

**HOWARD UNIVERSITY**

**Random Walks, Trees and Extensions  
of  
Riordan Group Techniques**

A Dissertation  
Submitted to the Faculty of the  
Graduate School

of

**HOWARD UNIVERSITY**

in partial fulfillment of  
the requirements for the  
degree

**DOCTOR OF PHILOSOPHY**

Department of Mathematics

by

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May 2002

## Abstract

The Catalan number sequence

$$1, 1, 2, 5, 14, 42, \dots, \frac{1}{n+1} \binom{2n}{n}, \dots$$

is one of the most important sequences in all of enumerative combinatorics. Richard Stanley cites at least 66 combinatorial settings where the sequence appears [7]. Among the numerous interpretations of the Catalan numbers, we have the number of paths restricted to the first quadrant of the  $x, y$ -plane starting at  $(0, 0)$  and ending at  $(2n, 0)$  using “up” steps  $(1, 1)$  or “down” steps  $(1, -1)$  (known as Dyck paths). This research will focus on a generalization of the Catalan numbers which can be interpreted as taking Dyck paths and perturbing the length of the down step. One particular example of this generalization is given by the sequence of numbers known as the ternary numbers,

$$1, 1, 3, 12, 55, 273, 1428, 7752, 43263, \dots, \frac{1}{2n+1} \binom{3n}{n}, \dots$$

This sequence arises in many natural contexts and extensions of known results related to combinatorial objects such as paths, trees, permutations, partitions, Young tableaux and dissections of convex polygons. Hence, we use this sequence as an important example of something which lies on the boundary of what is known and what is new. Much of this research is an attempt to extend many of the known results for the Catalan numbers to ternary and  $m$ -ary numbers.

In this dissertation, we establish, in the setting of the ternary numbers, several analogues to the better known Catalan numbers setting. We present an analogue to the Chung-Feller Theorem which says that for paths from  $(0, 0)$  to  $(3n, 0)$  with step set  $\{(1, 1), (1, -2)\}$ , the number of up  $(1, 1)$  steps above the  $x$ -axis is uniformly distributed. We also present analogues to the Binomial, Motzkin and Fine generating functions and discuss combinatorial interpretations of each. In addition, we establish some computational results regarding the area bounded by ternary paths and the number of returns to the  $x$ -axis. We also present generating function proofs for the area under Dyck and ternary paths, an interesting connection to weighted trees, and a conjecture for a closed form generating function for the area under ternary paths.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Basic Definitions and Overview . . . . .	1
1.1.1	Definitions . . . . .	1
1.1.2	Overview . . . . .	7
1.2	The Catalan Numbers: Some Interpretations and Related Results . . . . .	9
1.2.1	Combinatorial Interpretations of the Catalan Numbers . . . . .	9
1.2.2	Statistics Related to the Catalan Numbers . . . . .	11
1.2.3	Functions Related to $C(z)$ . . . . .	12
1.3	The Ternary numbers and Generalized Dyck Paths . . . . .	15
1.3.1	Combinatorial Interpretations of the Ternary Numbers . . . . .	15
1.3.2	Generalized $t$ -Dyck paths . . . . .	20
1.4	The Riordan Group . . . . .	20
<b>2</b>	<b>Analogues of Catalan Properties</b>	<b>28</b>
2.1	$T(z)$ is to $N(z)$ as $C(z)$ is to $B(z)$ . . . . .	28
2.2	The Chung-Feller Theorem for $t$ -Dyck paths . . . . .	31
2.3	The Motzkin Analogue . . . . .	34
2.3.1	The Balls on the Lawn Problem . . . . .	35
2.3.2	The Euler Transform . . . . .	38
2.4	The Fine Analogue . . . . .	41
<b>3</b>	<b>Returns and Area</b>	<b>45</b>
3.1	Expected Number of Returns . . . . .	45
3.1.1	Dyck Paths . . . . .	46
3.1.2	Ternary Paths . . . . .	46
3.2	Techniques for Finding Area Under Paths . . . . .	51
3.2.1	Dyck Paths . . . . .	52

3.2.2 Ternary Paths . . . . .	57
<b>4 Conclusions and Open Questions</b>	<b>63</b>
4.1 Summary and Work to be Done . . . . .	63
4.2 More Open Questions . . . . .	64
4.2.1 Elements of Pseudo Order 2 . . . . .	65
4.2.2 Determinant Sequences . . . . .	68
4.2.3 Narayana Analogue . . . . .	72

# List of Figures

1.1	The preorder of a plane tree $T$ . . . . .	5
1.2	Pictorial representation of $C(z) = 1 + zC^2(z)$ . . . . .	7
1.3	The three ternary paths from $(0,0)$ to $(6,0)$ . . . . .	15
1.4	The dot diagram of $R$ . . . . .	24
1.5	The dot diagram of $C$ . . . . .	25
1.6	Generating function for Dyck paths with $k$ returns is $z^k C^k(z)$ . . . . .	27
3.1	A plane tree, $T$ , such that $\nu(T) = 22$ . . . . .	54
3.2	A plane binary tree, $B$ , such that $\omega(B) = 24$ . . . . .	55
3.3	Generating function for vertices at height 1 is $2zyC^2(z)$ . . . . .	56
3.4	Calculation of Area Under Strict Ternary Paths and Ternary Paths . . . . .	61
4.1	Elements of Pseudo Order 2 and Their Dot Diagrams . . . . .	67
4.2	The determinant sequences of the ternary numbers . . . . .	72
4.3	The Narayana polynomials, $N_n(x, y)$ . . . . .	73
4.4	The Narayana polynomials in terms of elementary functions . . . . .	74

# Chapter 1

## Introduction

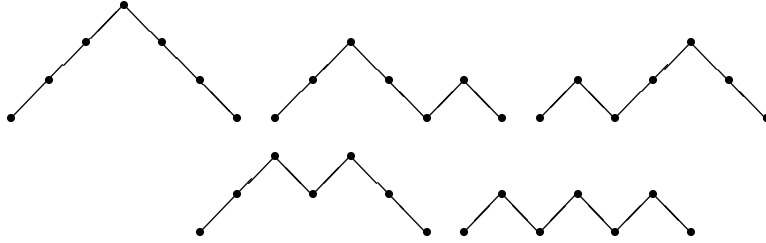
### 1.1 Basic Definitions and Overview

#### 1.1.1 Definitions

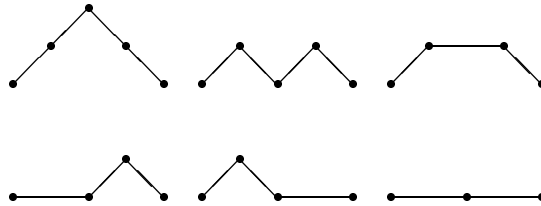
Throughout this dissertation, we will refer to objects known as paths and trees. There are many ways to view these objects, so here we set out the terminology as we intend to use it.

**Definition 1.1.** A **path**,  $P$ , of length  $n$  from  $(x_0, y_0)$  to  $(x, y)$  with step set  $S$  is a sequence of points in the plane,  $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) = (x, y)$  such that all  $(x_{i+1} - x_i, y_{i+1} - y_i) \in S$ . These points are called **vertices**. We define the **height** of a vertex,  $v$ , to be the ordinate of that point. We say that a path,  $P$ , is **positive** if each of its vertices has nonnegative height.

**Example 1.1.** *Dyck paths are positive paths from  $(0,0)$  to  $(2n,0)$  with  $S = \{(1,1), (1,-1)\}$ . Below are the five Dyck paths of length 6.*



**Example 1.2.** *Schröder paths are positive paths from  $(0,0)$  to  $(2n,0)$  with  $S = \{(1,1), (2,0), (1,-1)\}$ . Schröder paths are counted by the big Schröder numbers,  $1, 2, 6, 22, 90, \dots$ . Below are the six Schröder paths ending at  $(4,0)$ .*



**Definition 1.2.** *A (simple) graph is a triple,  $G = (V, E, \phi)$ , where  $V$  is a finite set of vertices,  $E$  is a finite set of edges and  $\phi$  is a function which assigns a unique 2-element set of vertices  $\{u, v\}$ ,  $u \neq v$ , to each edge  $e$ . Since  $\phi(e) = \{u, v\}$  is unique for each  $e$ , we write  $e = uv$  and say  $e$  joins  $u$  and  $v$ . A vertex  $u$  is **adjacent** to  $v$  if there exists an edge  $e$  such that  $e = uv$ .*

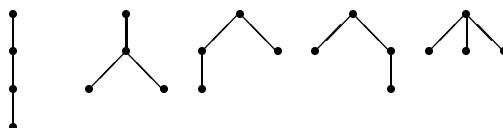
**Definition 1.3.** *A graph  $G$  is **connected** if for any two distinct vertices  $u$  and  $v$ , there exists a sequence  $v_0 v_1 \cdots v_n$  of vertices such that  $v_0 = u$ ,*



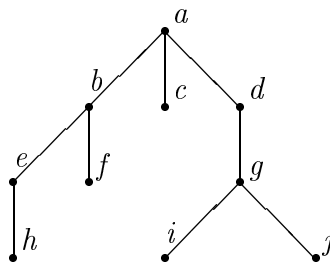
$v_n = v$  and any two consecutive terms  $v_i$  and  $v_{i+1}$  are adjacent. A **cycle** is a sequence  $v_0v_1 \cdots v_n$ ,  $n \geq 1$ , of vertices such that  $v_0 = v_n$ , all other  $v_i$ 's are distinct and any two consecutive terms are adjacent.

**Definition 1.4.** A (rooted) **plane tree** or **tree**,  $T$ , is a connected graph with no cycles, where one vertex is designated as the root.

**Example 1.3.** We take the convention of computer scientists and draw our trees with the root at the top of the tree. Pictured are the five plane trees on 4 vertices.



**Example 1.4.** Below is an example of a plane tree,  $T$ , consisting of a root labelled  $a$ , and subtrees  $T_1 = \{b, e, f, h\}$ ,  $T_2 = \{c\}$ , and  $T_3 = \{d, g, i, j\}$ .



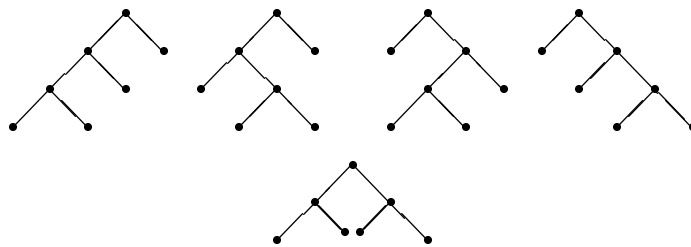
We say that  $b$  has two **successors**. Similarly,  $a$  has three successors and  $g$  has two. We say that  $g$  is the **predecessor** of  $i$  and  $j$ . Every non-root vertex

has a unique predecessor. We say that  $e$  and  $f$  are **siblings**. Similarly,  $j$  and  $i$  are siblings, as are  $b$ ,  $c$  and  $d$ . A vertex with no successors is called a **leaf** or **endpoint**. Vertices  $i$ ,  $j$ ,  $h$ ,  $f$  and  $c$  are leaves.

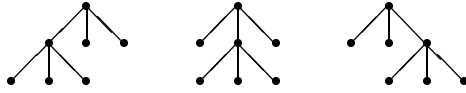
We define the **length** of a tree,  $T$ , denoted  $l(T)$  to be the number of edges of  $T$ . For any vertex,  $v$ , we define the **degree** of  $v$ , denoted  $deg(v)$ , to be the number of successors of  $v$ . We define the **height** of  $v$ , denoted  $hgt(v)$ , to be the length of the unique “path” from the root to  $v$ ; that is,  $hgt(v) = n$ , where  $v_0v_1 \cdots v_n$  is the unique sequence of vertices such that  $v_0$  is the root,  $v_n = v$  and any two consecutive terms  $v_i$  and  $v_{i+1}$  are adjacent. The height of  $v$  is 0 if and only if  $v$  is the root.

A plane  $m$ -ary tree is a plane tree in which every vertex (including the root) has degree 0 or  $m$ .

**Example 1.5.** *The five 2-ary (or binary) trees with 6 edges.*



**Example 1.6.** *The three 3-ary (or ternary) trees with 6 edges.*



Given a tree,  $T$ , it is useful to have a canonical linear ordering of its vertices. One such ordering is called the **preorder** of  $T$ . The preorder of  $T$  can be described (informally yet intuitively) in the following way. Imagine the edges of  $T$  as wooden sticks and imagine a worm starting at the root of  $T$  and crawling down. It begins toward the leftmost edge adjacent to the root and continues climbing along the edges of the sticks until it reaches the root again (for the last time). Recording each vertex as the worm reaches it for the first time gives us the preorder of  $T$ . See figure 1.1.

