

# Two Enumerative Functions

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## Abstract

We define enumerative functions  $F(n, k, m, P, Q)$  and  $G(n, k, m, P)$ . The function  $G$  is a particular case of  $F$  and is a natural generalization of binomial coefficients. Special cases of these functions are also power functions, factorials, rising factorials and falling factorials.

The first section of the paper is an introduction.

In the second section we first derive an explicit formula for  $F$  and  $G$ . From the expression for the power function we obtain a number theory result.

Then we derive a formula which shows that the case of arbitrary  $m$  may be reduced to the case  $m = 0$ . Well-known Vandermonde convolution is a particular case of this formula.

In the third section the functions  $F$  and  $G$  are described by recurrence relations with respect to each of the arguments  $k$ ,  $n$ ,  $P$  and  $Q$ . As a consequence we obtain the recurrence relation for coefficients of Chebyshev polynomials of both kind. The last formula in the paper is an extension of Newton binomial formula.

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## 1 Introduction

For the set  $P = \{p_1, p_2, \dots, p_n\}$  of positive integers and nonnegative integers  $m$  and  $k$  we consider the set  $X$  consisting of  $n$  blocks  $X_i$ , ( $|X_i| = p_i$ ,  $i = 1, 2, \dots, n$ ), which we shall call the main blocks of  $X$ , and an additional  $m$ -block  $X_{n+1}$ . We shall denote  $|X| = N = p_1 + \dots + p_n + m$ . Take next  $Y_i$ , ( $i = 1, 2, \dots, n$ ) to be nonempty  $q_i$ -subsets of  $X_i$ , ( $i = 1, 2, \dots, n$ ), and  $Q = \{q_1, q_2, \dots, q_n\}$ .

**Definition 1.** *By an  $s$ -inset of  $X$  we shall mean an  $s$ -subset of  $X$  such that*

$$Y_i \not\subseteq U \cap X_i, \quad (i = 1, 2, \dots, n).$$

**Definition 2.** *We define the function  $F(n, k, m, P, Q)$  to be the number of  $k$ -insets of  $X$ .*

If  $Y_i = X_i$ , ( $i = 1, 2, \dots, n$ ), then  $U$  is an inset of  $X$  if and only if  $X \setminus U$  intersects all main blocks. It implies that  $X \setminus U$  has at least  $n$  elements. This leads to the following:

**Definition 3.** For  $k \geq 0$  we define the function  $G(n, k, m, P)$  to be the number of  $n + k$ -subsets of  $X$  intersecting each main block  $X_i$ , ( $i = 1, 2, \dots, n$ ). We shall also use the name *inset* for such subsets.

The function  $G(n, k, m, P)$  seems to be a natural generalization of binomial coefficients, since in the case that all main blocks have only one elements obviously holds

$$G(n, k, m, P) = \binom{m}{k}. \quad (1)$$

It is also obvious that

$$G(n, 0, m, P) = p_1 \cdots p_n. \quad (2)$$

This equation shows that power function  $p^n$ , factorial, rising factorial, falling factorial are also particular cases of  $G(n, k, m, P)$ .

In the second section of this paper we prove that  $F$  and  $G$  may be expressed as an alternating sum of the binomial coefficients. This formula takes a simpler form when all main blocks have the same number of elements. By the expression of the power function we obtain a result in number theory.

In this section we also prove that  $F(n, k, m, P, Q)$  may be expressed in terms of  $F(n, k_1, 0, P, Q)$ , ( $k_1 \leq k$ ), that is, in terms of sets with no additional block. The same is true for  $G$ . The case  $n = 1$  in thus obtained formula is the well-known Vandermonde convolution.

The third section is devoted to the recurrence relations of  $F$  and  $G$ . We prove recurrence relation with respect to each parameter  $n$ ,  $k$ ,  $P$ ,  $Q$ . As a consequence of the recurrence relation with respect to  $k$  some binomial identities are obtained.

