmetics, the nonzero natural y can be unique way written under the form $y = 2^n(2k+1)$ with $(n,k) \in \mathbb{N}^2$, where $2k+1$ is the product of all odd factors. Therefore $x = y - 1$ can be unique way written under the form $x = 2^n(2k+1) - 1$ and finally $\forall x \in \mathbb{N} \quad \exists!\, (n,k) \in \mathbb{N}^2 \quad x = \mathrm{M}_n^k \quad \square$

3.3 Property 3.3

From definition 3.1 we get:

$$\forall (n,k) \in \mathbb{N}^2 \quad 1 + 3 \times \mathrm{M}_{n+1}^k = 2 \times \mathrm{M}_n^{3k+1}$$

4 Intrinsic algorithm for odd numbers

In this section we compute iterations of the intrinsic algorithm starting from any odd natural, and prove the existency of nodes between trajectories.

4.1 Lemma 4.1:

Let the function $f_1 : \mathcal{O} \longrightarrow \mathbb{N} \quad x \mapsto \frac{3x+1}{2}$ as defined in 1.4
The following statement holds:

$$\forall\, (n,k) \in \mathbb{N}^2 \quad f_1^n\left(\mathrm{M}_n^k\right) = 3^n\,(2k+1) - 1$$

Proof: Let $(n,k) \in \mathbb{N}^2$ such that M_n^k is an odd number.
By definition 3.1 we have: $\mathrm{M}_n^k \equiv 1 \pmod 2 \Longleftrightarrow n \neq 0$
By property 3.3 we get: $(3 \times \mathrm{M}_n^k + 1)/2 = \mathrm{M}_{n-1}^{3k+1}$
Let $\sigma : \mathbb{N} \to \mathbb{N} \quad i \longmapsto \sigma(i) = 3i + 1$ Then: $f_1\left(\mathrm{M}_n^k\right) = \mathrm{M}_{n-1}^{\sigma(k)}$
After n iterations: $f_1^n\left(\mathrm{M}_n^k\right) = \mathrm{M}_0^{\sigma^n(k)} = 2 \times \sigma^n(k) = 3^n(2k+1) - 1$
and this result holds also for $n = 0 \quad \square$

4.2 Theorem 4.2:

$$\forall (n,k) \in \mathbb{N}^2 \quad \begin{cases} \mathrm{M}_{2n}^{2k} \boxtimes \mathrm{M}_{2n+1}^{2k} \\ \\ \mathrm{M}_{2n+1}^{2k+1} \boxtimes \mathrm{M}_{2n+2}^{2k+1} \end{cases}$$

Proof: The lemma 4.1 computed the first n iterations of intrinsic algorithm.
We compute now the next iteration, according to definition 1.4
There are four cases, following the parity of rank and class.
Let $(n, k) \in \mathbb{N}^2$

• Case : rank even, class even:
$f^{2n}\left(\mathrm{M}_{2n}^{2k}\right) = 3^{2n}(4k+1) - 1 \equiv 0 \pmod 4 \implies$ following step is f_2
$f^{2n+1}\left(\mathrm{M}_{2n}^{2k}\right) = \left(3^{2n+1}(4k+1) - 3\right)/4$

• Case : rank even, class odd:
$f^{2n+1}\left(\mathrm{M}_{2n+1}^{2k}\right) = 3^{2n+1}(4k+1) - 1 \equiv 2 \pmod 4 \implies$ following step is f_3
$f^{2n+2}\left(\mathrm{M}_{2n+1}^{2k}\right) = \left(3^{2n+1}(4k+1) - 3\right)/4$

• Case : rank odd, class odd:
$f^{2n+1}\left(\mathrm{M}_{2n+1}^{2k+1}\right) = 3^{2n+1}(4k+3) - 1 \equiv 0 \pmod 4 \implies$ following step is f_2
$f^{2n+2}\left(\mathrm{M}_{2n+1}^{2k+1}\right) = \left(3^{2n+2}(4k+3) - 3\right)/4$

• Case : rank odd, class even:
$f^{2n+2}\left(\mathrm{M}_{2n+2}^{2k+1}\right) = 3^{2n+2}(4k+3) - 1 \equiv 2 \pmod 4 \implies$ following step is f_3
$f^{2n+3}\left(\mathrm{M}_{2n+2}^{2k+1}\right) = \left(3^{2n+2}(4k+3) - 3\right)/4$

