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Moments of inertia of Archimedean solids

Frédéric Perrier

Equipe Physique des Sites Naturels, Institut de Physique du Globe de Paris UMR7154, Université Paris Diderot, Sorbonne Paris Cité, 1, rue Jussieu, F-75005 Paris, France. Email : perrier@ipgp.fr

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Abstract

Rotational symmetries lead to an isotropic tensor of inertia for regular solids. The moment of inertia around any axis can then be obtained from the central moment of inertia without cumbersome calculations. Using this trick of the central moment, the moments of inertia are explicitly calculated here for all thirteen Archimedean solids. Among all Archimedean solids inscribed in the unit sphere, the Great Rhombicosidodecahedron has the highest ratio of moment of inertia by mass, 93.4 % of the corresponding ratio for the sphere, for 90 % of its volume. The highest Moment of Inertia Quotient (0.9964) is however obtained for the Small Rhombicosidodecahedron, and the second highest (0.9955) for the Snub Dodecahedron.

I. INTRODUCTION

Physics likes simplicity and harmony, and physicists enjoy symmetries because they bring spectacular simplifications in numerous physical quantities, a fact that can be illustrated using moments of inertia (MI). The condition of two axes of threefold or higher rotational symmetry crossing at the center of gravity is sufficient to produce an isotropic tensor of inertia.[1,2] The MI around any axis passing by the center of gravity are then identical. The explicit calculation was carried out for the five Perfect (or Platonic) Solids (PS), which are the regular solids assembled from identical equilateral triangles (Tetrahedron, Octahedron, Icosahedron), squares (Cube) or pentagons (Dodecahedron).[1,3]

However, the previously reported calculation becomes cumbersome in the case of the 13 Archimedean Solids (AS), which are the regular solids formed from two or three kinds of *n*-sided polygons with the same edge.[4] The AS were known to the Ancient Greeks but rediscovered during the XVth and XVIth centuries.[5] They have similar symmetries as the PS, namely the T, T_d, T_h, O, O_h, Y and Y_h point groups, and consequently have isotropic tensors of inertia. The AS are an interesting and heuristic family of solids whose importance has been increasingly pointed out recently in nanocrystals, water molecular assemblies or lattive packings.[6,7,8] The MI, however, was only reported for the Truncated Octahedron, the Rhombicuboctahedron and the Truncated Isocahedron.[2,3]

In this paper, the method previously used for the PS or the Tricontahedron is revised to avoid cumbersome calculations.[2,3] As an example of application, the MI for all 13 AS are derived and discussed.

II. PRINCIPLE OF THE CALCULATION

The basic tool of the calculation is the set of moments of inertia of a right pyramid of mass *m*, height *r* having a *n*-sided polygon of side *a* as base. Consider a coordinate system with the origin O at the apex of the pyramid and the *z* axis around the axes of *n*-rotational symmetry of the pyramid. The direction of the *x* and *y* axis parallel to the plane of the base of the pyramid is arbitrary. It is then elementary to establish that the inertia tensor is diagonal in this coordinate system.[1] Its principal components are:

$$\begin{cases} I_{xx} = I_{yy} = \dfrac{3}{5}mr^2 + \dfrac{1}{80}\left(3\cot^2\dfrac{\pi}{n}+1\right)ma^2 \\ I_{zz} = \dfrac{1}{40}\left(3\cot^2\dfrac{\pi}{n}+1\right)ma^2 \end{cases} . \quad (1)$$

The central moment is then:

$$I_O = \frac{I_{xx} + I_{yy} + I_{zz}}{2} = \frac{3}{5} m r^2 + \frac{1}{40}\left(3\cot^2\frac{\pi}{n} + 1\right) m a^2 . \quad (2)$$

Consider now a homogeneous solid of density 1 having sufficient rotational axes to have an isotropic tensor of inertia made of such pyramids, allowing for various kinds of pyramids, with F_i pyramids having as base a n_i-sided polygon. This will in particular cover the variety of all AS. For example (see Appendix A), the AS Cuboctahedron (CO) is made of F_1=8 pyramids with equilateral triangular base (n_1=3) and F_2=6 pyramids with square base (n_2=4). The Great Rhombicosidodecahedron (GRD) is made of F_1=30 pyramids with square base (n_1=3), F_2=20 pyramids with hexagonal base (n_2=6), and F_3=12 pyramids with decagonal base (n_3=10). Each family i of pyramids gives a contribution S_i to the surface area S of the solids and V_i to the volume V of the solid, given by:

$$\begin{cases} \dfrac{S_i}{a_i^2} = \dfrac{F_i n_i}{4}\cot\dfrac{\pi}{n_i} \\ \dfrac{V_i}{a_i^3} = \dfrac{1}{3}\dfrac{S_i}{a_i^2}\dfrac{r_i}{a_i} \end{cases} . \quad (3)$$

