Graph Algorithms

Chromatic Polynomials

Chromatic Polynomials – Definition

- ⋆ G – a simple labelled graph with n vertices and m edges.

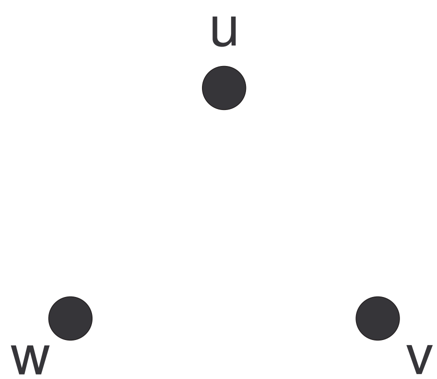
- ⋆ k – a positive integer.

- ⋆ $P_G(k)$ – number of different ways of coloring the vertices of G with k colors.

- ⋆ $P_G(k)$ is an integer function (polynomial) of k:
  - If $\chi(G) > k$ then $P_G(k) = 0$.
  - If $\chi(G) \leq k$ then $P_G(k) > 0$.

$\Rightarrow$ $\chi(G)$ is the smallest k such that $P_G(k) > 0$.

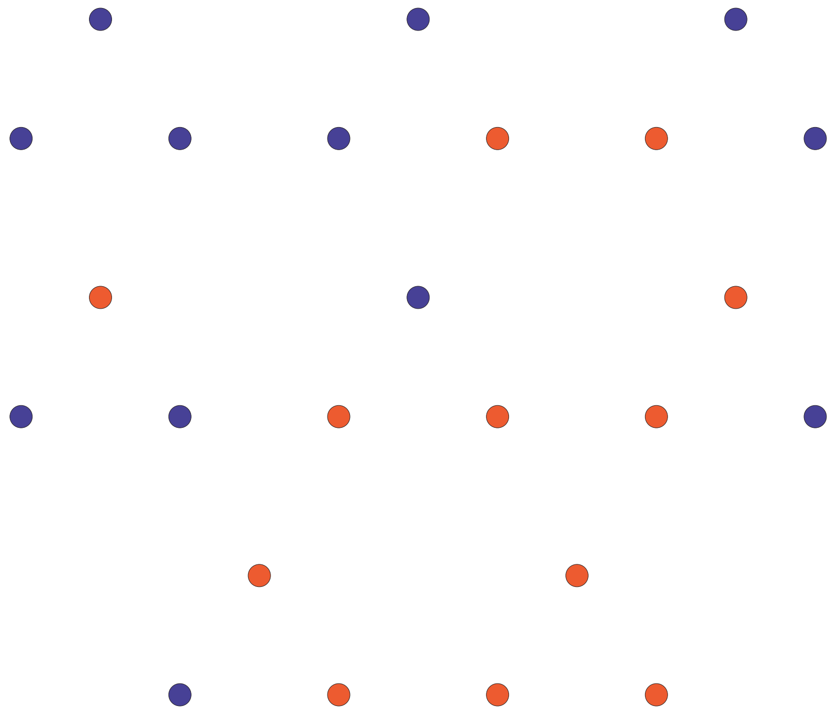
3 Vertices and 0 Edges



⋆ k ways to color independently each of the vertices u, v, w.