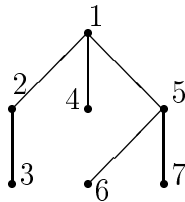
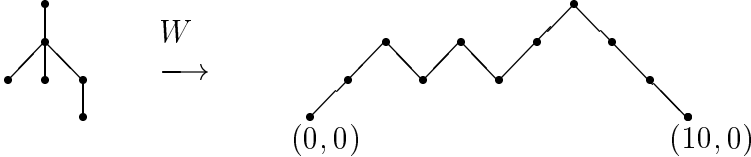


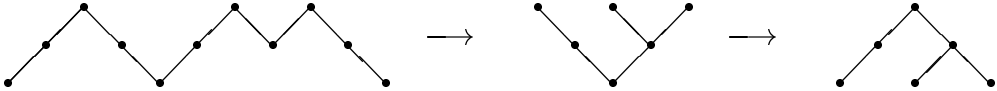
Figure 1.1: The preorder of a plane tree  $T$

There is a standard bijection between Dyck paths of length  $2n$  and plane trees of length  $n$ . We will call it the “worm bijection” since it is closely related to the preorder. Given a plane tree,  $T$ , of length  $n$ , we can map to a unique Dyck path,  $D$ , of length  $2n$ , by traversing  $T$  in preorder and

associating an up  $(1, 1)$  step with the act of traversing an edge downward away from the root and a down  $(1, -1)$  step with traversing an edge upward toward the root. Call this map  $W$ . As an illustration,



To see that the map is reversible, take a Dyck path,  $D$ , of length  $2n$  and “collapse”  $D$  to a plane tree,  $T$ , with  $n$  edges in the following way. Imagine squeezing  $D$  between your hands so that an up step meets its corresponding down step to form an edge of  $T$ . Then flip  $T$  upside down.



We let  $D_n$  denote the set of all Dyck paths of length  $2n$  and  $A_n$  denote the set of all plane trees with  $n$  edges. It can be shown that

$$|D_n| = |A_n| = \frac{1}{n + 1} \binom{2n}{n} \tag{1.1}$$

We let  $c_n$  denote the  $n$ th Catalan number,  $\frac{1}{n+1} \binom{2n}{n}$ , and we call  $C(z) = \sum_{n=0}^{\infty} c_n z^n$  the Catalan generating function. One way to see (1.1) is the following: Take the setting of Dyck paths. A Dyck path consists of either the empty path only *or* it consists of an up step, followed by a Dyck path

at height 1, followed by a down step, followed by a Dyck path at height 0. (This is illustrated in figure 1.2.) The result of this observation is that the generating function for Dyck paths,  $C(z)$ , satisfies

$$C(z) = 1 + zC^2(z).$$

The quadratic formula yields  $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ , and an application of the Lagrange Inversion formula [37, 4] or Taylor series provides the  $n$ -th coefficient,  $\frac{1}{n+1} \binom{2n}{n}$ .

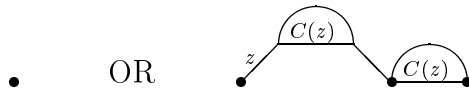


Figure 1.2: Pictorial representation of  $C(z) = 1 + zC^2(z)$

### 1.1.2 Overview

Chapter 1 of this dissertation is intended to lay the foundation and paint the background for the remainder of the document. We hope to (1) give the reader the language and notation used throughout the paper, (2) motivate the questions asked in the later chapters, and (3) present the tools used to answer these questions. In the last section, we presented some basic definitions and examples and established the language used throughout this dissertation.

In section 1.2, we discuss some combinatorial interpretations of the Catalan numbers and some of the interesting results related to those settings. In section 1.3, we describe the sequence of numbers known as the ternary numbers. We will discuss the ternary numbers as a base case generalization of the Catalan numbers and discuss the combinatorial similarities between the two. In section 1.4, we define the Riordan group and demonstrate its utility as a combinatorial tool for solving enumeration problems.

In Chapter 2, we ask the following question: How can the results relating to the Catalan numbers (i.e., the results of section 1.2) be extended to the generalized setting represented by the ternary numbers? What about  $m$ -ary numbers? Are there unusual complications involved in this extension, and, if so, to what can they be attributed? As is the case throughout the research, we will use the Riordan group where appropriate to obtain and interpret results.

In Chapter 3, we turn our attention to more quantitative aspects of the combinatorial structures studied so far. In particular, we want to know about quantities such as the number of returns to the  $x$ -axis (for paths), the degree of the root (for trees), and the area of the region bounded by a path and the  $x$ -axis. In this pursuit, as with others, the ideal goal is to have a bijective

proof of the results. It is always interesting to know to what extent the Riordan group can be used to provide insight for the construction of such bijections.

In Chapter 4, a summary of the work is presented along with unanswered questions. In addition, some new ideas with open questions are raised.

## 1.2 The Catalan Numbers: Some Interpretations and Related Results

### 1.2.1 Combinatorial Interpretations of the Catalan Numbers

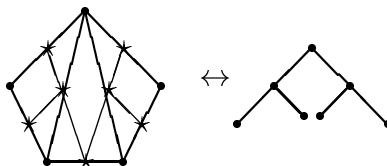
The Catalan number sequence,  $1, 1, 2, 5, 14, 42, \dots, \frac{1}{n+1} \binom{2n}{n}, \dots$  is probably the second best known sequence (after the Fibonacci numbers) in all of enumerative combinatorics. There have been numerous papers written about them. In fact, in [37], Richard Stanley lists 66 combinatorial interpretations of the Catalan number sequence. More interpretations have been listed on his home page. In the last section, we saw that  $C(z)$  counts Dyck paths from  $(0, 0)$  to  $(2n, 0)$  and plane trees with  $n$  edges. In this section, we will

introduce a few more interpretations and related results in order to motivate the work to be done in chapters 2 and 3.

**Theorem 1.1.**  $c_n = \frac{1}{n+1} \binom{2n}{n}$  is the number of:

- (i.) triangulations of a convex  $(n + 2)$ -gon into  $n$  triangles with  $n - 1$  non-intersecting (except at a vertex) diagonals,
- (ii.) planted (i.e., degree of the root is 1) plane trees with  $2n + 2$  vertices where every nonroot vertex has degree 0 or 2,
- (iii.) Dyck paths from  $(0, 0)$  to  $(2n + 2, 0)$  with no peak at height 2,
- (iv.) noncrossing partitions of  $[n]$ . (See [37])

**Proof:** (i.) [Proof by bijection to plane binary trees.] The picture should make clear the map which takes a triangulation of a polygon to a plane binary tree.

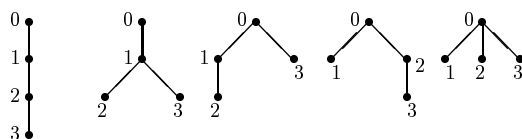


Note that the edges of the convex polygon  $P$  are thick solid lines and the vertices of  $P$  are dots, while the edges of the corresponding binary tree,  $T$ , are thin solid lines and the vertices of  $T$  are stars. Then we rotate the picture



of  $T$  and redraw it according to our convention. The root of  $T$  corresponds to the "bottom" edge of  $P$ . (ii.) [Proof by bijection to plane binary trees.] (iii.) [Proof by generating functions and by bijection provided in [26].] (iv.) [Proof by bijection to plane trees.] Each element of  $[n]$  represents a vertex of a plane tree with  $n + 1$  vertices labelled in preorder starting with the root labelled 0. Make elements of the same block siblings and place the remaining vertices according to the preordering of the tree.

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(This correspondence is due to Dershowitz and Zaks [6].)     $\square$

## 1.2.2 Statistics Related to the Catalan Numbers

Much work has been done with respect to the enumeration of combinatorial structures related to the Catalan numbers according to various statistics. As we will see in Chapter 3, the negative binomial probability distribution with parameters  $r$  and  $p$ , denoted  $negbin(r, p)$ , shows up often with, in many cases,  $r = 2$ . The following theorem [32] presents further evidence of this fact.

**Theorem 1.2.**

(i.) *The limiting distribution of Dyck paths enumerated by number of hills is  $\text{negbin}(2, \frac{2}{3})$ .*

(ii.) *The limiting distribution of noncrossing partitions enumerated by number of visible blocks is  $\text{negbin}(2, \frac{1}{2})$ .*

Another theorem related to the statistics of Catalan numbers is the Chung-Feller Theorem [5, 11]. This theorem addresses the following question. Of all paths from  $(0, 0)$  to  $(2n, 0)$ , how many paths have the property that exactly  $2k$  ( $0 \leq k \leq n$ ) of their steps lie above the  $x$ -axis? The Chung-Feller Theorem asserts that the surprising answer is  $c_n$ , regardless of  $k$ . We will revisit this theorem in Chapter 2.

**1.2.3 Functions Related to  $C(z)$** 

There are some generating functions which form interesting relationships with  $C(z)$ . In particular, they are the Central Binomial function,  $B(z)$ , the Fine function,  $F(z)$ , and the Motzkin function,  $M(z)$ .

In [7], Shapiro and Deutsch list and prove several identities involving the Catalan, Fine, and Central Binomial functions. Here, we describe these functions and discuss a few identities with their proofs.

**Definition 1.5.** *The Central Binomial function,  $B(z)$ , is the generating function for the central binomial coefficients,  $\binom{2n}{n}$ . That is,*

$$B(z) = \sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}$$

$B(z)$  counts paths from  $(0,0)$  to  $(2n,0)$  with  $S = \{(1,1), (1,-1)\}$ .

**Identity 1.1.**  $B(z) = 1 + 2zC(z)B(z)$

**Definition 1.6.** *The Fine function,  $F(z)$ , is the generating function for the Fine numbers,  $1, 0, 1, 2, 6, 18, 57, 186, \dots$ , (named after information theorist Terrence Fine of Cornell University).*

$$F(z) = \sum_{n=0}^{\infty} f_n z^n = \frac{1 - \sqrt{1-4z}}{z(3 - \sqrt{1-4z})}$$

The Fine numbers count Dyck paths with no hills. A hill is defined as an up step (starting on the  $x$ -axis) followed immediately by a down step. They also count plane trees with no leaf at height 1.

**Lemma 1.**

(i.)  $f_n \sim \frac{4}{9}c_n$

(ii.)  $F(z) = \frac{C(z)}{1+zC(z)}$

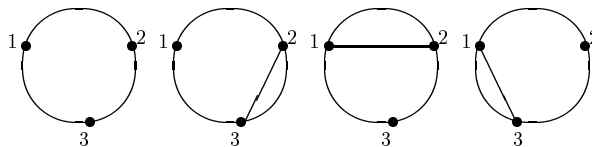
**Definition 1.7.** *The Motzkin function,  $M(z)$ , is the generating function for the Motzkin numbers 1, 1, 2, 4, 9, 21, 51, 127, 323, ...*

$$M(z) = \sum_{n=0}^{\infty} m_n z^n = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$$

The Motzkin numbers have a number of combinatorial interpretations.

They include:

1. the number of ways of drawing any number of nonintersecting chords on  $n$  points on a circle, ( $n = 3$  pictured below)



2. the number of positive paths from  $(0, 0)$  to  $(n, 0)$  using  $S = \{(1, 1), (1, 0), (1, -1)\}$ ,
3. the number of plane trees with  $n$  edges where every vertex has degree  $\leq 2$ .
4. the number of noncrossing partitions  $\pi = \{B_1, \dots, B_k\}$  of  $[n]$  such that if  $B_i = \{b\}$  and  $a < b < c$ , then  $a$  and  $c$  are in different blocks of  $\pi$ .

**Identity 1.2.**  $M(z) = \frac{1}{1-z} C\left(\frac{z^2}{(1-z)^2}\right)$

See section 2.3 for more on the Motzkin numbers.

## 1.3 The Ternary numbers and Generalized Dyck Paths

### 1.3.1 Combinatorial Interpretations of the Ternary Numbers

Let  $\mathcal{T}$  denote the set of all positive paths from  $(0,0)$  to some  $(x,y)$  with  $S = \{(1,1), (1,-2)\}$ . Let  $\mathcal{T}_{n,k}$  denote the paths in  $\mathcal{T}$  of length  $n$  in which the terminal vertex has ordinate  $k$ , so that  $\mathcal{T} = \bigcup_{n,k \geq 0} \mathcal{T}_{n,k}$ . We will refer to paths in  $\mathcal{T}_{n,0}$  as ternary paths. See figure 1.3.

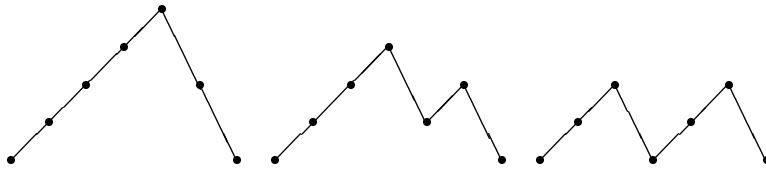


Figure 1.3: The three ternary paths from  $(0,0)$  to  $(6,0)$

Let  $t_{n,k} = |\mathcal{T}_{n,k}|$  and  $t_{3n,0} = t_n$ .

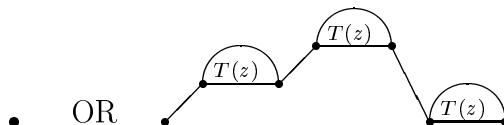
**Theorem 1.3.** *Let  $T(z) = \sum_{n=0}^{\infty} t_n z^n$ . Then:*

(i.)  $T(z) = 1 + zT^3(z)$

(ii.)  $t_{n,k} = [z^n]T^{k+1}(z) = \frac{k+1}{3n+k+1} \binom{3n+k+1}{n}$

$$(iii.) t_n = \frac{1}{2n+1} \binom{3n}{n}$$

**Proof:** (i) Associate a  $z$  to each down step. Every ternary path can be characterized as either the empty path or a path which starts with an up step followed by a ternary path at height 1, another up step, a ternary path at height 2, a down step, and, finally, a ternary path at height 0.



Hence,  $T(z) = 1 + zT^3(z)$ .

(ii.) Characterizing paths ending at height  $k$  as in (i), we have that the generating function for  $t_{n,k}$  is in fact given by  $T^{k+1}$ . The rest follows from the Lagrange Inversion formula. See [4].