The central MI of the solid is then the sum of the contributions of the pyramid families:

$$I_O = \sum_i V_i\left[\frac{3}{5} r_i^2 + \frac{1}{40}\left(3\cot^2\frac{\pi}{n_i} + 1\right) a_i^2\right]. \quad (4)$$

In the case of the AS, we have the additional condition that all polygon edges are equal. The MI I with respect to any axis passing by the center of symmetry is then given by:

$$\frac{I}{a^2} = \frac{2}{3}\frac{I_O}{a^2} = \frac{2}{3}\sum_i V_i\left[\frac{3}{5}\frac{r_i^2}{a^2} + \frac{1}{40}\left(3\cot^2\frac{\pi}{n_i} + 1\right)\right]. \quad (5)$$

The MI can then easily be derived once the properties of the solid are established. Recall that when the solids are inscribed in a sphere of radius R, then the radius r of the sphere tangent to each face, which is also the height of the associated pyramid, is related to R and the radius r' of the circle containing the vertices of the face by:

$$R^2 = r^2 + r'^2 . \quad (6)$$

In this case, there is no loss of generality to consider the solids inscribed in the unit sphere (R=1). As we have $a = 2r'\sin\frac{\pi}{n_i}$, then follows the property:

$$\frac{r_i^2}{a^2} = \frac{1}{a^2} - \frac{1}{4\sin^2\frac{\pi}{n_i}} . \quad (7)$$

Once a is known, all quantities necessary for the calculation of the MI in Eq. (5) are obtained from Eq. (6) and Eq. (3). The calculation of a for the AS is recalled in Appendix A.

III. RESULTS

The obtained expressions for the MI are given in Table 1 for the AS, whose complete set is presented here for the first time. The results reported previously for the Truncated Octahedron, the Rhombicuboctahedron by Satterly and for the Truncated Isocahedron by Aravind are confirmed.[2,3] For completion, the values of the MI are recalled in Table 2 for the PS. These expressions for the PS were already known before.[1] They can be reproduced easily in a few lines of calculation with our method.

The parameters of the various solids can be compared, for example taking all the solids inscribed in the unit sphere, using the expressions of Tables 1 and 2. The solid giving the smallest surface area is then the AS Snub Cube (~36 % for the sphere), while the solid giving the smallest volume (~12 % of the sphere) is the Tetrahedron. The AS GRD mentioned above gives the largest surface area (~96 % of the sphere) and the largest volume (~90 % of the sphere). The MI are shown in Fig. 1 versus the volume fraction of the sphere. In this figure, the MI are normalized to the mass (giration ratio) normalized to the same ratio for a homogeneous sphere (giration ratio of 2/5). The various PS and AS provide normalized IM values rather evenly spread between the minimum (~0.33, obtained for the Tetrahedon) and the maximum (~0.93, obtained for the GRD). Values smaller by a few % than the value given by the GRD are obtained for the Great Rhombicosidodecahedron (SRD) and the Snub Dodecahedron (SD).

