The Taylor Series for *e* and the Primes 2, 5, 13, 37, 463, . . ., A Surprising Connection

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1. Introduction In [**7**], [**8**] we studied arithmetic properties of the Taylor series for *e*

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

In the process, we discovered a surprising connection with certain prime numbers. To describe it, let N_n be the numerator of the *n*th partial sum in lowest terms,

$$N_n := \text{numerator of } \sum_{k=0}^{n} \frac{1}{k!} \qquad (n \geq 0). \tag{1}$$

Setting R_n equal to the greatest common divisor

$$R_n := \gcd(N_n, N_{n+2}), \tag{2}$$

the sequence $R_0, R_1, \ldots$ begins

$$1, 2, 5, \{1\}^7, 13, \{1\}^{23}, 37, \{1\}^{425}, 463, 1, 1, \ldots,$$

where $\{1\}^k$ stands for a string of ones of length *k*. Notice that the terms 2, 5, 13, 37, and 463 are primes. In fact, we prove the following result.

Theorem 1. *The sequence* $R_0, R_1, \ldots$ *consists of ones and all primes in the set*

$$P^* := \left\{ p \textit{ prime} : p \textit{ divides } 0! - 1! + 2! - 3! + 4! - \cdots + (-1)^{p-1}(p-1)! \right\}.$$

More precisely, for $n \geq 0$, *we have*

$$R_n = \begin{cases} 2 & \textit{if } n = 1, \\ p & \textit{if } n = p - 3 \textit{ and } p \in P^* \textit{ is odd}, \\ 1 & \textit{otherwise.} \end{cases}$$

Michael Mossinghoff [**5**] has calculated that 2, 5, 13, 37, 463 are the only elements of P^* less than 150 million. However, in Section 6 we use Mertens' theorem on the series of prime reciprocals to argue heuristically that the set P^* should be infinite, but very sparse. For this problem, and a related one on primes and alternating sums of factorials, see [**2**, B43] (where the set P^* is denoted instead by S) and [**9**]. Also, see [**6**, sequences A061354, A064383, A064384, A124779, A129924].

In Sections 2, 3, and 4, we establish some preliminary results before proving Theorem 1 in Section 5.

2. A formula for N_n For $n \geq 0$, let A_n denote the *unreduced* numerator of the *n*th partial sum

$$\sum_{k=0}^{n} \frac{1}{k!} = \frac{A_n}{n!}. \tag{3}$$

(It is easy to see that the recursion

$$A_0 = 1, \quad A_n = nA_{n-1} + 1 \quad (n \geq 1) \tag{4}$$

is equivalent to (3).) In terms of A_n, the *reduced* numerator N_n is

$$N_n = \frac{A_n}{\gcd(A_n, n!)}. \tag{5}$$

3. An alternate characterization of P^* We use A_n to give an alternate description of the primes in P^*.

Lemma 1. *A prime p is in P^* if and only if p divides A_{p-1}.*

Proof. We show that the congruence

$$0! - 1! + 2! - 3! + 4! - \cdots + (-1)^{n-1}(n-1)! \equiv A_{n-1} \bmod n \quad (n \geq 1)$$

holds. The lemma follows by setting *n* equal to a prime *p*.

We multiply (3) by $n!$, and replace *n* with $n-1$. Re-indexing the sum by changing *k* to $n-1-k$, we obtain

$$A_{n-1} = \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} = \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!} = \sum_{k=0}^{n-1} (n-1)(n-2)\cdots(n-k) \equiv \sum_{k=0}^{n-1} (-1)^k k! \pmod{n}. \bullet$$

4. A criterion for primality We will need the following simple fact.

Lemma 2. *Given $n > 1$, if $n!$ is* not *divisible by $n+3$, then $n+3$ is prime.*

Proof. We show that if $n+3$ is *not* prime, say $n+3=ab$ with $2 \le a \le b$, then $(n+3) \mid n!$.

Using $n>1$, we have $2b \le n+3 < 2n+2$. Hence $a \le b \le n$. In case $a<b$, clearly $ab \mid n!$. In case $a=b$, we get $a^2 = n+3 > 4$, so $a \ge 3$. Then $0 \le (a+1)(a-3)$ $= a^2 - 2a - 3 = n - 2a$. Now $a < 2a \le n$, implying $a^2 \mid n!$. Thus, in each case, $(n+3) \mid n!$. •

5. Proof of Theorem 1 First, note that the recursion (4) implies the relation

$$A_{n+2} = (n+2)(n+1)A_n + (n+3) \qquad (n \ge 0). \tag{6}$$

Now, to begin the proof, we use (1) to compute $N_0 = 1$, $N_1 = 2$, $N_2 = 5$, and $N_3 = 8$. Then (2) gives $R_0 = 1$ and $R_1 = 2 \in P^*$.

Next, fix $n>1$ and assume $R_n \ne 1$. By (2) and (5), the positive integer R_n divides A_n and A_{n+2} but not $n!$. From (6), we see that R_n divides $n+3$. Using Lemma 2, it follows that $R_n = n+3$ is prime. Since $R_n \mid A_{n+2}$, Lemma 1 gives $R_n \in P^*$.

It remains to show, conversely, that for all odd $p \in P^*$ we have $R_{p-3} = p$. Setting $n = p-3$, Lemma 1 yields $p \mid A_{n+2}$. Then, as $n \ge 0$ and $p = n+3$, relation (6) implies $p \mid A_n$. On the other hand, since $p > n$, the prime p does not divide $n!$. By (5) and (2), it follows that $p \mid R_n$. Recalling that $R_n \ne 1$ implies R_n is prime, we conclude that $R_n = p$. •

6. A heuristic argument that P^* is infinite but very sparse The following heuristics are naive. The prime 463 looks ``random,'' so a naive model might be that $0!-1!+2!-3!+4!-\cdots+(p-1)!$ is a ``random'' number modulo a prime p. If it is, the probability that it is divisible by p would be about $1/p$. Now let's also make the hypothesis that the events are independent. Then the expected number of elements of P^* which do not exceed a bound x would be approximately

$$\#\left(P^* \cap [0,x]\right) \approx \sum_{\text{prime } p \le x} \frac{1}{p}.$$

For this sum of prime reciprocals, Mertens in 1874 proved the estimate (see [**1**, p. 94], [**3**, Theorem 427])

$$\sum_{\text{prime } p \le x} \frac{1}{p} = \log\log x + 0.2614972128\ldots + o(1).$$

(Here $o(1)$ is a quantity that tends to zero as x becomes arbitrarily large.) Since $\log\log x$ approaches infinity with x, but very slowly, the set P^* should be infinite, but very sparse.

In particular, the sum of $1/p$ for primes p between 463 and 150,000,000 is about 1.12. Since this is greater than 1, one might expect to find the next prime in P^* soon.

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