(iii.) Follows from (ii.)  $\square$

Emeric Deutsch has provided the following explicit expression for  $T(z)$ .

$$T(z) = \frac{2 \sin \left( \frac{\arcsin \left( \frac{3\sqrt{3z}}{2} \right)}{3} \right)}{\sqrt{3z}}$$

However, in most cases, the equation  $T(z) = 1 + zT^3(z)$  proves more useful in solving problems involving the ternary numbers.

The Catalan-like sequence given in Theorem 1.1(iii) is known as the ternary numbers. There are many combinatorial objects counted by the

ternary numbers. We list some of those objects here and provide a few of the corresponding bijections. The ternary numbers count:

- (1) positive paths starting at  $(0, 0)$  and ending at  $(2n, 0)$  using steps in

$$\{(1, 1), (1, -1), (1, -3), (1, -5), (1, -7), \dots, \};$$

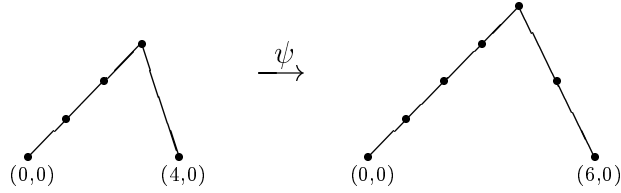
- (2) rooted plane trees with  $2n$  edges where every node (including the root) has even degree (from this point on, referred to as “even trees”);
- (3) rooted plane trees with  $3n$  edges where every node (including the root) has degree zero or three (referred to here as “ternary trees”);
- (4) dissections of a convex  $2(n+1)$ -gon into  $n$  quadrangles by drawing  $n-1$  diagonals;
- (5) noncrossing trees on  $n+1$  points;
- (6) recursively labeled forests on  $[n]$ .

For more detail regarding these objects, see pages 92-93, 168-170 of [37].

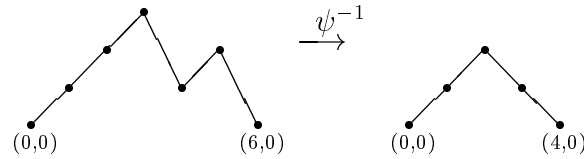
**Proof:** (1) There exists a bijection between these paths and ternary paths.

It is possible to take each one of these paths and convert it to a unique ternary path with the following procedure. Given a path,  $P$ , of type (1),

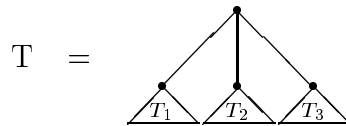
create a new path,  $\psi(P)$ , by replacing each down  $(1, -(2s + 1))$  step of  $P$  with an up  $(1, 1)$  step immediately followed by  $s + 1$  down  $(1, -2)$  steps. For example,



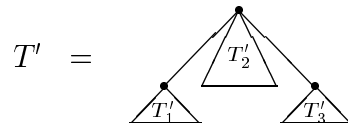
Then  $\psi(P)$  is a positive path from  $(0, 0)$  to  $(3n, 0)$  consisting of steps in  $\{(1, 1), (1, -2)\}$ . It is easy to see that  $\psi$  is invertible. For example,



(2) (due to Deutsch, Feretić, and Noy [8]) There exists a bijection between even trees and ternary trees. Given a nonempty ternary tree  $T$ ,



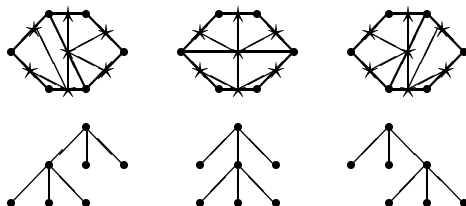
recursively define an even tree  $T'$  by





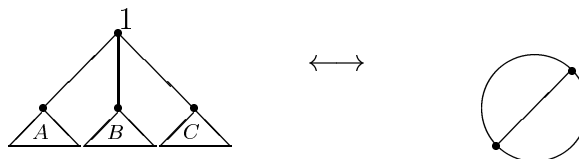
(3) There exists a bijection from ternary paths to ternary trees. First, take a ternary tree and label the children of every vertex of degree 3, U, U, D, from left to right. Then read the labels of the tree in preorder. Let U correspond to  $(1, 1)$  and D correspond to  $(1, -2)$ .

(4) Proof by bijection to ternary trees. (See proof of Theorem 1.1(i).)



(5) Proof by bijection to ternary trees. A noncrossing tree is defined as a tree on a linearly ordered vertex set where whenever  $i < j < k < l$ , then  $ik$  and  $jl$  are not both edges. Equivalently, a noncrossing tree is a tree whose vertices are  $n$  points on a circle labelled  $\{1, 2, \dots, n\}$  counterclockwise and whose edges are rectilinear and do not cross.

The picture below represents the bijection between noncrossing trees and ternary trees. Note that  $i$ ,  $1 < i \leq n$ , is the smallest labelled vertex adjacent to the vertex labelled 1.



(6) See [37], exercise 5.45, p. 92, and solution 5.45, p. 137.  $\square$

### 1.3.2 Generalized $t$ -Dyck paths

A **generalized  $t$ -Dyck path** is a positive path from  $(0, 0)$  to  $((t + 1)n, 0)$  with  $S = \{(1, 1), (1, -t)\}$ . A generalized 2-Dyck path is a ternary path, while a 1-Dyck path is a Dyck path. To a large extent, this research focuses on the following question: Of all the known properties of Dyck paths (plane trees) and the techniques for proving them, which ones can be "easily" extended to generalized  $t$ -Dyck paths? Which ones can not and why?

## 1.4 The Riordan Group

The Riordan group is used extensively throughout this research as a combinatorial tool for solving enumeration problems. As the recent work of Sprugnoli et al. demonstrates, there are also many interesting questions to be answered about the Riordan group itself, a few which are explored here.

**Definition 1.8.** *An infinite lower triangular matrix,  $L = (l_{n,k})_{n,k \geq 0}$  is a **Riordan matrix** if there exist generating functions  $g(z) = \sum g_n z^n$ ,  $f(z) = \sum f_n z^n$ ,  $f_0 = 0$ ,  $f_1 \neq 0$  such that  $l_{n,0} = g_n$  and  $\sum_{n \geq k} l_{n,k} z^n = g(z) (f(z))^k$ .*





$$= \begin{bmatrix} 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ 132 \\ 429 \\ \dots \end{bmatrix} \quad (1.3)$$

$$= \frac{C(z)}{1 - zC(z)} \quad (1.4)$$

$$= C^2(z) \quad (1.5)$$

*Equality (1.5) follows from the fact that  $C(z) = 1 + zC^2(z)$  implies  $C(z) = \frac{1}{1 - zC(z)}$ .*

Given two Riordan matrices,  $L_1 = (g_1(z), f_1(z))$  and  $L_2 = (g_2(z), f_2(z))$ , it turns out that the product of  $L_1$  and  $L_2$  (under usual matrix multiplication) is also Riordan. In fact,

$$L_1 L_2 = (g_1(z)g_2(f_1(z)), f_2(f_1(z)))$$

Furthermore, (i) matrix multiplication is associative, (ii) the identity matrix  $I = (1, z)$  is Riordan, and (iii) every Riordan matrix  $L = (g(z), f(z))$  has a Riordan inverse,  $L^{-1} = (\frac{1}{g(\bar{f}(z))}, \bar{f}(z))$ , where  $\bar{f}(z)$  is the compositional inverse of  $f(z)$ . Hence, the set of Riordan matrices,  $\mathcal{R}$ , forms a group under matrix multiplication.

Rogers [29] showed that every element,  $x_{n+1, k+1}$ , such that  $n \geq 0$ , of a

Riordan matrix,  $R$ , can be expressed as

$$x_{n+1,k+1} = a_0 x_{n,k} + a_1 x_{n,k+1} + a_2 x_{n,k+2} + a_3 x_{n,k+3} + \dots$$

where  $\{a_j\}_{j=0}^{\infty}$  is a sequence that is independent of  $n$  and  $k$ . The sequence  $\{a_j\}_{j=0}^{\infty}$  is referred to as the  **$A$ -sequence of  $R$**  and its generating function is denoted  $A_R(z)$ . Similarly, we have that the elements of the first column of  $R$  (excluding  $x_{0,0}$ ), can be expressed as

$$x_{n+1,0} = z_0 x_{n,0} + z_1 x_{n,1} + z_2 x_{n,2} + z_3 x_{n,3} + \dots$$

where  $\{z_j\}_{j=0}^{\infty}$  is a sequence that is independent of  $n$  and  $k$ . The sequence  $\{z_j\}_{j=0}^{\infty}$  is referred to as the  **$Z$ -sequence of  $R$**  and its generating function is denoted  $Z_R(z)$ .

We can view the  $A$ - and  $Z$ -sequences of a Riordan matrix,  $R$ , pictorially as what we refer to as the **dot diagram** of  $R$ . See figure 1.4.

$$\left( \begin{array}{cccc} z_0 & z_1 & z_2 & \dots \\ \circ & \circ & \circ & \dots \\ \times & & & \end{array} , \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 & \dots \\ \circ & \circ & \circ & \circ & \dots \\ & \times & & & \end{array} \right)$$

Figure 1.4: The dot diagram of  $R$

**Example 1.10.** *The dot diagram of  $C = (C(z), zC(z))$  is given in figure 1.5.*

$$\left( \begin{array}{cccc} 1 & 1 & 1 & \dots \\ \circ & \circ & \circ & \dots \\ \times & & & \end{array} , \begin{array}{cccc} 1 & 1 & 1 & \dots \\ \circ & \circ & \circ & \dots \\ \times & & & \end{array} \right)$$

Figure 1.5: The dot diagram of  $C$ 

**Theorem 1.5.** (Merlini et al. [22]). *Let  $R = (g, f)$  and  $S = (h, k)$  be Riordan matrices and  $V = RS$ . Then*

$$A_V(z) = A_S(z)[A_R(y)|_{z=k(y)}]$$

There are several subgroups of the Riordan group which are of particular combinatorial interest:

The **Appell subgroup**,  $\{L \in \mathcal{R} : L = (g(z), z) \text{ for some } g\}$

The **Bell subgroup**,  $\{L \in \mathcal{R} : L = (g(z), zg(z)) \text{ for some } g\}$

The **Associated subgroup**,  $\{L \in \mathcal{R} : L = (1, f(z)) \text{ for some } f\}$

The **Checkerboard subgroup**,  $\{(g, f) \in \mathcal{R} : g \text{ is an even function, } f \text{ is an odd function}\}$

**Theorem 1.6.** *The Appell subgroup is normal.*

**Theorem 1.7.** *The Checkerboard subgroup is the centralizer of  $(1, -z)$ .*

**Theorem 1.8.**  *$\mathcal{R}$  is the semi-direct product of the Appell subgroup and the Associated subgroup and also of the Appell subgroup and the Bell subgroup.*

**Proof:**

$$\begin{aligned}
 (g, f) &= (g, z)(1, f) \\
 &= \left(\frac{gz}{f}, z\right)\left(\frac{f}{z}, f\right) \\
 &= \left(\frac{gz}{f} \cdot \frac{f}{z}, f\right).
 \end{aligned}$$

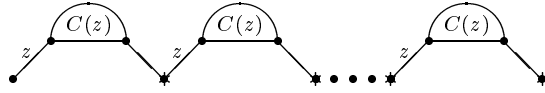
□

**Example 1.11.** Consider Dyck paths from  $(0, 0)$  to  $(2n, 0)$ . Suppose we ask for the average number of returns to the  $x$ -axis. (A "return" is defined as a nonorigin vertex having ordinate 0.) One way to calculate an answer is to use the Riordan group. We take an initial count by letting  $\langle x_{n,k} \rangle$  be the number of paths having  $k$  returns. Then

$$\langle x_{n,k} \rangle = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 2 & 2 & 1 & & \\ 0 & 5 & 5 & 3 & 1 & \\ & & \dots & & & \end{bmatrix},$$

which looks like it could be the Riordan element  $(1, zC(z))$ . But how do we prove it? Figure 1.6 shows that the generating function for the number of paths with  $k$  returns is  $z^k C^k(z)$ , which is precisely the  $k$ th column of  $(1, zC(z))$ . To compute the total number of returns, we multiply by the column vector  $[0, 1, 2, 3, \dots]^T$  whose generating function is  $\frac{z}{(1-z)^2}$ .





A Dyck path with  $k$  returns  
(a return is denoted by  $*$ )

Figure 1.6: Generating function for Dyck paths with  $k$  returns is  $z^k C^k(z)$ .

$$\begin{aligned}
 \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 2 & 2 & 1 & \\ 0 & 5 & 5 & 3 & 1 \\ & & \dots & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \dots \end{bmatrix} &= (1, zC(z)) * \left( \frac{z}{(1-z)^2} \right) \\
 &= \frac{zC(z)}{(1-zC(z))^2} \\
 &= zC^3(z) \\
 &\leftrightarrow 0, 1, 3, 9, \dots
 \end{aligned}$$

Hence, the generating function for total number of returns is  $zC^3(z)$ . To find the average, we divide by the total number of paths.

$$\begin{aligned}
 \frac{[z^n]zC^3(z)}{[z^n]C(z)} &= \frac{\frac{3}{2(n-1)+3} \binom{2(n-1)+3}{n-1}}{\frac{1}{n+1} \binom{2n}{n}} \\
 &= \frac{3n}{n+2} \\
 &\rightarrow 3
 \end{aligned}$$

## Chapter 2

# Analogues of Catalan

## Properties

### 2.1 $T(z)$ is to $N(z)$ as $C(z)$ is to $B(z)$

In this section, we are concerned with comparing the function  $T(z)$  with the famous Catalan function,  $C(z)$ . We consider  $T(z)$  as combinatorially analogous to  $C(z)$ . As two illustrations of this, we have  $C(z)$  counting Dyck paths from  $(0, 0)$  to  $(2n, 0)$  and rooted plane trees with  $n$  edges; analogously,  $T(z)$  counting the objects described in section 1.3. Deutsch and Shapiro presented several basic identities involving  $C(z)$  and the generating function

for the central binomial coefficients, [7]. As discussed, combinatorial proofs exist for the following identities:

- $B(z) = 1 + 2zC(z)B(z)$
- $B(z)C^s(z) = \sum_{n=0}^{\infty} \binom{2n+s}{n} z^n$

We would like to present combinatorial proofs of two analogous identities for  $T(z)$ . First, we introduce  $N(z) = \sum \binom{3n}{n} z^n$ , the analogue of  $B(z)$ .