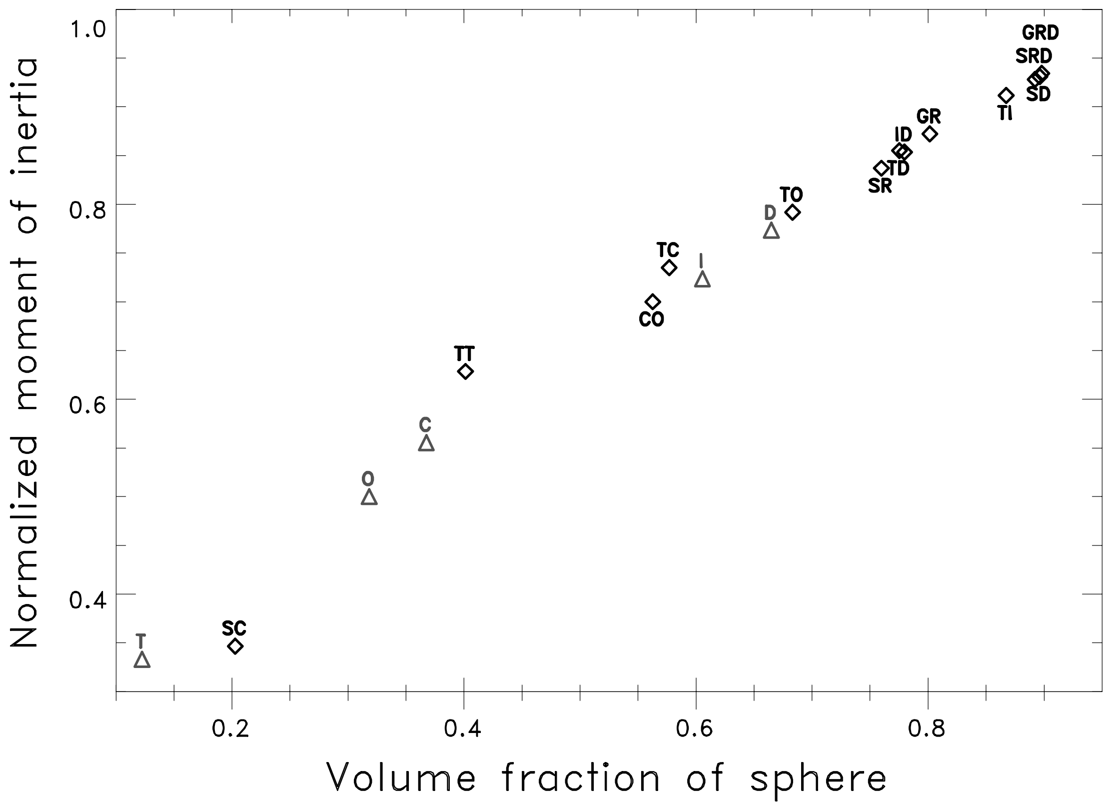

Fig. 1. Moment of inertia divided by mass and normalized to 2/5 (value for a homogeneous sphere) versus volume fraction of sphere, for PS and AS inscribed in the unit sphere.

Another customary way of comparing these solids is the Isoperimetric Quotient (IQ) of George Polya defined as:

$$\mathrm{IQ} = \left(\frac{4\pi}{S_1} \right)^3 . \qquad (8)$$

where S_1 is the surface area of the solid having the same volume as the unit sphere.[1] Analogously, Aravind defined the dimensionless Moment of Inertia Quotient (MIQ) as:

$$\mathrm{MIQ} = \left(\frac{I_{\mathrm{sph}}}{I} \right)^3 , \qquad (8)$$

where I_{sph} is the MI of the sphere having the same mass and the same density as the considered polyhedron.

The values of IQ and MIQ can be calculated from the obtained expressions for a, S/a^2, V/a^3 and I/Ma^2. The numerical values are given in Tables 1 and 2, and 1-MIQ is shown versus 1-IQ in Fig. 2. Note that the values of IQ for AS were already reported by Aravind.[8] A logarithmic scale is used in Fig. 2 as MIQ are close to 1 for numerous AS. The solid with the smallest IQ (~0.3023) is the Tetrahedron who has also the smallest MIQ ((~0.4053), and thus is the regular solid most different from a sphere. The SD has the largest IQ (~0.947), but it is the Small Rhombicosidodecahedron (SRD) which exhibits the largest MIQ of ~0.9964. The second largest MIQ (~0.9955) belongs to the SD. Thus, depending on the way the matter is considered, the AS most similar to a sphere is GRD, SD or SRD.