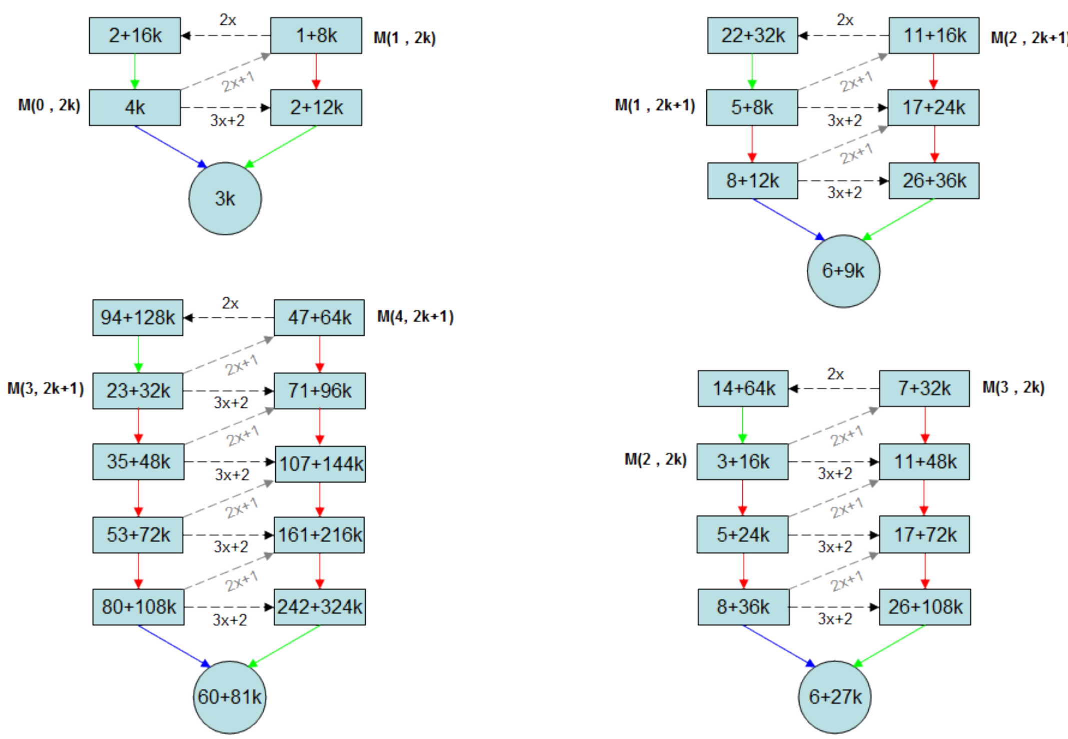
$$\text{Therefore we get:} \begin{cases} f^{2n+1}\left(\mathrm{M}_{2n}^{2k}\right) = f^{2n+2}\left(\mathrm{M}_{2n+1}^{2k}\right) \\ \\ f^{2n+2}\left(\mathrm{M}_{2n+1}^{2k+1}\right) = f^{2n+3}(\mathrm{M}_{2n+2}^{2k+1}) \end{cases}$$

and the definition 2.2 achieves the result. □

4.3 Interpretation of theorem 4.2

Let $(n, k) \in \mathbb{N}^2$ The naturals $x = \mathrm{M}_{2n}^{2k}$ and $y = 1 + 2x = \mathrm{M}_{2n+1}^{2k}$ have a common node in their trajectories by the intrinsic algorithm. Because of the existency of this node, to prove $y \in \mathbb{E}$ we do not have to find a number i of iterations such that $f^i(y) < y$. The value of the node can be bigger than the considered number y, this number is nevertheless connected to the smaller number x. Same finding for the numbers M_{2n+1}^{2k+1} and M_{2n+2}^{2k+1} .

Figure 3: Illustration of Theorem 4.2



5 Conclusions

The intrinsic algorithm allows to build by induction the tree $\mathbb{E}$ which is much more regular than the Collatz tree. If we prove $\mathbb{E} = \mathbb{N}$ then the Collatz conjecture will be proved. Theorem 4.2 makes us believe in a proof by induction, where the base $(\forall x \leq 2^{n_0} \quad x \in \mathbb{E})$ can be proved with the help of computer, the number $n_0 \in \mathbb{N}$ being a constant such that we can prove the induction:

$$\forall n \geq n_0 \qquad (\forall x \leq 2^n \quad x \in \mathbb{E}) \Longrightarrow \big(\forall x \leq 2^{n+1} \quad x \in \mathbb{E}\big)$$

References

[1] J.C. Lagarias: *The 3x + 1 problem: An annotated bibliography* (1963–1999), arXiv:math.NT/0309224

[2] J.C. Lagarias: *The 3x + 1 problem: An annotated bibliography, II* (2000–), arXiv:math.NT/0608208