**Theorem 2.1.** *The function  $N(z) = \sum \binom{3n}{n} z^n$  counts*

- (i.) *paths in  $Z \times Z$  starting at  $(0,0)$  and ending at  $(3n,0)$  using steps in  $\{(1,1), (1,-2)\}$ , and*
- (ii.) *even trees with  $2n$  edges and one node colored red.*

We want to explore the observation that  $B(z)$  is to  $C(z)$  as  $N(z)$  is to  $T(z)$ . Note the following identities:

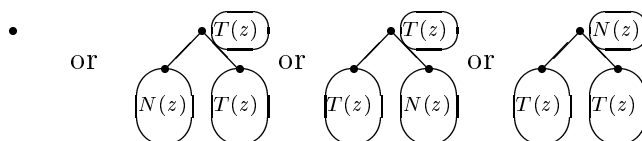
**Theorem 2.2.**

- (i.)  $N(z) = 1 + 3zT^2(z)N(z)$
- (ii.)  $N(z)T^s(z) = \sum_{n=0}^{\infty} \binom{3n+s}{n} z^n$

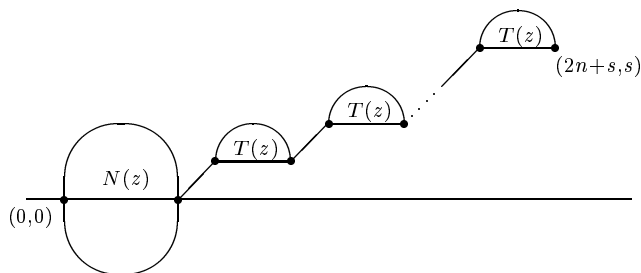
**Proof:** (i.) Consider any even tree with one red node (counted by  $N(z)$ ).

What does it look like? It is either the “empty” even tree consisting of just

the root, or it has two edges adjacent to the root with an even tree (possibly empty) growing from each edge and an even tree (possibly empty) growing from the root. The red node is located in one of the three locations, as illustrated below.



Hence, the generating function for even trees with one red node is given by  $1 + 3zT^2(z)N(z)$ . (ii.) Consider paths starting at  $(0,0)$  and ending at  $(3n+s, s)$ , using steps in  $\{(1, 1), (1, -2)\}$  and consisting of  $2n+s$  “up” steps and  $n$  “down” steps. Clearly, the generating function that counts these paths is given on the right hand side of this equation. On the other hand, below is a picture of an arbitrary path of this kind.



From the picture, it is easy to see that the generating function for these paths is also given by  $N(z)T^s(z)$ .  $\square$

Note that the identity in Theorem 2.2(i) is equivalent to the observation that

$$\frac{d}{dz}(zT(z^2)) = N(z^2).$$

## 2.2 The Chung-Feller Theorem for $t$ -Dyck paths

We have seen that the  $n$ th Catalan number,  $c_n = \frac{1}{n+1} \binom{2n}{n}$  counts the number of paths from  $(0,0)$  to  $(2n,0)$  using steps in the set  $\{(1,1), (1,-1)\}$  and staying above the  $x$ -axis (called Dyck paths). If we remove the latter “above the ground” restriction, we have what we shall refer to as binomial paths which are counted by the  $n$ th binomial coefficient,  $\binom{2n}{n}$ .

**Theorem 2.3.** (*Chung, Feller [11]*) *Let  $c_{n,k}$  denote the number of paths from the origin to the point  $(2n,0)$  such that  $2k$  of its edges lie above the  $x$ -axis and  $2n - 2k$  below ( $k = 0, 1, \dots, n$ ). Then  $c_{n,k} = c_n$ , independently of  $k$ .*

This theorem makes the assertion that for binomial paths the number of “up” steps above the  $x$ -axis has a uniform distribution. It turns out that we have the same result for generalized Dyck paths.

**Theorem 2.4.** *Let  $u_{n,k}$  be the number of paths from  $(0,0)$  to  $(3n,0)$  consisting of steps in  $\{(1,1), (1,-2)\}$  with  $k$  up steps above the  $x$ -axis. Then*

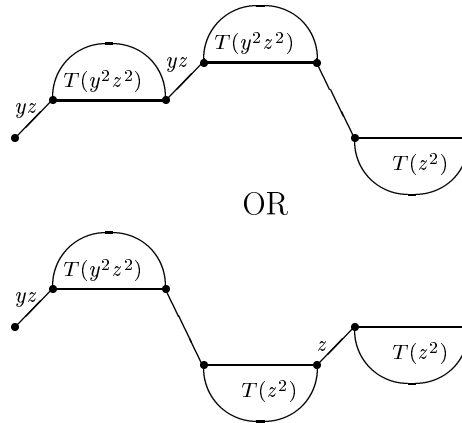
$$u_{n,k} = t_n = \frac{1}{2n+1} \binom{3n}{n}.$$

**Proof:** Associate a  $z$  with each up step and a  $y$  with each up step above the  $x$ -axis. Define  $U(y, z)$  to be the bivariate generating function for  $\{u_{n,k}\}_{n,k \geq 0}$ .

That is,

$$U(y, z) = \sum_{0 \leq k \leq n} u_{n,k} y^k z^n$$

Let us examine  $U = U(y, z)$  and write it in terms of  $T(z)$ . We can characterize any path  $P$  counted by  $u_{n,k}$  in the following way. First,  $P$  is allowed to make an initial descent (possibly empty) below the  $x$ -axis before it makes its first step from height 0 to height 1. This trip below the  $x$ -axis is counted by  $T(z^2)$ . Then, at that point,  $P$  can be decomposed into any number of repetitions of two connected components (pictured below), whose generating function we denote  $\Psi(y, z)$ .



As the picture illustrates, we have

$$\Psi(y, z) = y^2 z^2 T^2(y^2 z^2) T(z^2) + y z^2 T(y^2 z^2) T^2(z^2)$$

It follows that all  $P$  are counted by  $\frac{T(z^2)}{1-\Psi(y,z)}$ . Hence,

$$U(y, z) = \frac{T(z^2)}{1 - \Psi(y, z)} \quad (2.1)$$

$$= \frac{T(z^2)}{1 - y^2 z^2 T^2(y^2 z^2) T(z^2) - y z^2 T(y^2 z^2) T^2(z^2)} \quad (2.2)$$

$$= \frac{H(z)}{1 - y^2 z^2 H^2(yz) H(z) - y z^2 H(yz) H^2(z)}, \quad (2.3)$$

where, for brevity, we let  $H(z) = T(z^2)$ . (Note that  $H(z) = 1 + z^2 H^3(z)$ .)

Now, on the other hand, suppose  $u_{n,k} = \frac{1}{2n+1} \binom{3n}{n}$  (depending on  $n$ , independent of  $k$ ). Then let  $U^*(y, z)$  be its bivariate generating function. It follows that:

$$\begin{aligned} U^*(y, z) &= 1 + (1 + y + y^2)z^2 + (1 + y + y^2 + y^3 + y^4)3z^4 + \dots + \\ &\quad + (1 + y + \dots + y^{2k})u_{n,k}z^{2k} + \dots \end{aligned} \quad (2.4)$$

$$= \frac{1}{1-y} H(z) - \frac{y}{1-y} H(yz) \quad (2.5)$$

$$= \frac{H(z) - yH(yz)}{1-y} \quad (2.6)$$

To prove the theorem, it suffices to show  $U(y, z) = U^*(y, z)$ .

$$[H(z) - yH(yz)][1 - y^2 z^2 H^2(yz) H(z) - y z^2 H(yz) H^2(z)] \quad (2.7)$$

$$\begin{aligned}
&= H(z) - y^2 z^2 H^2(yz) H^2(z) - y z^2 H(yz) H^3(z) \\
&\quad - y H(yz) + y^3 z^2 H^3(yz) H(z) + y^2 z^2 H^2(yz) H^2(z) \tag{2.8}
\end{aligned}$$

$$= H(z) - y H(yz) - y z^2 H(yz) H^3(z) + y^3 z^2 H^3(yz) H(z) \tag{2.9}$$

$$= H(z) - y H(yz) - y(z^2 H(yz) H^3(z) - y^2 z^2 H^3(yz) H(z)) \tag{2.10}$$

$$= H(z) - y H(yz) - y[(H(z) - 1)H(yz) - (H(yz) - 1)H(z)] \tag{2.11}$$

$$= H(z) - y H(z) - y(H(z) - H(yz)) \tag{2.12}$$

$$= H(z)(1 - y) \tag{2.13}$$

And we have our result.  $\square$

It can be shown that this theorem and its proof generalize to paths with steps in  $\{(1, 1), (1, -s)\}$  for  $s \in N$ .

## 2.3 The Motzkin Analogue

In the previous sections, we discovered Catalan-like properties and identities for the ternary numbers. When considering many of the Catalan settings, one often encounters their closely related “cousin,” the Motzkin numbers,  $1, 1, 2, 4, 9, 21, 51, \dots$

One nice illustration of the Motzkin numbers and their close relationship to the Catalan numbers is the following. Take Dyck paths, which are



counted by the Catalan numbers, and create a new set of paths by allowing, in addition to the up (1,1) and down (1,-1) steps, a level step (1,0).

Motzkin numbers count [37]:

- (1) paths from  $(0,0)$  to  $(n,0)$  using  $\{(1,1), (1,-1), (1,0)\}$  which never go below the  $x$ -axis,
- (2) plane trees with  $n$  edges where every node has degree at most 2.

The generating function for the Motzkin numbers is given by  $M(z) = 1 + zM(z) + z^2M^2(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$  where  $M(z) = \sum_{n=0}^{\infty} m_n z^n$  and  $m_n$  is the  $n$ th Motzkin number.

A natural question to ask is: What is the Motzkin analogue for  $T(z)$ ? In other words,  $M(z)$  is to  $C(z)$  as    ? is to  $T(z)$ . It should be noted that the answer to this may not be unique.

### 2.3.1 The Balls on the Lawn Problem

Consider plane trees as one of the many settings of Motzkin and Catalan numbers. Catalan numbers count plane binary trees with  $2n$  edges ( $2n + 1$  vertices). Motzkin numbers count plane trees with  $n$  edges where every node has outdegree  $\leq 2$ . There is another less known interpretation of the Catalan

and Motzkin numbers which connects nicely to the plane tree setting. We refer to it as the “Balls on the Lawn Problem” (TBOTLP).

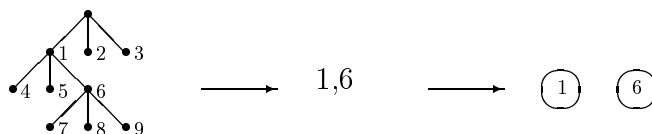
Imagine a gnome living in a basket on a lawn. You toss two balls labelled “1” and “2” into the basket. The gnome tosses back one of the two balls onto the lawn. You then toss two more balls labelled “3” and “4” into the basket. The gnome tosses back one of the three remaining balls. The process continues. The number of possible combinations of balls on the lawn after  $n$  tosses is the  $(n + 1)$ -st Catalan number,  $c_{n+1}$ .

\*\*\*\*VARIATION: If we vary TBOTLP by requiring that all balls tossed in at toss  $j$  are labelled “ $j$ ”, then the number of combinations of balls on the lawn is the  $n$ -th Motzkin number,  $m_n$ .

If we slightly change TBOTLP by having three balls per toss, the answer is the  $(n + 1)$ -st ternary number,  $t_{n+1}$ .

We can prove these results through a bijection to plane  $m$ -ary trees, where  $m$  is the number of balls per toss. For instance, consider ternary trees. Define a map,  $\mathcal{L}$ , in the following manner. Label all nonroot vertices from top to bottom, left to right, in ascending order starting with 1. Then record only nonroot branch vertices (i.e., nonterminal vertices), again in ascending order.

For example,



This gives each ternary tree,  $T$ , with  $3(n + 1)$  edges, a unique image  $\mathcal{L}(T)$ , which corresponds to a unique combination of balls on the lawn after  $n$  tosses.