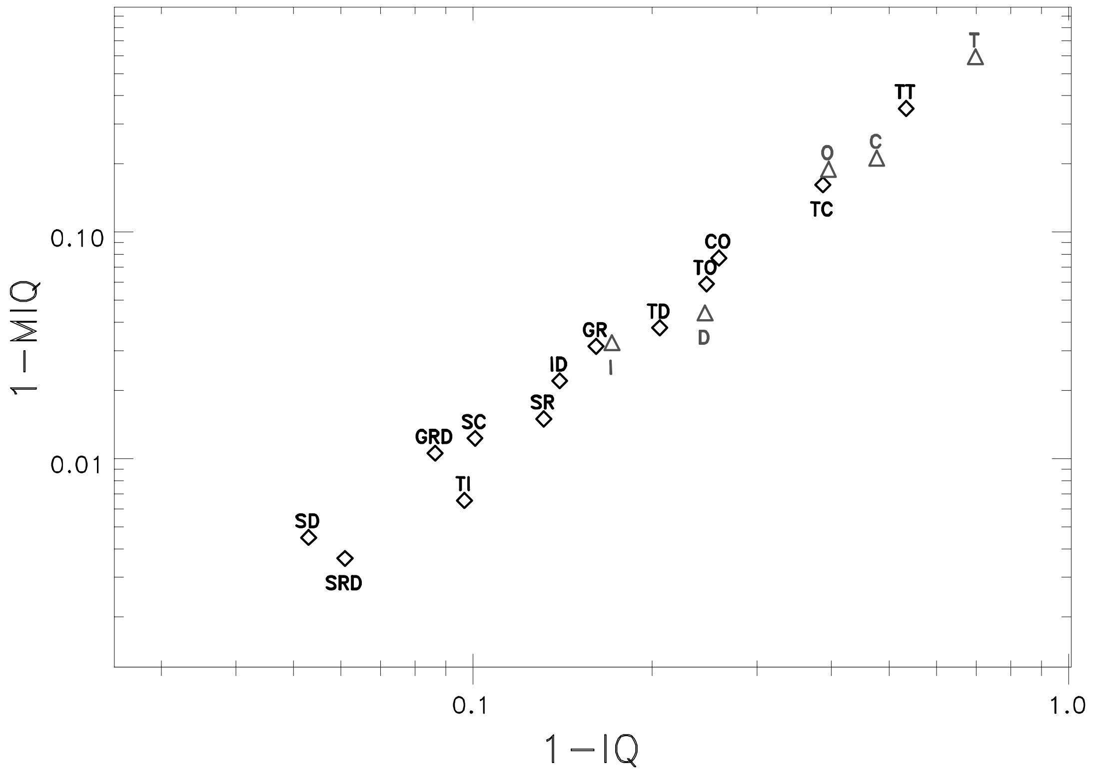

Fig. 2. Moment of Inertia Quotient deficit to 1 versus Isoperimetric Quotient deficit to 1.

IV. CONCLUSION

Calculating the MI of AS, as presented in this paper, appears elementary once clear and simple steps are established and followed comfortably. Such systematic practice can be of great pedagogical value, and thus the calculation of AS MI provides beautiful exercises to train students in computational elegance combined with spatial visualization and representation.

Exact analytical expressions for the MI of AS can also be useful to check numerical codes, in particular in technological issues involving rotational properties, or, for example, optical properties of nanocrystals and quasicrystals. Some biological properties of viruses, which can show fivefold symmetries, could also be interpreted in terms of mechanical properties. Further solids could also be studied using the method illustrated here in the case of AS, for example stellations of regular polytopes or combined forms. Much remains to be explored in the powerful and wonderful emergence of abstract mathematical harmony in the actual or possible realizations of our physical world.[9,10]

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Table 1. Overview of AS results: surface area *S*, volume *V* and ratio of the MI *I* to mass *M*, with normalization to the proper exponent of the edge a. The number $U \cong 1.19149$ is defined by $U^3 = U + 1/2$. It is related to the Tribonacci number $t = 1 + 1/U$ which satisfies $t^3 = 1 + t + t^2$. The number $\theta \cong 1.71556$ is defined by $\theta^3 = \varphi + 2\theta$ where $\varphi = (1+\sqrt{5})/2$ is the golden ratio.

Archimedean Solid	a	S/a^2	V/a^3	I/Ma^2	IQ	MIQ
TT : Truncated Tetrahedron	$2\sqrt{\frac{2}{11}}$	$7\sqrt{3}$	$\frac{23\sqrt{2}}{12}$	$\frac{159}{460}$	0.4662	0.6489
TO : Truncated Octahedron	$\sqrt{\frac{2}{5}}$	$6\left(1+2\sqrt{3}\right)$	$8\sqrt{2}$	$\frac{19}{24}$	0.7534	0.9410
TI : Truncated Icosahedron	$\sqrt{\frac{2}{109}\left(29-9\sqrt{5}\right)}$	$15\left(2\sqrt{3}+\sqrt{1+\frac{2}{5}\sqrt{5}}\right)$	$\frac{125+43\sqrt{5}}{4}$	$\frac{118625+42771\sqrt{5}}{95700}$	0.9032	0.9935
TC : Truncated Cube	$2\sqrt{\frac{7-4\sqrt{2}}{17}}$	$2\left(6+6\sqrt{2}+\sqrt{3}\right)$	$\frac{7}{3}\left(3+2\sqrt{2}\right)$	$\frac{17+11\sqrt{2}}{35}$	0.6130	0.8382
TD : Truncated Dodecahedron	$\sqrt{\frac{2\left(37-15\sqrt{5}\right)}{61}}$	$5\left(\sqrt{3}+6\sqrt{5+2\sqrt{5}}\right)$	$\frac{5}{12}\left(99+47\sqrt{5}\right)$	$3\frac{3195+1367\sqrt{5}}{6220}$	0.7940	0.9622
SR : Small Rhombicuboctahedron	$\sqrt{2\frac{10-4\sqrt{2}}{17}}$	$2\left(9+\sqrt{3}\right)$	$\frac{2}{3}\left(6+5\sqrt{2}\right)$	$\frac{100+59\sqrt{2}}{280}$	0.8685	0.9850
CO : Cuboctahedron	1	$2\left(3+\sqrt{3}\right)$	$\frac{5\sqrt{2}}{3}$	$\frac{7}{25}$	0.7412	0.9231
ID : Icosidodecahedron	$\sqrt{\frac{3-\sqrt{5}}{2}}$	$5\sqrt{3}+3\sqrt{25+10\sqrt{5}}$	$\frac{45+17\sqrt{5}}{6}$	$\frac{690+271\sqrt{5}}{1450}$	0.8602	0.9779
GR : Great Rhombicuboctahedron	$2\sqrt{\frac{13-6\sqrt{2}}{97}}$	$12\left(2+\sqrt{2}+\sqrt{3}\right)$	$2\left(11+7\sqrt{2}\right)$	$\frac{7+3\sqrt{2}}{6}$	0.8390	0.9687
GRD : Great Rhombicosidodecahedron	$2\sqrt{\frac{31-12\sqrt{5}}{241}}$	$30\left(1+\sqrt{3}+\sqrt{5+2\sqrt{5}}\right)$	$5\left(19+10\sqrt{5}\right)$	$\frac{11828+4785\sqrt{5}}{4170}$	0.9136	0.9894
SRD : Small Rhombicosidodecahedron	$2\sqrt{\frac{11-4\sqrt{5}}{41}}$	$5\left(6+\sqrt{3}+3\sqrt{1+\frac{2}{5}\sqrt{5}}\right)$	$20+\frac{29}{3}\sqrt{5}$	$3\frac{3885+1601\sqrt{5}}{12100}$	0.9390	0.9964
SC : Snub Cube	$2\sqrt{\frac{U-1}{2U-1}}$	$6+8\sqrt{3}$	$\frac{1}{\sqrt{U-1}}+\frac{4}{3}\sqrt{\frac{2U+1}{U-1}}$	$\frac{1}{10}\frac{\sqrt{2U+1}}{U-1}\frac{4U+\sqrt{2U+1}}{3+4\sqrt{2U+1}}$	0.8992	0.9877
SD : Snub Dodecahedron	$2\sqrt{\frac{\theta-\varphi}{2\theta-\varphi}}$	$20\sqrt{3}+3\sqrt{25+10\sqrt{5}}$	$\frac{20\theta^{\frac{3}{2}}+3\sqrt{5}\sqrt{8+13\varphi-2\theta-4\theta\varphi}}{6\sqrt{\theta-\varphi}}$	$\frac{1}{50(\theta-\varphi)}\frac{100\theta^{\frac{5}{2}}+\sqrt{5}\sqrt{8+13\varphi-2\theta-4\theta\varphi}\left(6+13\theta+8\varphi-6\theta\varphi\right)}{20\theta^{\frac{3}{2}}+3\sqrt{5}\sqrt{8+13\varphi-2\theta-4\theta\varphi}}$	0.9470	0.9955