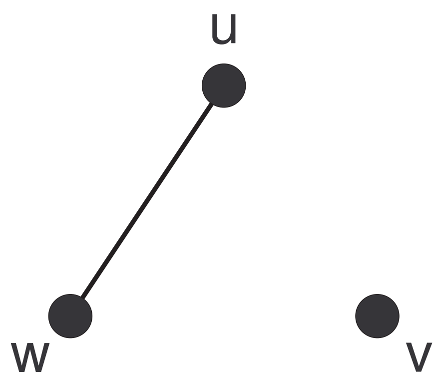
$$P_G(k) = k^3$$

3 Vertices, 0 Edges, and 2 colors



$$P_G(2) = 2^3 = 8$$

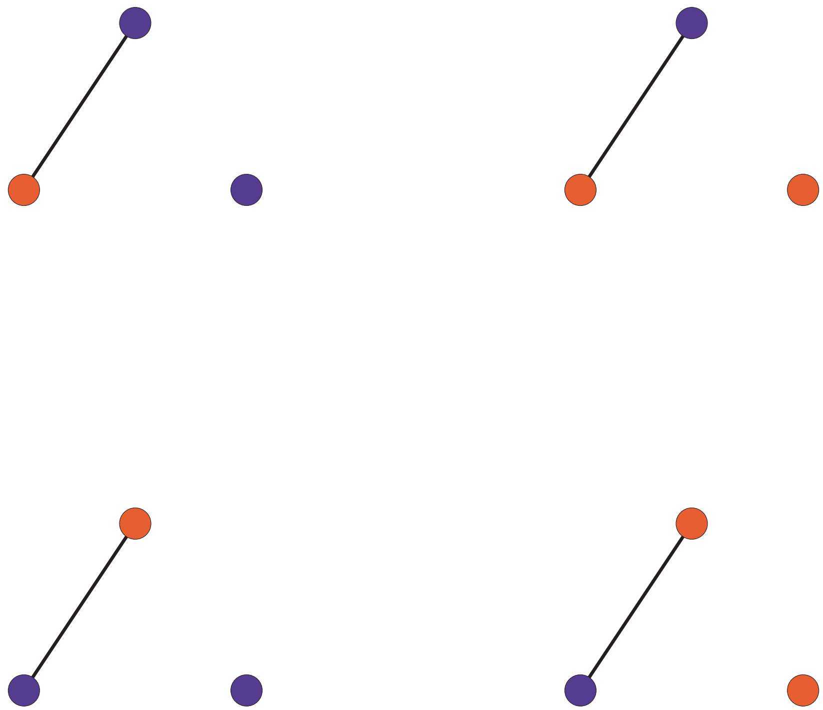
3 Vertices and 1 Edge



- k ways to color v; k ways to color u; $k-1$ ways to color w that cannot get the color of u.

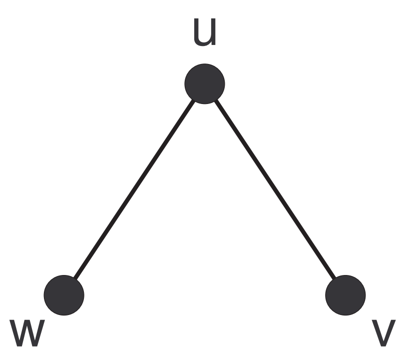
$$\begin{aligned} P_G(k) &= k^2(k-1) \\ &= k^3 - k^2 \end{aligned}$$

3 Vertices, 1 Edge, and 2 colors



$$P_G(2) = 2^3 - 2^2 = 4$$

3 Vertices and 2 Edges



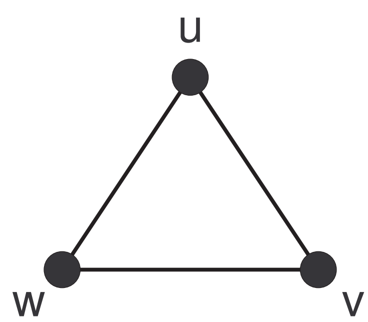
⋆ k ways to color u; $k-1$ ways to color v that cannot get the color of u; $k-1$ ways to color w that cannot get the color of u.

$$\begin{aligned} P_G(k) &= k(k-1)^2 \\ &= k^3 - 2k^2 + k \end{aligned}$$

3 Vertices, 2 Edges, and 2 colors

$$P_G(2) = 2^3 - 2 \cdot 2^2 + 2 = 2$$

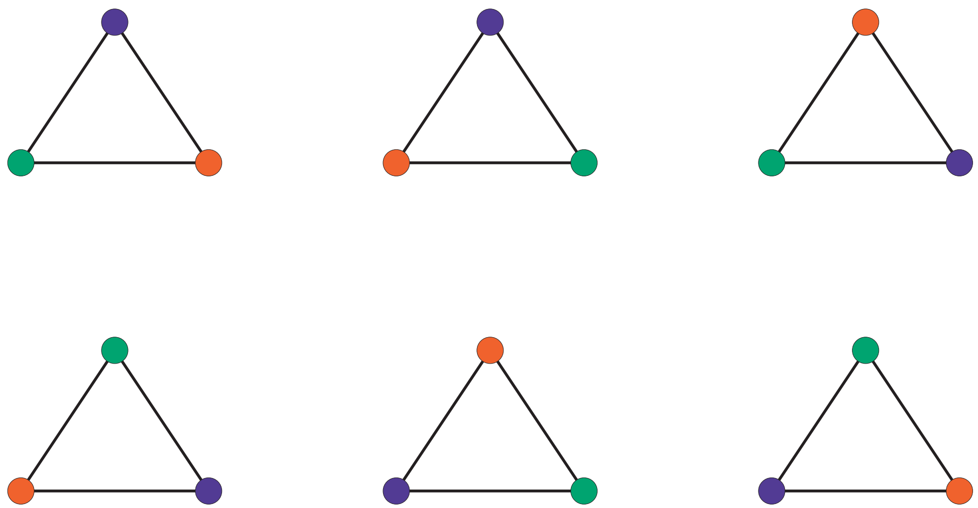
3 Vertices and 3 Edges



- ⋆ k ways to color u; $k-1$ ways to color v that cannot get the color of u; $k-2$ ways to color w that cannot get the colors of u and v.