Now, if we take the variation of TBOPLP (mentioned in the box above) and say that all balls tossed in at toss  $j$  are labelled “ $j$ ”, we obtain our Motzkin analogue,  $M_T(z)$ , for the ternary numbers. In terms of plane trees, the variation is equivalent to plane trees with  $n$  edges where every vertex has degree  $\leq 3$ . It is easy to see that  $M_T(z)$  satisfies  $M_T(z) = 1 + zM_T(z) + z^2M_T^2(z) + z^3M_T^3(z)$ .

$$M_T(z) = 1 + z + 2z^2 + 5z^3 + 13z^4 + 36z^5 + 104z^6 + \dots$$

In terms of paths,  $M_T(z)$  counts paths from  $(0, 0)$  to  $(n, 0)$  using  $\{(1,1), (1,-1), (1,-2), (1,0)\}$  which never go below the  $x$ -axis. In particular, the  $(n, k)$ th entry of the Bell subgroup Riordan matrix

$$\begin{aligned} M_T &= (M_T(z), zM_T(z)) \\ &= \langle x_{n,k} \rangle_{n,k \geq 0} \end{aligned}$$



the Motzkin and Catalan numbers. The expression in terms of Riordan matrices is:  $M = PC$  where  $M = (M(z), zM(z))$ ,  $C = (C(z^2), zC(z^2))$ , and  $P = (\frac{1}{1-z}, \frac{z}{1-z})$ . The relationship is proven combinatorially by taking Dyck paths, with generating function  $C(z^2)$  where  $z$  corresponds to any non-origin vertex, and placing any number of level (1,0) steps, with generating function  $\frac{1}{1-z}$ , at the beginning and/or at any non-origin points on the paths.

Now that we have a Motzkin analogue,  $M_T(z)$ , we would like to know whether there is a relationship between  $T(z)$  and  $M_T(z)$  which mimics the Euler transform discussed above. We do know immediately that the following relationship exists:  $PA = M_T$  or, equivalently,  $M_T(z) = \frac{1}{1-z}A(\frac{z}{1-z})$ , where  $M_T = (M_T(z), zM_T(z))$ ,  $P = (\frac{1}{1-z}, \frac{z}{1-z})$ , and

$$(i.) \quad A(z) = 1 + z^2A^2(z) + z^3A^3(z)$$

(ii.)

$$A = (A(z), zA(z))$$

$$= \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 2 & 3 & 0 & 1 \\ 5 & 5 & 3 & 4 & 0 & 1 \\ 8 & 12 & 9 & 4 & 5 & 0 & 1 \\ 21 & 21 & 21 & 14 & 5 & 6 & 0 & 1 \\ & & & \dots & & & & \end{bmatrix}$$



## 2.4 The Fine Analogue

As mentioned in section 1.2.3, the Fine numbers  $\{f_n\}$  count the number of Dyck paths from  $(0,0)$  to  $(2n,0)$  with no hills. The generating function for the Fine numbers,  $F(z)$ , is related to the Catalan generating function by these identities (among others):

- $F(z) = \frac{C(z)}{1+zC(z)}$
- $F(z) = \frac{1}{1-z^2C^2(z)}$

In this section, we will use the Riordan group to find the analogue of the Fine numbers for  $T(z)$ . But, first, let us rediscover the Fine numbers themselves through a Riordan group technique known as “subgroup decomposition” [31]. We recall that there are several subgroups of the Riordan group which are of interest. We can view a chosen subgroup as a group acting on a set of vectors, or in particular, the vector  $\left[ 1 \ 1 \ 1 \ 1 \ \dots \right]^T \leftrightarrow \left( \frac{1}{1-z} \right)$ . What we get then is a decomposition of a targeted sequence with combinatorial significance into “pieces” equipped with a related combinatorial interpretation. For example, consider the Bell subgroup and let our targeted sequence be the Catalan numbers. We seek the function  $h(z)$  such that  $(h(z), zh(z)) * \left( \frac{1}{1-z} \right) = C(z)$ . It turns out that  $h(z)$  is in fact  $F(z)$ , the Fine

generating function. Hence, the element  $F = (F(z), zF(z))$  is the unique Bell subgroup element such that  $F * \left(\frac{1}{1-z}\right) = C(z)$ .

$$\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ 2 & 2 & 0 & 1 & & \\ 6 & 4 & 3 & 0 & 1 & \\ 18 & 13 & 6 & 4 & 0 & 1 \\ & & & \dots & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \dots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ \dots \end{bmatrix}$$

The matrix on the left has the property that the  $k$ -th column,  $k = 0, 1, 2, 3, \dots$ , is the generating function for the Dyck paths with  $k$  hills, its row sums being the Catalan numbers. Hence, we have decomposed the Catalan numbers into Dyck paths with  $k$  hills.


This method provides an ideal way to get the Fine analogue for  $T(z)$ . Let  $F_T = (F_T(z), zF_T(z))$  be such that  $F_T * \left(\frac{1}{1-z}\right)$ . Using Riordan multiplication, we get

$$F_T(z) = \frac{T(z)}{1 + zT(z)}.$$

So we have,

$$\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 2 & 0 & 1 & & & \\ 7 & 4 & 0 & 1 & & \\ 34 & 14 & 6 & 0 & 1 & \\ 171 & 72 & 21 & 8 & 0 & 1 \\ & & & \dots & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \dots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 12 \\ 55 \\ 273 \\ \dots \end{bmatrix}$$



The  $k$ -th column,  $k = 0, 1, 2, \dots$ , of the matrix on the left represents ternary paths with  $k$  bumps. A bump is defined as two up (1,1) steps followed by a down (1,-2) step (on the  $x$ -axis).  The 0-th column is, of course, our Fine analogue,  $F_T(z)$ , which counts ternary paths with no bumps.

Using  $F_T(z) = \frac{T(z)}{1+zT(z)}$  and  $T(z) - zT^3(z) = 1$ , we obtain the following identities.

**Identity 2.1.**

$$T(z) = \frac{F_T(z)}{1 - zF_T(z)}$$

**Proof:**

$$F_T(z) = \frac{T(z)}{1 + zT(z)}$$

$$F_T(z) = T(z) - zF_T(z)T(z)$$

$$F_T(z) = T(z)(1 - zF_T(z))$$

$$T(z) = \frac{F_T(z)}{1 - zF_T(z)}$$

□

**Identity 2.2.**

$$F_T(z) = \frac{1}{1 + z - zT^2(z)}$$

**Proof:**

$$\begin{aligned} F_T(z) &= \frac{T(z)}{1+zT(z)} \cdot \frac{1-zT^2(z)}{1-zT^2(z)} \\ &= \frac{T(z) - zT^3(z)}{(1+zT(z))(1-zT^2(z))} \\ &= \frac{1}{(1+zT(z))(1-zT^2(z))} \\ &= \frac{1}{1+z(T(z) - zT^3(z)) - zT^2(z)} \\ &= \frac{1}{1+z - zT^2(z)} \end{aligned}$$

□

# Chapter 3

## Returns and Area

### 3.1 Expected Number of Returns

In this section, we consider some statistical questions related to generalized Catalan numbers and their interpretations. For simplicity, we choose to focus on the setting of generalized  $t$ -Dyck paths. There are various substructures of interest in this setting, such as hills, peaks, returns, double descents, and valleys. Here, we will restrict our attention to returns. There are numerous questions one can ask. For instance, what is the probability of randomly choosing a path having  $k$  returns? How many returns can one expect to see? What is the limiting distribution for the number of returns?

Furthermore, how can these results be interpreted in other settings, such as plane trees, dissections of convex  $n$ -gons, and noncrossing partitions?

We begin our exploration with Dyck paths, then ternary paths, and finally generalized  $t$ -Dyck paths.

### 3.1.1 Dyck Paths

We saw in section 1.4 that the expected number of returns to the  $x$ -axis for Dyck paths is  $\frac{3n}{n+2}$ , which approaches 3 as  $n \rightarrow \infty$ .

### 3.1.2 Ternary Paths

Let us restrict  $\mathcal{T}$  (see section 1.3) to those paths which have terminal height 0, i.e.  $T_{n,0}$ , and count these paths by number of returns to the  $x$ -axis. Let  $r_{n,j}$  denote the number of paths of length  $n$  with  $j$  returns to the  $x$ -axis.

$$R = (r_{n,j})_{n,j \geq 0} = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 0 & 7 & 4 & 1 & & & & \\ 0 & 30 & 18 & 6 & 1 & & & \\ 0 & 143 & 88 & 33 & 8 & 1 & & \\ 0 & 728 & 455 & 182 & 52 & 10 & 1 & \\ & & & \dots & & & & \end{bmatrix} = (1, zT^2(z))$$

Note that  $R$  is in the Associated subgroup.

One might ask, on average, how many returns to the  $x$ -axis can we expect to see? To answer this, we calculate the associated probabilities.

**Question 1.** *What is the probability of  $m$  returns to the  $x$ -axis?*

We readily note that the generating function for the number of paths with  $m$  returns is given by the function corresponding to the  $m$ th column of  $R$ , that is,  $z^m T^{2m}(z)$ . Let  $p_m(n)$  denote the probability that a randomly chosen path of length  $3n$  has  $m$  returns. Then

$$\begin{aligned} p_m(n) &= \frac{[z^n] z^m T^{2m}(z)}{[z^n] T(z)} \\ &= \frac{\frac{2m}{3(n-m)+2m} \binom{3(n-m)+2m}{n-m}}{\frac{1}{3n+1} \binom{3n+1}{n}} \\ &= \frac{2m(2n+1)(n-m+1)(n-m+2) \cdots (n-1)}{3(3n-m)(3n-m+1)(3n-m+2) \cdots (3n-1)} \end{aligned}$$

It follows immediately that  $p_m(n) \rightarrow \frac{4m}{3^{m+1}}$  as  $n \rightarrow \infty$ . Thus, the limiting distribution is  $negbin(2, \frac{2}{3})$ .

**Question 2.** *What is the expected number of returns to the  $x$ -axis?*

To answer this, let  $Y(n)$  denote the random variable for the number of returns to the  $x$ -axis. Since we have calculated probability exactly above, we can form a probability generating function for  $Y(n)$ , call it  $P_{Y(n)}(z)$ , and use

it to calculate both the expected value and variance of  $Y(n)$ . So define

$$\begin{aligned} P_{Y(n)}(z) &:= \sum_{m=0}^{\infty} p_m(n) z^n \\ &= \sum_{m=0}^{\infty} \frac{2m(2n+1)(n-m+1)(n-m+2) \cdots (n-1)}{3(3n-m)(3n-m+1)(3n-m+2) \cdots (3n-1)} z^n \end{aligned}$$

It turns out that  $P_{Y(n)}(z)$  is close to being a hypergeometric function. In fact,

$$P_{Y(n)}(z) = \frac{2(2n+1)}{3(3n-1)} z F(2, 1-n; 2-3n; z)$$

where  $F(a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_m; z)$  is a hypergeometric function with upper parameters  $a_1, \dots, a_n$  and lower parameters  $b_1, \dots, b_m$  of the form

$$F(a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_m; z) = \sum_{j=0}^{\infty} \frac{a_1^{\bar{j}} a_2^{\bar{j}} \cdots a_n^{\bar{j}}}{b_1^{\bar{j}} b_2^{\bar{j}} \cdots b_m^{\bar{j}}} \cdot \frac{z^j}{j!}$$

where  $x^{\bar{j}} = x(x+1)(x+2) \cdots (x+j-1)$ . [15]

It follows from the definition of  $P_{Y(n)}(z)$  that the expected value of  $Y(n)$ ,  $EY(n)$ , is given by  $EY(n) = P'_{Y(n)}(1)$ .

There are two facts about hypergeometric functions we will use here. The first is the Chu-Vandermonde identity and the second is an identity for derivatives. [18]

- $F(a, -j; c; 1) = \frac{(a-c)^{\underline{j}}}{(-c)^{\underline{j}}}$
- $\frac{d^n}{dz^n} F(a, b; c; z) = \frac{a^{\bar{n}} b^{\bar{n}}}{c^{\bar{n}}} F(a+n, b+n; c+n; z)$

Using the latter identity and the product rule for differentiation, we have

$$P'_{Y(n)}(z) = \frac{2(2n+1)}{3(3n-1)} \left( \frac{2(1-n)}{2-3n} z F(3, 2-n; 3-3n; z) + F(2, 1-n; 2-3n; z) \right)$$

Then, using the Chu-Vandermonde identity, we calculate  $EY(n)$ .

$$\begin{aligned} EY(n) &= P'_{Y(n)}(1) \\ &= \frac{2(2n+1)}{3(3n-1)} \left( \frac{2(1-n)}{2-3n} F(3, 2-n; 3-3n; 1) + F(2, 1-n; 2-3n; 1) \right) \\ &= \frac{2(2n+1)}{3(3n-1)} \left( \frac{2(n-1)(3n)^{n-2}}{(3n-2)(3n-3)^{n-2}} + \frac{(3n)^{n-1}}{(3n-2)^{n-1}} \right) \\ &= \frac{2(2n+1)}{3(3n-1)} \left( \frac{2(n-1)(3n)(3n-1)(3n-2)}{(3n-2)(2n+2)(2n+1)(2n)} + \frac{(3n)(3n-1)}{(2n)(2n+1)} \right) \\ &= \frac{4n(n-1)(3n-2)}{(3n-2)(2n+2)(2n)} + 1 \\ &= \frac{2(n-1)}{2(n+1)} + 1 \\ &= \frac{2n}{n+1} \end{aligned}$$

and therefore,  $EY(n) \rightarrow 2$  as  $n \rightarrow \infty$ .