Table 2. Overview of (previously known) quantities of PS: surface area S, volume V and ratio of the MI I to mass M, with normalization to the proper exponent of the edge a.[1]

Platonic Solid	a	$\frac{S}{a^2}$	$\frac{V}{a^3}$	$\frac{I}{Ma^2}$	IQ	MIQ
T : Tetrahedron	$2\sqrt{\frac{2}{3}}$	$\sqrt{3}$	$\frac{\sqrt{2}}{12}$	$\frac{1}{20}$	0.3023	0.4053
O : Octahedron	$\sqrt{2}$	$2\sqrt{3}$	$\frac{\sqrt{2}}{3}$	$\frac{1}{10}$	0.6046	0.8106
C : Cube	$\frac{2}{\sqrt{3}}$	6	1	$\frac{1}{6}$	0.5236	0.7879
I : Icosahedron	$\sqrt{2}\sqrt{1-\frac{\sqrt{5}}{5}}$	$5\sqrt{3}$	$\frac{5}{12}\left(3+\sqrt{5}\right)$	$\frac{3+\sqrt{5}}{20}$	0.8288	0.9675
D : Dodecahedron	$\frac{\sqrt{5}-1}{\sqrt{3}}$	$3\sqrt{25+10\sqrt{5}}$	$\frac{15+7\sqrt{5}}{4}$	$\frac{95+39\sqrt{5}}{300}$	0.7547	0.9561

Appendix A
Parameters of Archimedean Solids

Here we briefly state one pedestrian method to derive the parameters of the set of 13 AS. These solids are obtained by covering the unit sphere by at least two spherical isosceles triangles (Fig. A1). When covering the sphere with identical isosceles triangles, the PS are constructed, and their parameters are recalled in Table A1.

Table A1. Overview of PS parameters.

Platonic Solid	n	F	a	$\frac{r}{a}$
Tetrahedron	3	4	$2\sqrt{\frac{2}{3}}$	$\frac{1}{2\sqrt{6}}$
Octahedron	3	8	$\sqrt{2}$	$\frac{1}{\sqrt{6}}$
Cube	4	6	$\frac{2}{\sqrt{3}}$	$\frac{1}{2}$
Icosahedron	3	20	$\sqrt{2}\sqrt{1-\frac{\sqrt{5}}{5}}$	$\sqrt{3}\frac{3+\sqrt{5}}{12}$
Dodecahedron	5	12	$\frac{\sqrt{5}-1}{\sqrt{3}}$	$\sqrt{\frac{25+11\sqrt{5}}{40}}$