The recurrence relation with respect to  $n$ , for the function  $G$ , in the case that each main block has exactly two elements, gives the coefficients of Chebyshev polynomials of both kind. This offers a possibility for a combinatorial definition of these polynomials.

We also prove that  $F(n, k, m, P, Q)$  may be expressed in terms of  $G(n, k, m, P)$  and then prove a recurrence relation for  $G$  with respect to the number of elements in main blocks. A particular case of this formula is Newton binomial formula.

Note that the functions  $F$  and  $G$  generate a number of sequences in the well-known Sloane's Encyclopedia of Integers Sequences [1].

## 2 The formula for $F(n, k, m, P, Q)$ and $G(n, k, m, P)$

We shall first derive an explicit formula for  $F$  and  $G$ .

**Theorem 1.** *It holds*

$$F(n, k, m, P, Q) = \sum_{I \subseteq [n]} (-1)^{|I|} \binom{|X| - \sum_{i \in I} |Y_i|}{k - \sum_{i \in I} |Y_i|}, \quad (3)$$

where the sum is taken over all subsets of  $[n]$ .

**Proof.** For  $i = 1, 2, \dots, n$  and a  $k$ -subset  $U$  of  $X$  define the property  $i$  to be:  $Y_i \subseteq U \cap X_i$ . By PIE method we obtain

$$F(n, k, m, P, Q) = \sum_{I \subseteq [n]} (-1)^{|I|} N(I),$$

where  $N(I)$  is the number of  $k$ -subsets  $U$  of  $X$  such that  $Y_i \subseteq U \cap X_i$ , ( $i \in I$ ). There are

$$A(I) = \binom{|X| - \sum_{i \in I} |Y_i|}{k - \sum_{i \in I} |Y_i|},$$

such subsets, and the theorem is proved.

In the particular case  $|Y_i| = 1$ , ( $i = 1, \dots, n$ ) we obtain the well-known binomial identity

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \binom{|X| - i}{k - i} = \binom{|X| - n}{k}.$$

If  $|X_i| = p$ ,  $|Y_i| = q < p$ , ( $i = 1, 2, \dots, n$ ) then the formula takes a simpler form

$$F(n, k, m, P, Q) = \sum_{i=0}^n (-1)^i \binom{n}{i} \binom{pn + m - iq}{k - iq}.$$

In the case  $p = q$  we have

$$G(n, k, m, P) = \sum_{i=0}^n (-1)^i \binom{n}{i} \binom{np + m - ip}{n + k}. \quad (4)$$

For  $k = 0$  we obtain an expression for the power function.

$$p^n = \sum_{i=0}^n (-1)^i \binom{n}{i} \binom{pn + m - pi}{n}.$$

If  $n$  is a prime, then it divides all terms in the sum on the right side except eventually the first and the last. The first term is  $\binom{pn+m}{n}$ , and the last is  $(-1)^n \binom{m}{n}$ . We thus obtain

**Corollary 1.** *If  $q$  is a prime then for integers  $r > 1$  and  $m \geq 0$  holds*

$$q \mid r^q - \binom{rq + m}{q} + (-1)^{q+1} \binom{m}{q}.$$

The next formula reduces the case of arbitrary  $m$  to the case  $m = 0$ .

**Theorem 2.** *The following formula is true*

$$F(n, k, m, P, Q) = \sum_{i=0}^{\min(m, k)} \binom{m}{i} F(n, k - i, 0, P, Q). \quad (5)$$

The same is also true for  $G$ .

**Proof.** Omitting the additional  $m$ -blocks of  $X$  we obtain the set  $Z$  with  $n$  main blocks and no additional block.

Each  $k$ -inset of  $Z$  is an  $k$ -inset of  $X$ . There are  $F(n, k, 0, P, Q)$  such insets.

The remaining  $k$ -insets of  $X$  are the union of a  $k - i$ -inset of  $Z$  and an  $i$ -subset of additional block, for some  $i$ , ( $1 \leq i \leq m$ ). This proves (5).