**Question 3.** *What is the variance of  $Y(n)$ ?*

It follows from the definition of  $P_{Y(n)}(z)$  that

$$\text{Var}Y(n) = P''_{Y(n)}(1) + P'_{Y(n)}(1) - (P'_{Y(n)}(1))^2$$

Differentiating  $P'_{Y(n)}(z)$ , we get

$$P''_{Y(n)}(z) = \frac{4(2n+1)(n-1)}{3(3n-1)(3n-2)} \left( \frac{3(2-n)}{3-3n} z F(4, 3-n; 4-3n; z) + 2F(3, 2-n; 3-3n; z) \right)$$

Then,

$$\begin{aligned}
P_{Y(n)}''(1) &= \frac{4(2n+1)(n-1)}{3(3n-1)(3n-2)} \left( \frac{3(2-n)}{3-3n} \frac{(3n)^{n-3}}{(3n-4)^{n-3}} + 2 \frac{(3n)^{n-2}}{(3n-3)^{n-2}} \right) \\
&= 2 \frac{(n-1)}{(n+1)} + \frac{3(n-1)(n-2)}{(n+1)(2n+3)} \\
&= \frac{7n(n-1)}{(n+1)(2n+3)}
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\text{Var}Y(n) &= \frac{7n(n-1)}{(n+1)(2n+3)} + \frac{2n}{n+1} - \left( \frac{2n}{n+1} \right)^2 \\
&= \frac{3n^3 - 2n^2 - n}{(n+1)^2(2n+3)}
\end{aligned}$$

and  $\text{Var}Y(n) \rightarrow \frac{3}{2}$  as  $n \rightarrow \infty$ .

So we have that the average number of returns for ternary paths approaches 2 with variance  $\frac{3}{2}$ . It can be shown that the limiting distribution is  $negbin(2, \frac{2}{3})$ . In the case of Dyck paths, the average number of returns is 3 with variance 4 and the limiting distribution is  $negbin(2, \frac{1}{2})$ . Naturally, it would be nice to know the limiting distribution for generalized Dyck paths. The calculation proceeds almost exactly as above and we get the following result.

**Theorem 3.1.** *The expected number of returns for generalized Dyck paths from  $(0, 0)$  to  $((t+1)n, 0)$  is  $\frac{(t+2)n}{tn+2}$  with variance  $\frac{2tn(n-1)((t+1)n+1)}{(tn+2)^2(tn+3)}$ .*



**Proof:** As before, let  $Y(n)$  denote the number of returns. Then

$$\begin{aligned} P_{Y(n)}(z) &= \sum_{m=0}^{\infty} \frac{tm(tn+1)(n-m+1)(n-m+2)\cdots n}{((t+1)n-m)((t+1)n-m+1)((t+1)n-m+2)\cdots((t+1)n)} z^m \\ &= \frac{t(tn+1)}{(t+1)((t+1)n-1)} zF(2, 1-n; 2-(t+1)n; z) \end{aligned}$$

Take  $P'_{Y(n)}(1)$  and  $P''_{Y(n)}(1)$ .  $\square$

Hence, the average number of returns for generalized Dyck paths approaches  $\frac{t+2}{t}$  with variance  $\frac{2(t+1)}{t^2}$ . The limiting distribution in this case is  $negbin(2, \frac{t}{t+1})$ .

We can also interpret these results in terms of trees and other settings. For example, we can prove bijectively that on average, the average degree of the root for even trees approaches 4, while the average length of the central path for ternary trees approaches 2. (It would be interesting to know the comparable results for, say, noncrossing trees or diagonally convex directed polyminoes [8].)

## 3.2 Techniques for Finding Area Under Paths

In the last section, we studied some statistical properties of a certain substructure of generalized Dyck paths. In this section, we study a more subtle substructure of paths, i.e. the area bounded by the path. This con-

cept has been studied before. In fact, the work of Pergola, Pinzani, Rinaldi, and Sulanke [27, 28] has inspired much of the work done here. A particularly fascinating aspect of the work done by Pergola et al. is their use of bijective proofs. However, as their results are mainly for Schröder paths and generalized Motzkin paths, it would be remarkable if the ideas and techniques presented in their work could produce bijective proofs in the case of the generalized Dyck paths we study here.

As before, we begin our exploration with Dyck paths, and then extend to ternary paths, and finally  $t$ -Dyck paths.

### 3.2.1 Dyck Paths

Much work has been done with respect to the area under Dyck paths, defined as the area of the region bounded by the path and the  $x$ -axis. [40, 5] In this section, we revisit some of these results and set the groundwork for calculating area under ternary paths.

The first result is equivalent to Woan et al. [40]

**Theorem 3.2.** *The sum of the areas of all strict Dyck paths of length  $2n$  is  $4^{n-1}$ .*

The next result is originally due to Kreweras, although both Merlini et

al. [22] and Chapman [5] have provided alternative proofs.

**Theorem 3.3.** *The sum of the areas of all Dyck paths of length  $2n$  is  $4^n - \frac{1}{2} \binom{2n+2}{n+1}$ .*

In this section, we find out that there is a direct correspondence between weighted plane binary trees and area of Dyck paths.

Let  $\mathcal{D}_n$  denote the set of Dyck paths of length  $2n$ . For  $D \in \mathcal{D}_n$ , define the **area** of  $D$ ,  $a(D)$ , to be the area of the region bounded by  $D$  and the  $x$ -axis.

One of the known correspondences between area of Dyck paths and weighted plane trees uses a weight  $\nu$  defined as follows. Given a plane tree  $T$ , we define  $\nu(T)$  in the following way:

$$\nu(T) = \sum_{v \in T} hgt(v) totdeg(v)$$

That is, we label each vertex,  $v$ , with its height times its total degree,  $totdeg(v)$ , which is the number of successors plus one, and then sum over all vertices in the tree. See figure 3.1.

It is easy to see this correspondence through the "worm" bijection as explained in Chapter 1. As we traverse the tree in preorder, each time we encounter a node, the height of that node corresponds to the height of a vertex on the path. The sum of the heights on the path is the area. Hence,

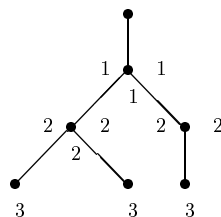


Figure 3.1: A plane tree,  $T$ , such that  $\nu(T) = 22$

it follows that

$$\nu(T) = a(D),$$

where  $D$  is the Dyck path which corresponds to  $T$  under the worm bijection.

It turns out that there is also a correspondence between weighted plane binary trees and the area of Dyck paths. Let  $\mathcal{B}_n$  denote the set of all plane binary trees with  $2n$  edges. Let  $B \in \mathcal{B}_n$ . Define the total weight of  $B$ ,  $\omega(B)$ , in the following way:

$$\omega(B) = \sum_{v \in B} \text{hgt}(v)$$

See figure 3.2.

In the following theorem, we have the somewhat surprising result that the total sum of the weights of binary trees with  $2n$  edges is twice the total sum of the areas of Dyck paths of length  $2n$ .

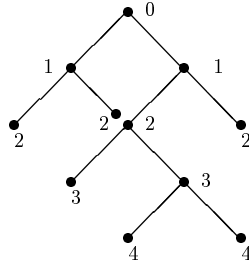


Figure 3.2: A plane binary tree,  $B$ , such that  $\omega(B) = 24$

**Theorem 3.4.** For all  $n \in \mathbb{N}$ ,

$$\sum_{B \in \mathcal{B}_n} \omega(B) = 2 \sum_{D \in \mathcal{D}_n} a(D)$$

*Proof.* (by generating functions) Let  $b(k, n)$  denote the total number of vertices at height  $k$  in  $\mathcal{B}_n$ . Let  $\beta(y, z) = \sum_{n, k \geq 0} b(k, n) y^k z^n$ . Then

$$\beta(y, z) = 1 + (2y)z + (4y + 4y^2)z^2 + (10y + 12y^2 + 8y^3)z^3 + (28y + 36y^2 + 32y^3 + 16y^4)z^4 + \dots$$

If we differentiate  $\beta(y, z)$  with respect to  $y$  and evaluate at  $y = 1$ , we get the generating function for the total weight of all binary trees. Hence, it would be helpful to have a closed form of  $\beta(y, z)$ . Claim: The generating function for the number of vertices at height  $k$  is  $(2zy)^k C^{k+1}(z)$ , where  $C(z)$  is the Catalan generating function. The above illustration makes the claim easy to see. Hence,

$$\beta(y, z) = 2zyC^2(z) + 4z^2y^2C^3(z) + 8z^3y^3C^4(z) + 16z^4y^4C^5(z) + \dots$$

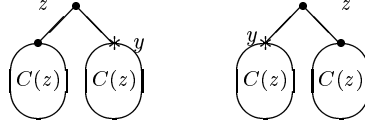


Figure 3.3: Generating function for vertices at height 1 is  $2zyC^2(z)$

$$= \frac{2zyC^2(z)}{1 - 2zyC(z)}$$

Differentiating, we have

$$\begin{aligned} \frac{d}{dy}(\beta(y, z))|_{y=1} &= \frac{(1 - 2zyC(z))(2zC^2(z)) - 2zyC^2(z)(-2zC(z))}{(1 - 2zyC(z))^2}|_{y=1} \\ &= \frac{2zC^2(z) - 4z^2yC^3(z) + 4z^2yC^3(z)}{(1 - 2zyC(z))^2}|_{y=1} \\ &= \frac{2zC^2(z)}{(1 - 2zC(z))^2} \\ &= \frac{1 - 2z - \sqrt{1 - 4z}}{z(1 - 4z)} \\ &= \frac{2z + (1 - 4z) - \sqrt{1 - 4z}}{z(1 - 4z)} \\ &= \frac{2}{1 - 4z} - \frac{\sqrt{1 - 4z} - (1 - 4z)}{z(1 - 4z)} \\ &= \frac{2}{1 - 4z} - \frac{1}{z} \left( \frac{1}{\sqrt{1 - 4z}} - 1 \right) \end{aligned}$$

Extracting coefficients, we have

$$\begin{aligned} [z^n] \left\{ \frac{d}{dy}(\beta(y, z))|_{y=1} \right\} &= [z^n] \left\{ \frac{2}{1 - 4z} \right\} - [z^n] \left\{ \frac{1}{z} \left( \frac{1}{\sqrt{1 - 4z}} - 1 \right) \right\} \\ &= 2 \cdot 4^n - \binom{2(n+1)}{n+1} \end{aligned}$$

which is twice the area under Dyck paths cited above.  $\square$

We readily observe that the correspondence between weighted binary trees and area of Dyck paths is only true for the total weight and the total area. The correspondence does not seem to hold for any individual trees and paths. Still, it seems reasonable that a bijective proof of the theorem exists.

### 3.2.2 Ternary Paths

We will take two different approaches to finding the area under ternary paths. The first approach involves finding the number of vertices at height  $k$ . The second approach recalls the correspondence between Dyck paths (previous section) and weighted binary trees, and sets up a similar correspondence between ternary paths and weighted ternary trees.

Let  $\mathcal{S}_n$  denote the set of all ternary paths in  $\mathcal{T}_{n,0}$  with exactly 1 return. We will refer to these paths as **strict** ternary paths. Let  $a_n^s$ , resp.  $a_n^t$ , denote the total area of the region bounded by the paths of  $\mathcal{S}_n$ , resp.  $\mathcal{T}_{n,0}$ , and the  $x$ -axis. Our goal is to find a generating function or recurrence for the sequence  $\{a_n^s\}_{n=0}^\infty$ , resp.  $\{a_n^t\}_{n=0}^\infty$ .

To approach this problem, we consider  $h_{n,k}^s$ , the number of points in  $\mathcal{S}_n \cap \mathbb{Z} \times \mathbb{Z}$  which have height  $k$ . (It will suffice to consider strict ternary paths for now.) The idea is to calculate area by multiplying the number of

vertices by their respective heights and summing over all heights. The first task is to find a generating function for  $h_{n,k}^s$ .

**Theorem 3.5.** *Let  $H_k^s(z) = \sum_{n=0}^{\infty} h_{n,k}^s z^n$  and  $H_k^t(z) = \sum_{n=0}^{\infty} h_{n,k}^t z^n$ .*

(i.)

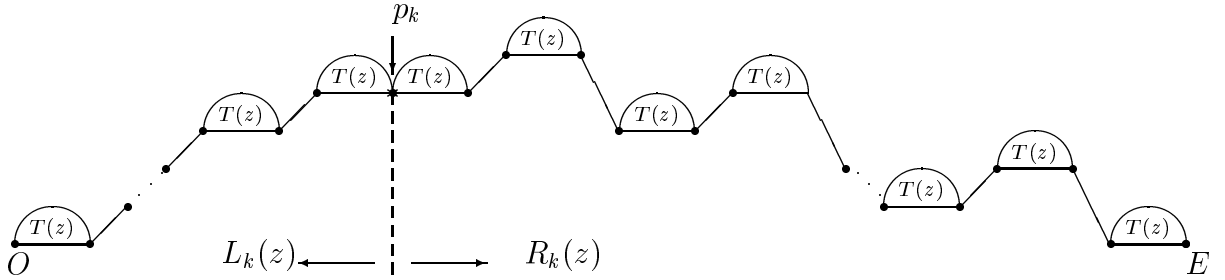
$$H_k^s(z) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} (zT^3(z))^{k-j} = (zT^3(z))^k \frac{\frac{z^2}{1-z} T^3\left(\frac{z^2}{1-z}\right)}{z^2 T^3\left(\frac{z^2}{1-z}\right) - 1}$$

(ii.)