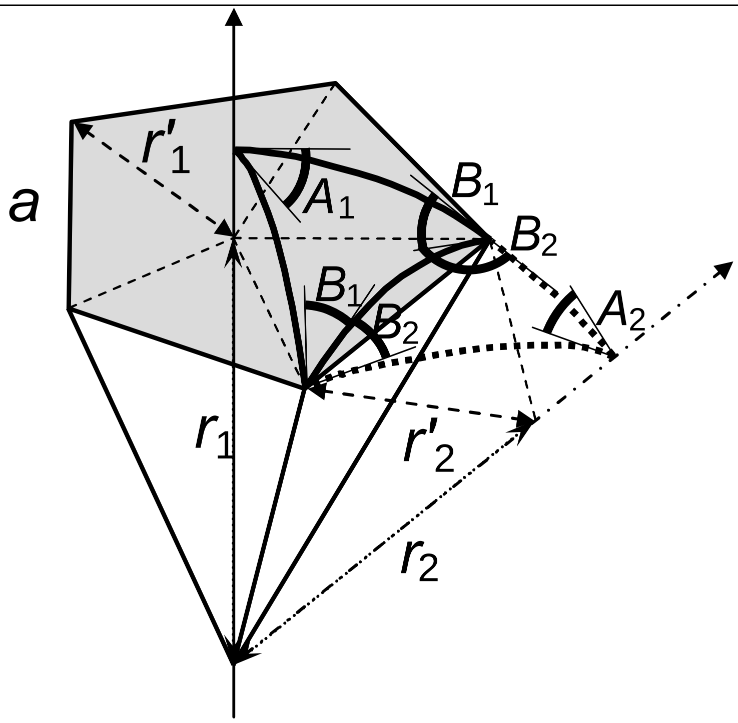

Fig. A1. Sketch of an n_1-gonal face generated by n_1 isosceles spherical triangles of summit angle A_1 and base angles B_1. Here the edge a is shared by another spherical triangle of summit angle A_2 and base angles B_2.

Let A_1 the spherical angle at the summit of spherical triangle 1, with B_1 being the other two equal angles (Fig. A1). An integer number n_1 of spherical triangles combine at their summit to generate, when keeping only the base edge of size a, an n_1-gonal face and we have $A_1 = 2\pi/n_1$. The construction of the AS results from combining 2 or 3 types of spherical triangles, and from three constraints. First, the covering of the full surface of the sphere (Euler condition):

$$4\pi = \sum_i F_1 n_1 \left(A_1 + 2B_1 - \pi \right). \tag{A1}$$

Second, the edge a must be the same for each type of face. The edge is given by the angle α at the center of the sphere ($a = 2\sin\alpha/2$), which is related to the angles of the spherical triangle by spherical trigonometry:

$$\cos\alpha = \frac{\cos A_i + \cos^2 B_i}{\sin^2 B_i} = \frac{\cos A_j + \cos^2 B_j}{\sin^2 B_j}, \tag{A2}$$

which can be rewritten:

$$\sin B_i \cos\frac{\pi}{n_j} = \sin B_j \cos\frac{\pi}{n_i}. \tag{A3}$$

The third constraint comes from the sum of all angles at the vertices of the base angles of the spherical triangles, leading to the existence of integers q_i, such that:

$$\pi = \sum_i q_i B_i . \tag{A4}$$

It is then a Diomantian problem to exhaust all possibilities for the integers q_i, n_i, and the number of faces F_i, which must obey the additional compatibility condition $n_i F_i / q_i = n_j F_j / q_j$. In each case, one can determine angle B_1, from which the edge can be obtained using Eq. (A2) and conveniently rewritten as:

$$a^2 = 4\sin^2\frac{\alpha}{2} = 2\left(2 - \frac{1+\cos A_1}{\sin^2 B_1} \right). \tag{A5}$$

We summarize the results leading to the complete set of 13 AS below, with parameters needed for the calculation of MI collected in Tables A2 and A3. We ignore below the prisms and the antiprisms, also given by this construction, but which we do not consider in this paper. Values of trigonometric quantities useful for the calculation of AS parameters including values of MI are collected in Table A4.

A1 Covering of the sphere by two spherical isosceles triangles

A1a. Case q_1=2 and q_2=1

Conditions (A3) and (A4) then give:

$$\cos B_1 = \frac{\cos\frac{\pi}{n_2}}{2\cos\frac{\pi}{n_1}}. \tag{A6}$$

The only possible integers n_1 and n_2 compatible with Eq. (A1) are then n_1=6 and n_2=3 (Truncated Tetrahedron TT), n_1=6 and n_2=4 (Truncated Octahedron TO), n_1=6 and n_2=5 (Truncated Icosahedron TI), n_1=8 and n_2=3 (Truncated Cube TC), and n_1=10 and n_2=3 (Truncated Dodecahedron TD).