In the special case  $p_1 = \dots = p_n = 1$  we obtain

$$\binom{|X| - n}{k} = \sum_{i=0}^{\min(m, k)} \binom{m}{i} \binom{|X| - n - m}{k - i},$$

which is the well-known Vandermonde convolution. Taking additionally  $m = 1$  we obtain the recurrence relation for binomial coefficients:

$$\binom{|X| - n}{k} = \binom{|X| - n - 1}{k} + \binom{|X| - n - 1}{k - 1}.$$

### 3 Recurrence Relations

The first recurrence relation is with respect to the parameter  $k$ .

**Theorem 3.** *For each  $s = 1, 2, \dots$  the following formula is true*

$$F(n, k, m, P, Q) = \sum_{i=0}^s (-1)^i \binom{s}{i} F(n, k + s, m + s - i, P, Q). \quad (6)$$

The same is also true for  $G$ .

**Proof.** Summing over all  $I \subseteq [n]$  in the recurrence of binomial coefficients

$$\binom{|X| - \sum_{i \in I} |Y_i|}{k - \sum_{i \in I} |Y_i|} + \binom{|X| - \sum_{i \in I} |Y_i|}{1 + k - \sum_{i \in I} |Y_i|} = \binom{1 + |X| - \sum_{i \in I} |Y_i|}{1 + k - \sum_{i \in I} |Y_i|},$$

we obtain

$$F(n, k, m, P, Q) = F(n, k + 1, m + 1, P, Q) - F(n, k + 1, m, P, Q),$$

and then (6) follows by induction.

In the case  $m = 0$ ,  $n = 1$  the formula (6), for each  $s = 0, 1, 2, \dots$ , produces the following binomial identities

$$\binom{p}{k + 1} = \sum_{i=0}^s (-1)^s \binom{s}{i} \binom{p + s - i}{k + s + 1}.$$

The following recurrence relation is with respect to the number of main blocks.

**Theorem 4.** For a fixed  $j \in [n]$  the following formula is true :

$$F(n, k, m, P, Q) = \sum_{i=0}^{p_j} \left( \binom{p_j}{i} - \binom{p_j - q_j}{i - q_j} \right) F(n-1, k-i, m, P \setminus \{p_j\}, Q \setminus \{q_j\}). \quad (7)$$

**Proof.** Omitting  $X_j$ -block of  $X$  we obtain a set  $Z$  with  $n-1$  main blocks, and additional  $m$ -block.

Each  $k$ -inset of  $Z$  is also  $k$ -inset of  $X$ . The remaining  $k$ -sets of  $X$  may be obtained as a union of a  $k-s$ -inset of  $Z$  and a  $s$ -subset of  $X_j$  not containing  $Y_j$ . There are

$$\binom{p_j}{s} - \binom{p_j - q_j}{s - q_j},$$

such subsets, which proves (7).

The equation (7), for the function  $G(n, k, m, P)$ , takes the form

$$G(n, k, P, m) = \sum_{i=1}^{p_j} \binom{p_j}{i} G(n-1, k-i+1, P \setminus \{p_j\}, m). \quad (8)$$

In the case  $p_1 = p_2 = \dots = p_n = 2$ , we shall denote  $F(n, k, m, P) = c(n, k, m)$ .

In this case the formula (8) gives

$$c(n, k, m) = 2c(n-1, k, m) + c(n-1, k-1, m). \quad (9)$$

We shall prove that this is in fact the recurrence relation for coefficients of Chebyshev polynomials.

**Theorem 5.** The numbers  $(-1)^k c(n, k, m)$ , ( $m = 0, 1$ ) are the coefficients of  $x^{n-k+m}$  in Chebyshev polynomial  $P_{n+k+m}(x)$ . If  $m = 0$  we obtain coefficients for the polynomials of the second, and for  $m = 1$  of the first kind.

In this way we may obtain almost all coefficients of Chebyshev polynomials of both kind.