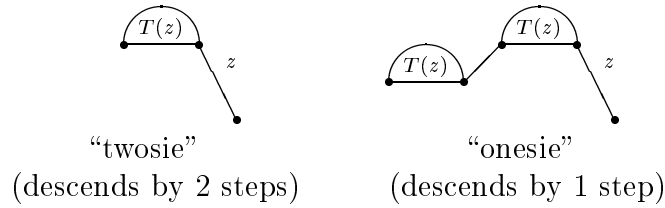
$$H_k^t(z) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} (zT^3(z))^{k-j} T^2(z) = (zT^3(z))^{k+2} \frac{\frac{z^2}{1-z} T^3\left(\frac{z^2}{1-z}\right)}{z^2 T^3\left(\frac{z^2}{1-z}\right) - 1}$$

**Proof:** Suppose we have a point,  $p_k$ , with ordinate  $k$  on an arbitrary path,  $P \in \bigcup_{n=0}^{\infty} \mathcal{S}_n$  with starting point  $O$  and terminal point  $E$ . Let  $L_k(z)$  and  $R_k(z)$  be the generating functions for the number of ways to reach  $p_k$  from  $O$  (i.e., from the left) and  $E$  from  $p_k$  (i.e., from the right), respectively. Then  $H_k^s(z) = L_k(z)R_k(z)$ . It is easy to see that  $L_k(z) = T^k(z)$ .  $R_k(z)$ , on the other hand, requires a little more analysis. As the picture below illustrates, to determine  $R_k$  we want to enumerate paths from  $p_k$  to  $E$  using the allowed steps.





There are really two “components”, combinations of which allow a descent from  $p_k$  to  $E$ , and each has a corresponding generating function. Those two components, which for obvious reasons we shall refer to as “onesies” and “twosies”, are illustrated below:



(These combinations are enumerated by the Fibonacci numbers.) If  $j$  is the number of twosies, there are  $\binom{k-j}{j}$  ways to reach  $E$  using  $k-j$  onesies and  $j$  twosies. Note that each time a onesie is used the corresponding generating function will be  $zT^2(z)$  and each time a twosie is used the generating function is  $zT(z)$ . Hence,

$$R_k(z) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} (zT(z))^j (zT^2(z))^{k-2j}$$

Therefore, we have that

$$H_k^s(z) = T^k(z) \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} (zT(z))^j (zT^2(z))^{k-2j} \quad (3.1)$$

$$= \begin{bmatrix} 1 & & & & & & & & & \\ 1 & & & & & & & & & \\ 1 & 1 & & & & & & & & \\ 1 & 2 & & & & & & & & \\ 1 & 3 & 1 & & & & & & & \\ 1 & 4 & 3 & & & & & & & \\ 1 & 5 & 6 & 1 & & & & & & \\ 1 & 6 & 10 & 4 & & & & & & \\ 1 & 7 & 15 & 10 & 1 & & & & & \\ & & \dots & & & & & & & \end{bmatrix} \begin{bmatrix} (zT^3(z))^k \\ (zT^3(z))^{k-1} \\ (zT^3(z))^{k-2} \\ (zT^3(z))^{k-3} \\ (zT^3(z))^{k-4} \\ (zT^3(z))^{k-5} \\ (zT^3(z))^{k-6} \\ (zT^3(z))^{k-7} \\ (zT^3(z))^{k-8} \\ \dots \end{bmatrix} \quad (3.2)$$

$$= \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} (zT^3(z))^{k-j} \quad (3.3)$$

Having the necessary generating functions for vertices at height  $k$ , we are now able to calculate area for ternary and strict ternary paths. See figure 3.4.

Currently, we do not have a “nice” closed form for the generating functions above. We would like to use these generating functions to form a manageable Riordan or Riordan-like matrix. Given such a matrix, we could multiply it by the Riordan vector  $(\frac{z}{1-z})$ , the product being the vectors  $(a_n^s)_{n=0}^\infty$  and  $(a_n^t)_{n=0}^\infty$ .

We saw in section 3.2.1 that there is a correspondence between the weight of binary trees and area under Dyck paths. It is natural to ask if there is a similar correspondence for ternary trees and ternary paths. It turns out that indeed such a correspondence exists.

Let  $\mathcal{W}_n$  denote the set of all ternary trees with  $3n$  edges.

$$\begin{aligned}
 & \begin{bmatrix} 1 & & & & & & & & & & \\ 0 & 1 & 1 & & & & & & & & \\ 0 & 3 & 4 & 2 & 1 & & & & & & \\ 0 & 12 & 18 & 13 & 9 & 3 & 1 & & & & \\ 0 & 55 & 88 & 75 & 61 & 31 & 15 & 4 & 1 & & \\ 0 & 273 & 455 & 427 & 383 & 238 & 140 & 58 & 22 & 5 & 1 \\ & & & & \dots & & & & & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \dots \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 21 \\ 144 \\ 981 \\ 6663 \\ \dots \end{bmatrix} \leftrightarrow a_n^s \\
 \\
 & \begin{bmatrix} 1 & & & & & & & & & & \\ 2 & 1 & 1 & & & & & & & & \\ 7 & 5 & 6 & 2 & 1 & & & & & & \\ 30 & 25 & 33 & 17 & 11 & 3 & 1 & & & & \\ 143 & 130 & 182 & 115 & 86 & 37 & 17 & 4 & 1 & & \\ 728 & 700 & 1020 & 728 & 598 & 321 & 177 & 66 & 24 & 5 & 1 \\ & & & & \dots & & & & & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \dots \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 27 \\ 207 \\ 1506 \\ 10692 \\ \dots \end{bmatrix} \leftrightarrow a_n^t
 \end{aligned}$$

Figure 3.4: Calculation of Area Under Strict Ternary Paths and Ternary Paths

**Conjecture 3.1.** *For all  $n \in \mathbb{N}$ ,*

$$\sum_{W \in \mathcal{W}_n} \omega(W) = \sum_{T \in \mathcal{T}_n} a(T)$$

If we let  $t(k, n)$  denote the total number of vertices at height  $k$  in  $\mathcal{W}_n$  and  $\tau(y, z) = \sum t(k, n)y^k z^n$ , then we can easily show (see proof of Theorem 3.3) that

$$\begin{aligned} \frac{d}{dy}(\tau(y, z))|_{y=1} &= \frac{3zT^3(z)}{(1 - 3zT^2(z))^2} \\ &= 3z + 27z^2 + 207z^3 + 1506z^4 + 10692z^5 + \dots \end{aligned}$$

which agrees with the area function for ternary paths up to the first several terms. Hence, it remains to prove that  $\frac{3zT^3(z)}{(1-3zT^2(z))^2}$  is, in fact, the area function.

# Chapter 4

## Conclusions and Open Questions

### 4.1 Summary and Work to be Done

In Chapter 2, we established, in the setting of the ternary numbers, several analogues to the better known Catalan numbers setting. We presented an analogue to the Chung-Feller Theorem [5, 11] which says that for paths from  $(0, 0)$  to  $(3n, 0)$  with step set  $\{(1, 1), (1, -2)\}$ , the number of up  $(1, 1)$  steps above the  $x$ -axis is uniformly distributed. We also presented analogues to the Binomial, Motzkin and Fine generating functions and discussed com-

binatorial interpretations of each. Further study of the functions should lead to more interpretations. In particular, further study into the existence of an Euler Transform analogue and analogues to the Fine/Catalan/Binomial identities [7] is needed.

In Chapter 3, we established some computational results regarding area and the number of returns in the context of ternary paths. We concluded that the number of returns for generalized  $t$ -Dyck paths approaches a  $negbin(2, \frac{t}{t+1})$  distribution. Further study will address remaining questions, such as the combinatorial significance of the parameter 2. We also presented generating function proofs for the area under Dyck and ternary paths, a connection to weighted trees, and a conjecture for a closed form generating function for the area under ternary paths. Future work regarding area under paths will include finding generating functions for the area under generalized  $t$ -Dyck paths and providing bijective proofs.

## 4.2 More Open Questions

The remainder of the dissertation addresses some open research questions which, for the sake of order, have been grouped apart from the preceding

results. The reason for the separation is that the material that follows, although related to the ideas covered so far, has connections to other topics in combinatorics not necessarily emphasized in the previous chapters.

### 4.2.1 Elements of Pseudo Order 2

An element  $R$  of the Riordan group is said to have pseudo order two, or order  $2*$ , if  $RM$  has order two, where  $M = (1, -z)$ . A prime example of an element of order  $2*$  is the generalized Pascal's triangle matrix,  $\Psi_b = \left(\frac{1}{1-2bz}, \frac{-z}{1-2bz}\right)$ . An initial survey shows that it is the case for some  $b$  that  $\Psi_b = BMB^{-1}M$  for some  $B$ . For instance,  $\Psi_2 = C_0MC_0^{-1}M$ , where  $C_0 = (C^2(z), zC^2(z))$ .

We would like to find combinatorial interpretations of such elements of order  $2*$ . We know for example that elements of the form  $BMB^{-1}M$  have order  $2*$ . One might ask, if  $R$  has order  $2*$ , is it the case that  $R = BMB^{-1}M$  for some  $B$ ? If so, what is the combinatorial relationship between  $R$  and  $B$ ? It turns out that we can affirmatively answer the first of these questions for a special class of matrices, and, in some cases, characterize  $B$  given a certain  $R$ .

**Theorem 4.1.** *Let*

$$\Theta_{b,\lambda,\epsilon,\delta} = \left( \frac{1 + \frac{\epsilon z}{1-bz} C \left( \frac{\lambda z^2}{(1-bz)^2} \right) - \frac{\delta z^2}{(1-bz)^2} C^2 \left( \frac{\lambda z^2}{(1-bz)^2} \right)}{1 - \frac{\epsilon z}{1-bz} C \left( \frac{\lambda z^2}{(1-bz)^2} \right) - \frac{\delta z^2}{(1-bz)^2} C^2 \left( \frac{\lambda z^2}{(1-bz)^2} \right)}, \frac{1}{1-2bz}, \frac{-z}{1-2bz} \right)$$

*Then*

(i.)  $\Theta_{b,\lambda,\epsilon,\delta}$  has order  $2^*$ .

(ii.) There exists a Riordan matrix  $B$  such that  $\Theta_{b,\lambda,\epsilon,\delta} = BMB^{-1}M$ .

**Corollary 4.1.**  $\Psi_b = BMB^{-1}M$  where

$$B = \left( \frac{2n}{2n - k - (2n - k)bz + k\sqrt{1 - 2bz + (b^2 - 4n)z^2}}, \frac{1 - bz - \sqrt{1 - 2bz + (b^2 - 4n)z^2}}{2nz} \right)$$

for some  $n$  and  $k$ .

**Corollary 4.2.**  $\Theta_{b,\lambda,\epsilon,0} = \left( \frac{(2\lambda - \epsilon b)z + \epsilon - \epsilon\sqrt{1 - 2bz + (b^2 - 4\lambda)z^2}}{(2\lambda + \epsilon b)z - \epsilon + \epsilon\sqrt{1 - 2bz + (b^2 - 4\lambda)z^2}}, \frac{1}{1-2bz}, \frac{-z}{1-2bz} \right) = BMB^{-1}M$ ,

where

$$B = \left( \frac{2}{1 - (2\epsilon + b)z + \sqrt{1 - 2bz + (b^2 - 4\lambda)z^2}}, \frac{1 - bz - \sqrt{1 - 2bz + (b^2 - 4\lambda)z^2}}{2nz} \right)$$

.

However, there exist elements of order  $2^*$  which are not of the class  $\Theta_{b,\lambda,\epsilon,\delta}$ .

An example is the Bell subgroup element  $(1 + zM(z), z(1 + zM(z)))$ , where  $M(z)$  is the Motzkin generating function. This matrix counts directed animals and the rows approach  $negbin(2, \frac{1}{3})$ . Other examples are Nkwanta's



$A = (g, f)$	Dot diagram of $A$	$A^* = AMA^{-1}M$	Dot diagram of $A^*$
$(M(z), zM(z))$	$\left( \begin{array}{ccc ccc} 1 & 1 & & 1 & 1 & 1 \\ \circ & \circ & & \circ & \circ & \circ \\ \times & & & & \times & \end{array} \right)$	$\left( \frac{1}{1-2z}, \frac{z}{1-2z} \right)$	$\left( \begin{array}{ccc ccc} 2 & & & 1 & & 2 \\ \circ & & & \circ & & \circ \\ \times & & & & \times & \end{array} \right)$
$(C^2(z), zC^2(z))$	$\left( \begin{array}{ccc ccc} 2 & 1 & & 1 & 2 & 1 \\ \circ & \circ & & \circ & \circ & \circ \\ \times & & & & \times & \end{array} \right)$	$\left( \frac{1}{1-4z}, \frac{z}{1-4z} \right)$	$\left( \begin{array}{ccc ccc} 4 & & & 1 & & 4 \\ \circ & & & \circ & & \circ \\ \times & & & & \times & \end{array} \right)$
$(S(z), zS(z))$	$\left( \begin{array}{ccc ccc} 3 & 2 & & 1 & 3 & 2 \\ \circ & \circ & & \circ & \circ & \circ \\ \times & & & & \times & \end{array} \right)$	$\left( \frac{1}{1-6z}, \frac{z}{1-6z} \right)$	$\left( \begin{array}{ccc ccc} 6 & & & 1 & & 6 \\ \circ & & & \circ & & \circ \\ \times & & & & \times & \end{array} \right)$
$\left( \frac{1}{\sqrt{z^2-6z+1}}, zS(z) \right)$	$\left( \begin{array}{ccc ccc} 3 & 4 & & 1 & 3 & 2 \\ \circ & \circ & & \circ & \circ & \circ \\ \times & & & & \times & \end{array} \right)$	$\left( \frac{1}{1-6z}, \frac{z}{1-6z} \right)$	$\left( \begin{array}{ccc ccc} 6 & & & 1 & & 6 \\ \circ & & & \circ & & \circ \\ \times & & & & \times & \end{array} \right)$

Note:  $S(z) = \frac{1-3z-\sqrt{z^2-6z+1}}{4z^2}$  is the generating function for the little Schröder numbers, 1,3,11,45,...