A1b. Case q_1=3 and q_2=1

Conditions (A3) and (A4) here give:

$$\sin B_1 = \frac{1}{2}\sqrt{3 - \frac{\cos\frac{\pi}{n_2}}{\cos\frac{\pi}{n_1}}}. \tag{A7}$$

The only possible integers n_1 and n_2 are n_1=4 and n_2=3 (Small Rhombicuboctahedron SR).

A1c. Case q_1=2 and q_2=2

Conditions (A3) and (A4) here give:

$$\tan B_1 = \frac{\cos\dfrac{\pi}{n_1}}{\cos\dfrac{\pi}{n_2}}. \qquad \text{(A8)}$$

The possible integers n_1 and n_2 are n_1=3 and n_2=4 (Cuboctahedron CO), and n_1=3 and n_2=5 (Icosidodecahedron ID).

A1d. Case q_1=4 and q_2=1

In this case, conditions (A3) and (A4) here lead to a cubic equation for $u = \cos B_1$:

$$2u^3 - u - \frac{\cos\dfrac{\pi}{n_2}}{4\cos\dfrac{\pi}{n_1}} = 0. \qquad \text{(A9)}$$

The possible integers n_1 and n_2 are then n_1=3 and n_2=4 (Snub Cube SC), and n_1=3 and n_2=5 (Snub Decahedron SD).

In the case of the SC, we define the number $U \cong 1.19149$ by $U^3 = U + 1/2$, and we have $\cos B_1 = U/\sqrt{2}$. Number U is related to the Tribonacci number t, which satisfies $t^3 = 1 + t + t^2$ and which is sometimes used to express the parameters of the SC, by t=1+1/U.

In the case of the SD, we define the number $\theta \cong 1.71556$ by $\theta^3 = \varphi + 2\theta$ where $\varphi = \left(1+\sqrt{5}\right)/2$ is the golden ratio, and we have $\cos B_1 = \theta/2$.

A2 Covering of the sphere by three spherical isosceles triangles

In this case, Eq. (A3) gives two conditions:

$$\begin{cases} \sin B_1 \cos\dfrac{\pi}{n_2} = \sin B_2 \cos\dfrac{\pi}{n_1} \\ \sin B_1 \cos\dfrac{\pi}{n_3} = \sin B_3 \cos\dfrac{\pi}{n_1} \end{cases}. \qquad \text{(A10)}$$

A2a. Case q_1=1, q_2=1, and q_3=1

Elimination of B_2 and B_3 from (A10) and (A4) then gives:

$$\cos B_1 = \frac{\cos^2\dfrac{\pi}{n_3} + \cos^2\dfrac{\pi}{n_2} - \cos^2\dfrac{\pi}{n_1}}{2\cos\dfrac{\pi}{n_2}\cos\dfrac{\pi}{n_3}}, \qquad \text{(A11)}$$

and the possible integers n_1, n_2 and n_3 are n_1=4, n_2=6 and n_3=8 (Great Rhombicuboctahedron GR), and n_1=4, n_2=6 and n_3=10 (Great Rhombicosidodecahedron GRD).

A2b. Case q_1=2, q_2=1, and q_3=1

Elimination of B_2 and B_3 from (A10) and (A4) in this case gives:

$$\cos B_1 = \frac{1}{2}\sqrt{\frac{\cos^2\frac{\pi}{n_3} + \cos^2\frac{\pi}{n_2} + 2\cos\frac{\pi}{n_2}\cos\frac{\pi}{n_3}}{\cos^2\frac{\pi}{n_1} + \cos\frac{\pi}{n_2}\cos\frac{\pi}{n_3}}}\,, \qquad \text{(A12)}$$

and the only possible integers n_1, n_2 and n_3 are $n_1=4$, $n_2=3$ and $n_3=5$ (Small Rhombicosidodecahedron SRD).

Table A2. Overview of AS parameters for AS solids generated from two different spherical isosceles triangles. See text for definition of U and θ.