**Proof.** We deal here with polynomials whose degree are  $n+k+m$ ,  $n \geq 1$ ,  $k \geq 0$ . Thus, if  $m = 0$  we start with polynomials of the first degree, and with its  $a(1, 1)$  coefficient. If  $m = 1$  then we start with polynomial of the second degree, and its  $a(2, 2)$  coefficient.

Multiplying (9) by  $(-1)^k$  we obtain

$$(-1)^k c(n, k, p) = 2(-1)^k c(n-1, k, m) - (-1)^{k-1} c(n-1, k-1, m). \quad (10)$$

Denote  $(-1)^k c(n, k, m) = a(r, s)$ , where

$$\begin{aligned} n + k + m &= r, \\ n - k + m &= s \end{aligned} \quad (11)$$

This system has the solution for any  $n, k$  if and only if  $r$  and  $s$  are of the same parity. This will be enough since only such Chebyshev coefficients are not zero. The equation (10) becomes

$$a(n+k+m, n-k+m) = 2a(n-1+k+m, n-1-k+m) - a(n-2+k+m, n-k+m),$$

that is

$$a(r, s) = 2a(r-1, s-1) - a(r-2, s),$$

which is the well-known recurrence relation for Chebyshev coefficients.

For  $m = 0$  the initial condition is  $a(1, 1) = 2$ , which is the coefficient of Chebyshev polynomials of the second kind, while in the case  $m = 1$  the initial condition is  $a(2, 2) = 2$ , which is the coefficient of the polynomials of the first kind.

**Remark 1.** *The system (11) has the solution by  $n$  and  $k$  if and only if  $r$  and  $s$  are of the same parity, which is the case with nonzero coefficients of Chebyshev polynomials. This means that in the preceding way we may obtain all Chebyshev coefficients except coefficients of polynomials  $U_0(x), T_0(x), T_1(x), T_2(x)$ .*

The following result reduces the case  $q < p$  to the case  $p = q$ .

**Theorem 6.** *Suppose  $I_0 \subseteq [n]$ ,  $|I_0| = n_0$  such that  $q_i < p_i$ , ( $i \in I_0$ ). The following formula is true :*

$$F(n, k, m, P, Q) = \sum_{i=0}^{n_0} \binom{n_0}{i} f(n, k-i, m, P', Q), \quad (12)$$

where

$$P' = \{p'_1, \dots, p'_n\}, \quad p'_i = p_i, \quad (i \notin I_0), \quad p'_i = p_i - 1, \quad (i \in I_0).$$

**Proof.** Omitting fixed  $x_i \in X_i \setminus Y_i$ , ( $i \in I_0$ ) we obtain the set  $Z$  whose parameters are  $n, k, m, P', Q$ . Each  $k$ -inset of  $Z$  is a  $k$ -inset of  $X$ . Each of the remaining  $k$ -inset of  $X$  may be obtained as a union of some  $k-i$ -inset of  $Z$  and an  $i$ -subset of the set of omitting elements. This proves (12).

According to the preceding theorem we may restrict our attention to the case  $p = q$ .

If  $p_1 = \dots = p_n = 1$  then

$$G(n, k, P, m) = \binom{m}{k}.$$

We thus may suppose that there are  $i \in [n]$  such that  $p_i > 1$ . Take  $I_0 \subseteq [n]$ , ( $I_0 \neq \emptyset$ ) such that  $p_i > 1$ , ( $i \in I_0$ ). For each  $i \in I_0$  choose an element  $x_i \in X_i$  and form the set  $X(I_0)$  consisting of these elements. Denote further  $Z = X \setminus X(I_0)$ . Form the set  $P' = \{p'_1, p'_2, \dots, p'_n\}$ , putting  $p'_i = p_i - 1$  if  $i \in I_0$ , and  $p'_i = p_i$  otherwise. The set  $Z$  has also  $n$  main blocks. For  $I \subseteq I_0$  denote by  $P'(I) = P' \setminus \{p'_i : i \in I\}$ .