Figure 4.1: Elements of Pseudo Order 2 and Their Dot Diagrams

RNA triangle

$$R = \left( \frac{1-z+z^2-\sqrt{1-z-3z^2}}{2z^2}, z \frac{1-z+z^2-\sqrt{1-z-3z^2}}{2z^2} \right)$$

which counts secondary RNA structures [32] and  $\tilde{C} = (2C(z) - 1, z(2C(z) - 1))$

which counts Dyck paths with hills that are one of two colors. In the case

of the latter, we have that  $\tilde{C} = BMB^{-1}M$  where  $B = (C(z), zC(z))$ . (See

section 1.4) As for  $R$ , it is still unknown whether there exists  $B$  such that

$$R = BMB^{-1}M.$$

## 4.2.2 Determinant Sequences

Yet another remarkable property of the Catalan numbers has to do with its determinant sequences. Given a sequence  $\{a_n\}_{n=0}^{\infty}$ , we define its determinant sequence in the following way

$$A_n^k := \begin{vmatrix} a_k & a_{k+1} & a_{k+2} & \cdots & a_{n+k-1} \\ a_{k+1} & a_{k+2} & a_{k+3} & \cdots & a_{n+k} \\ a_{k+2} & a_{k+3} & a_{k+4} & \cdots & a_{n+k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+k-1} & a_{n+k} & a_{n+k+1} & \cdots & a_{2n+k-2} \end{vmatrix}$$

Both Shapiro [30] and Aigner [1] showed that in the case of the Catalan number sequence,  $A_n^0 = A_n^1 = 1$ , for all  $n$ . Furthermore, there is a combinatorial interpretation of the determinant sequence in terms of paths. The determinant sequences for the Catalan numbers  $\{c_n\}$  represent vertex disjoint Dyck path systems [21].

Naturally, it would be interesting to know about the determinant sequences for the ternary numbers. An initial investigation immediately reveals that in the case of the ternary numbers  $A_n^0 \neq A_n^1$ . However, both  $A_n^0$  and  $A_n^1$  are sequences which have shown up in the literature, particularly [2, 35], and have connections to alternating sign matrices and plane partitions, as

we shall see below.

As a result of the following theorem of Gronau et al. [16], the origins of which are traced to Gessel and Viennot [12, 20], we have a combinatorial interpretation of the determinant sequences for the ternary numbers.

Let  $G$  be an acyclic directed graph, i.e.  $G$  has no directed cycles. Let  $A = \{a_1, a_2, \dots, a_n\}$  be a fixed set of sources, and  $B = \{b_1, b_2, \dots, b_n\}$  be a fixed set of sinks in  $G$ . A path system in  $(G, A, B)$  is a set  $W = \{w_1, w_2, \dots, w_n\}$  of paths in  $G$  such that there exist a permutation  $\sigma = \sigma(W) \in S_n$  so that  $w_i$  leads from  $a_i$  to  $b_{\sigma(i)}$  for every  $i \in \{1, 2, \dots, n\}$ .  $W$  is disjoint if for every  $i$  and  $j$  ( $1 \leq i < j \leq n$ ),  $w_i$  and  $w_j$  have disjoint sets of vertices.

**Theorem 4.2.** *Let  $p_{ij}$  be the number of paths leading from  $a_i$  to  $b_j$  in  $G$ , let  $p^+$  be the number of disjoint path systems  $W$  in  $(G, A, B)$  for which  $\sigma(W)$  is an even permutation, and let  $p^-$  be the number of such systems with for which  $\sigma(W)$  is odd. Then  $\det(p_{ij}) = p^+ - p^-$ .*

Let  $\mathcal{G}$  be the infinite directed graph with vertex set  $\mathbb{Z} \times \mathbb{Z}$  and directed edges from  $(i, j)$  to  $(i + 1, j + 1)$  and to  $(i + 1, j - 2)$ ,  $\forall i, j \in \mathbb{Z}$ . Let  $Q_n^k$  be the family of all sets of pairwise vertex disjoint paths in  $\mathcal{G}$ ,  $\xi_0, \xi_1, \dots, \xi_{n-1}$ , such that  $\xi_i$  joins  $(-3i, 0)$  with  $(3(i + k), 0)$ ,  $i = 0, 1, \dots, n - 1$ . Observe that the number of directed paths in  $G$  from  $(-3i, 0)$  to  $(3(i + k), 0)$  is the ternary

number,  $t_{2i+k}$ . The theorem implies that

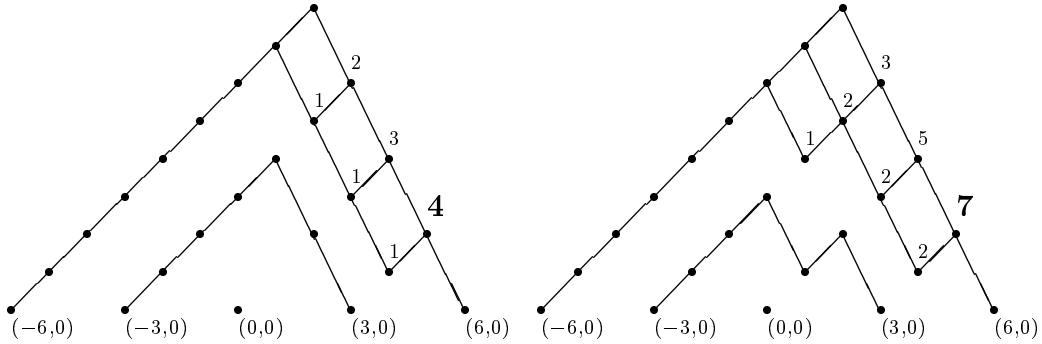
$$A_n^k = |Q_n^k|$$

(Note that there are no disjoint path systems  $W$  in  $\mathcal{G}$  for which  $\sigma(W)$  is an odd permutation. Hence,  $p^- = 0$ .)

For example, let  $n = 3$ . We shall illustrate the case when  $k = 0$  and  $k = 1$ . When  $k = 0$ ,

$$|Q_3^0| = A_3^0 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 12 \\ 3 & 12 & 55 \end{vmatrix} = 11$$

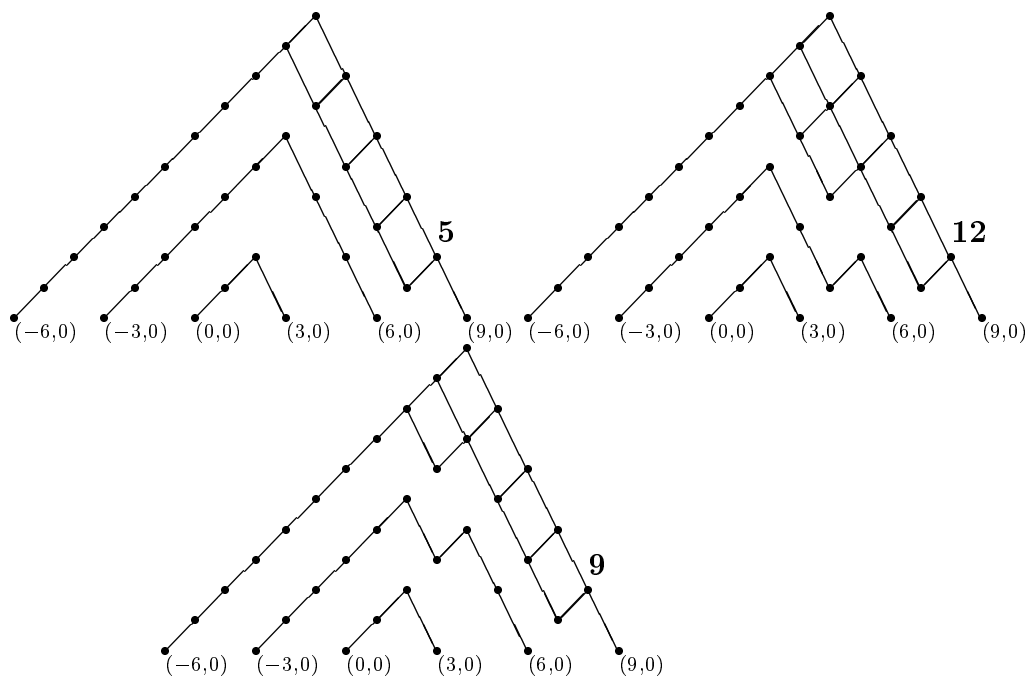
Counting vertex disjoint path systems, we have



When  $k = 1$ , we have

$$|Q_3^1| = A_3^1 = \begin{vmatrix} 1 & 3 & 12 \\ 3 & 12 & 55 \\ 12 & 55 & 273 \end{vmatrix} = 26$$

and the corresponding vertex disjoint path systems are



For more values of  $A_n^0$  and  $A_n^1$ , see figure 4.2. The sequence  $A_n^1$  agrees with the conjecture for the number,  $F_{2n+3}$ , of  $(2n+3) \times (2n+3)$  alternating sign matrices which are invariant under a reflection about the vertical axis, given by  $F_1 = 1$ ,  $F_{2n} = 0$ , and

$$\frac{F_{2n+1}}{F_{2n-1}} = \frac{\binom{6n-2}{2n}}{2 \binom{4n-1}{2n}}$$

See [35, 23]. This observation leads us to the following question.

**Question 4.** *Is it the case that  $A_n^1 = F_{2n+3}$ ? If so, is there a bijection between vertex disjoint ternary path systems and alternating sign matrices invariant under vertical reflection?*

$n$	$A_n^0$	$A_n^1$
0	1	1
1	2	3
2	11	26
3	170	646
4	7429	45,885
5	920,460	9,304,650
6	323,801,820	5,382,618,658

Figure 4.2: The determinant sequences of the ternary numbers

### 4.2.3 Narayana Analogue

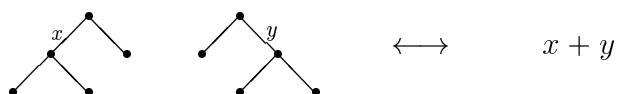
The Narayana numbers,  $N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$ , are yet another sequence with strong ties to the Catalan numbers. As is often the case, there are many ways to interpret the Narayana numbers. Here, we have chosen the setting of plane binary trees.

Consider the plane binary trees with  $2n$  edges. Associate an “ $x$ ” with each branch vertex which is a left successor and a “ $y$ ” with each branch vertex which is a right successor. (It will suffice to view a branch vertex as the edge which joins it to its parent.) For example, in the case where  $n = 2$ ,

$n$	$N_n(x, y)$
0	1
1	1
2	$x + y$
3	$x^2 + 3xy + y^2$
4	$x^3 + 6x^2y + 6xy^2 + y^3$
5	$x^4 + 10x^3y + 20x^2y^2 + 10xy^3 + y^4$
6	$x^5 + 15x^4y + 50x^3y^2 + 50x^2y^3 + 15xy^4 + y^5$
7	$x^6 + 21x^5y + 105x^4y^2 + 175x^3y^3 + 105x^2y^4 + 21xy^5 + y^6$

Figure 4.3: The Narayana polynomials,  $N_n(x, y)$ 

we have



$N(n, k)$  is the number of plane binary trees with  $2n$  edges and  $k - 1$  branch right successors. Let  $N_n(x, y) = \sum_{k=1}^n N(n, k)x^{n-k}y^{k-1}$ . Note that  $N_n(1, 1) = c_n$ . It turns out that  $N_n(x, y)$  is a symmetric function  $\forall n$  and we can write it in terms of elementary functions  $e_1 = x + y$  and  $e_2 = xy$ . See figure 4.3 and figure 4.4.

Suppose we now consider ternary trees with  $3n$  edges. We have the same

$n$	$N_n(x, y)$ in terms of $e_1$ and $e_2$
0	1
1	1
2	$e_1$
3	$e_1^2 + e_2$
4	$e_1^3 + 3e_1e_2$
5	$e_1^4 + 6e_1^2e_2 + 2e_2^2$
6	$e_1^5 + 10e_1^3e_2 + 10e_1e_2^2$
7	$e_1^6 + 15e_1^4e_2 + 30e_1^2e_2^2 + 5e_2^3$

Figure 4.4: The Narayana polynomials in terms of elementary functions



setup as for binary trees, except that now  $y$  corresponds to center branch successors and we have another parameter  $z$  corresponding to the right branch successors. We let  $S(n, k, j)$  denote the number of ternary trees with  $3n$  edges and  $k - 1$  right branch successors and  $j - 1$  center branch successors. Let  $S_n(x, y, z) = \sum_{k=1}^n S(n, k, j)x^{n-k-j+1}y^{j-1}z^{k-1}$ . It turns out that we can still write our polynomials  $S_n(x, y, z)$  in terms of elementary functions  $e_1 = x + y + z$ ,  $e_2 = xy + xz + yz$ , and  $e_3 = xyz$ . In fact, we have

$n$	$S_n(x, y, z)$ in terms of $e_1, e_2,$ and $e_3$
0	1
1	1
2	$e_1$
3	$e_1^2 + e_2$
4	$e_1^3 + 3e_1e_2 + e_3$
5	$e_1^4 + 6e_1^2e_2 + 2e_2^2 + 4e_1e_3$
6	$e_1^5 + 10e_1^3e_2 + 10e_1e_2^2 + 10e_1^2e_3 + 5e_2e_3$

**Question 5.** *What can we say about  $S(n, k, j)$  in general? How can we interpret  $S(n, k, j)$  in other settings, such as paths and partitions? What can we say about the coefficients of  $S_n(x, y, z)$  when written in terms of elementary functions?*

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