Archimedean Solid	q_1	q_2	n_1	n_2	F_1	F_2	$\cos B_1$	a	$\frac{r_1}{a}$	$\frac{r_2}{a}$
Truncated Tetrahedron	2	1	6	3	4	4	$\frac{1}{2\sqrt{3}}$	$2\sqrt{\frac{2}{11}}$	$\sqrt{\frac{3}{8}}$	$\sqrt{\frac{25}{24}}$
Truncated Octahedron	2	1	6	4	8	6	$\frac{1}{\sqrt{6}}$	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{2}$
Truncated Icosahedron	2	1	6	5	20	12	$\frac{1+\sqrt{5}}{4\sqrt{3}}$	$\sqrt{\frac{2}{109}\left(29-9\sqrt{5}\right)}$	$\frac{\sqrt{3}}{4}\left(3+\sqrt{5}\right)$	$\frac{1}{2}\sqrt{\frac{125+41\sqrt{5}}{10}}$
Truncated Cube	2	1	8	3	6	8	$\frac{\sqrt{2-\sqrt{2}}}{2\sqrt{2}}$	$2\sqrt{\frac{7-4\sqrt{2}}{17}}$	$\frac{1+\sqrt{2}}{2}$	$\frac{3+2\sqrt{2}}{2\sqrt{3}}$
Truncated Dodecahedron	2	1	10	3	12	20	$\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{10}}$	$\sqrt{\frac{2\left(37-15\sqrt{5}\right)}{61}}$	$\frac{\sqrt{25+11\sqrt{5}}}{2\sqrt{2}}$	$\frac{\sqrt{103+45\sqrt{5}}}{2\sqrt{6}}$
Small Rhombicuboctahedron	3	1	4	3	18	8	$\frac{\sqrt{2+\sqrt{2}}}{2\sqrt{2}}$	$\sqrt{2\frac{10-4\sqrt{2}}{17}}$	$\frac{1+\sqrt{2}}{2}$	$\frac{3+\sqrt{2}}{2\sqrt{3}}$
Cuboctahedron	2	2	3	4	8	6	$\sqrt{\frac{2}{3}}$	1	$\sqrt{\frac{2}{3}}$	$\frac{\sqrt{2}}{2}$
Icosidodecahedron	2	2	3	5	20	12	$\frac{\sqrt{5+\sqrt{5}}}{\sqrt{10}}$	$\sqrt{\frac{3-\sqrt{5}}{2}}$	$\frac{3+\sqrt{5}}{2\sqrt{3}}$	$\sqrt{1+\frac{2\sqrt{5}}{5}}$
Snub Cube	4	1	3	4	32	6	$\frac{U}{\sqrt{2}}$	$2\sqrt{\frac{U-1}{2U-1}}$	$\sqrt{\frac{2U+1}{12\left(U-1\right)}}$	$\frac{1}{2\sqrt{\left(U-1\right)}}$
Snub Dodecahedron	4	1	3	5	80	12	$\frac{\theta}{2}$	$2\sqrt{\frac{\theta-\varphi}{2\theta-\varphi}}$	$\frac{\theta^{\frac{3}{2}}}{2\sqrt{3\left(\theta-\varphi\right)}}$	$\frac{1}{2}\sqrt{\frac{4+2\theta+7\varphi-4\theta\varphi}{5\left(\theta-\varphi\right)}}$

Table A3. Overview of AS parameters for AS solids generated from three different spherical isosceles triangles.

Archimedean Solid	q_1	q_2	q_3	n_1	n_2	n_3	F_1	F_2	F_3	$\cos B_1$	a	$\frac{r_1}{a}$	$\frac{r_2}{a}$	$\frac{r_3}{a}$
Great Rhombicuboctahedron	1	1	1	4	6	8	12	8	6	$\frac{\sqrt{10+\sqrt{2}}}{2\sqrt{6}}$	$2\sqrt{\frac{13-6\sqrt{2}}{97}}$	$\frac{3+\sqrt{2}}{2}$	$\sqrt{3}\frac{1+\sqrt{2}}{2}$	$\frac{4+\sqrt{2}}{2\sqrt{2}}$
Great Rhombicosidodecahedron	1	1	1	4	6	10	30	20	12	$\frac{\sqrt{25+2\sqrt{5}}}{2\sqrt{15}}$	$2\sqrt{\frac{31-12\sqrt{5}}{241}}$	$\frac{3+2\sqrt{5}}{2}$	$\sqrt{3}\left(\frac{2+\sqrt{5}}{2}\right)$	$\frac{\sqrt{5(5+2\sqrt{5})}}{2}$
Small Rhombicosidodecahedron	2	1	1	4	3	5	30	20	12	$\frac{\sqrt{5+2\sqrt{5}}}{2\sqrt{5}}$	$2\sqrt{\frac{11-4\sqrt{5}}{41}}$	$\frac{2+\sqrt{5}}{2}$	$\frac{3+2\sqrt{5}}{2\sqrt{3}}$	$\frac{3}{2}\sqrt{5+\frac{2\sqrt{5}}{5}}$

Table A4. Useful values of trigonometric quantities.

n	3	4	5	6	8	10
$\cos\frac{\pi}{n}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\sqrt{\frac{3+\sqrt{5}}{8}}=\frac{1+\sqrt{5}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\sqrt{\frac{5+\sqrt{5}}{8}}=\sin\frac{2\pi}{5}$
$\sin\frac{\pi}{n}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\sqrt{\frac{5-\sqrt{5}}{8}}$	$\frac{1}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\sqrt{\frac{3-\sqrt{5}}{8}}=\frac{-1+\sqrt{5}}{4}=\cos\frac{2\pi}{5}$
$\cot\frac{\pi}{n}$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{1+\frac{2}{5}\sqrt{5}}$	$\sqrt{3}$	$1+\sqrt{2}$	$\sqrt{5+2\sqrt{5}}$