**Theorem 7.** *For each  $n_0$ , ( $1 \leq n_0 \leq n$ ) the following formula is true*

$$G(n, k, m, P) = G(n, k, m, P') + \sum_{I \subseteq I_0, I \neq \emptyset} \sum_{J \subseteq I, J \neq \emptyset} G(n-|J|, k-|I|+|J|, m, P'(J)), \quad (13)$$

where sums are taken over all nonempty subsets  $I$  of  $I_0$  and over all nonempty subsets  $J$  of  $I$ .

**Proof.** It is clear that each  $n + k$ -inset of  $Z$  is also an  $n + k$ -inset of  $X$ . In such a way we obtain all  $n + k$ -insets of  $X$  which do not meet  $X(I_0)$ . There are  $G(n, k, m, P')$  such insets.

The remaining  $n + k$ -insets of  $X$  intersect  $X(I_0)$ . Take nonempty  $I \subseteq I_0$ , and count the number of  $n + k$ -insets  $U$  of  $X$  such that  $U \cap X(I_0) = X(I)$ .

First, one such inset may be obtained as the union of an  $n + k - |I|$ -inset of  $Z$  and  $X(I)$ . There are  $G(n, k - |I|, m, P')$  such insets. Further,  $U$  may be obtained as the union of some  $n + k - |I|$ -subset of  $Z$  not intersecting one, two, ...,  $|I|$  main blocks which indices belong to  $I$ , and  $X(I)$ .

Take  $J \subseteq I$ ,  $J \neq \emptyset$  and count  $n + k - |I|$ -subset  $V$  of  $Z$  which do not intersect main blocks whose indices belong to  $J$ . It follows that  $V$  is an  $n + k - |I|$ -inset of the set  $Z_1$  obtained from  $Z$  by removing main blocks whose indices lying in  $J$ . Hence, the set  $Z_1$  has  $P'(J)$  main blocks which yields that the number of  $V$  is

$$G(n - |J|, k - |I| + |J|, m, P'(J)).$$

Note that this number depends not of  $I$ , but only on  $|I|$ , the number of elements of  $I$ .

Summing over all nonempty subsets  $J$  of  $I$  and then over all nonempty subsets  $I$  of  $I_0$  we finally obtain the formula (13).

In the special case  $k = 0$  we have

$$p_1 \cdot p_2 \cdots p_n = (p_1 - 1) \cdot (p_2 - 1) \cdots (p_n - 1) + 1 + \sum_{I \subseteq [n], I \neq [n], \emptyset \neq I} \prod_{i \in I} (p_i - 1),$$

that is,

$$p_1 \cdot p_2 \cdots p_n = (p_1 - 1) \cdot (p_2 - 1) \cdots (p_n - 1) + 1 + \sum_{J \subseteq [n], J \neq [n], \emptyset \neq J} \prod_{j \in J} (p_j - 1).$$

If  $p_1 = p_2 = \cdots = p_n = 2$  then we simply count all subsets of  $[n]$ . If  $p_1 = p_2 = \cdots = p_n = p$  then

$$p^n = \sum_{i=0}^n \binom{n}{i} (p-1)^{n-i},$$

which is in fact Newton binomial formula.

The formula (13) takes a simpler form if main blocks have the same number of elements. In this case the value of  $G(n - |J|, k - |I| + |J|, m, P'(J))$  depends only on the number of elements of  $I$  and  $J$ .

**Theorem 8.** *If, in the conditions of the preceding theorem, all main blocks have the same number of elements then*

$$G(n, k, P, m) = \sum_{i=0}^{n_0} \sum_{j=0}^i \binom{n_0}{i} \binom{i}{j} G(n - j, k - i + j, m, P'(j)).$$

**Remark 2.** *Applying Theorem 7. several times we may reduce the general case to the case when all main blocks have exactly one element.*

### **References**

[1] N. J. A. Sloane, (2007), The On-Line Encyclopedia of Integer Sequences, published electronically at [www.research.att.com/~njas/sequences/](http://www.research.att.com/~njas/sequences/).