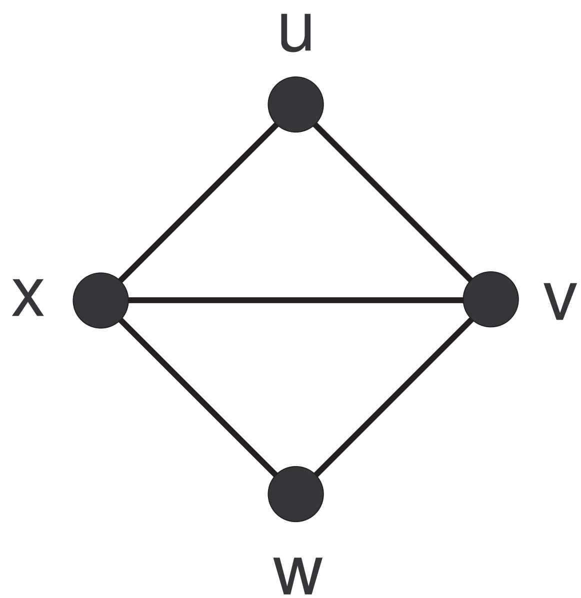
$$\begin{aligned} P_G(k) &= k(k-1)(k-2) \\ &= k^3 - 3k^2 + 2k \\ &= (k-1)^3 - (k-1) \end{aligned}$$

3 Vertices, 3 Edges, and 3 colors



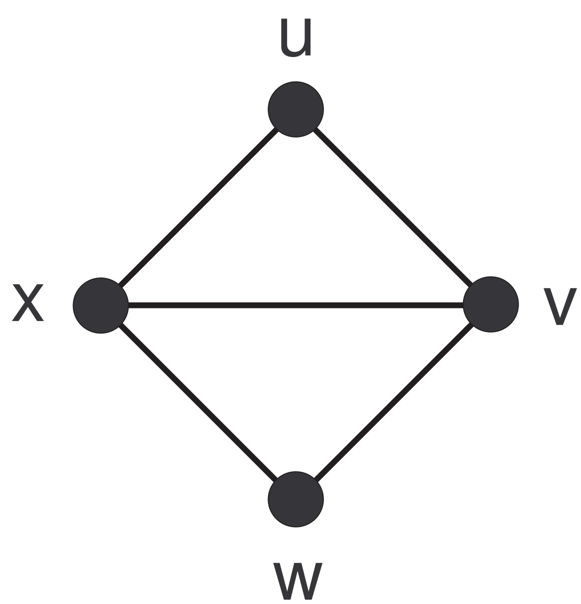
$$P_G(3) = 3 \cdot 2 \cdot 1 = 6$$

4 Vertices and 5 Edges



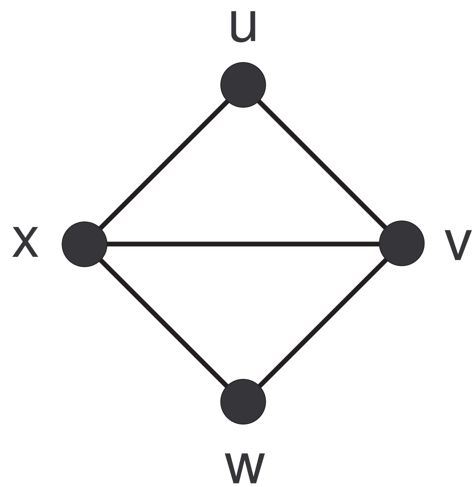
⋆ k ways to color v; $k-1$ ways to color x that cannot get the color of v; $k-2$ ways to color u that cannot get the colors of v and x; $k-2$ ways to color w that cannot get the colors of v and x.

4 Vertices and 5 Edges



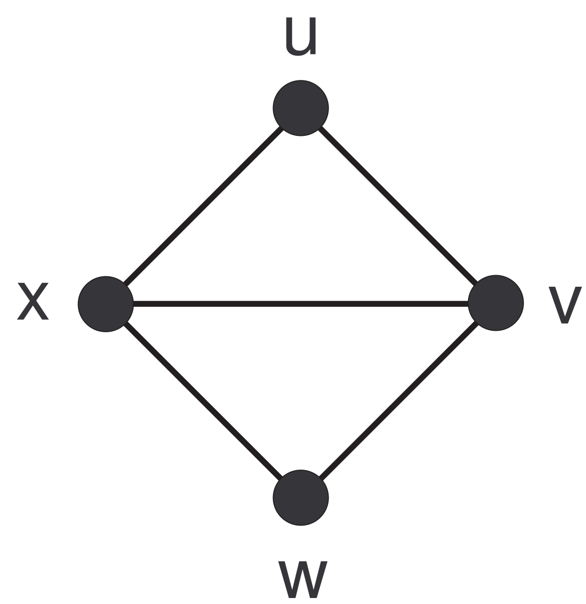
$$\begin{aligned} P_G(k) &= k(k-1)(k-2)^2 \\ &= k^4 - 5k^3 + 8k^2 - 4k \end{aligned}$$

4 Vertices and 5 Edges



- $k(k-1)$ ways to color u and w with different colors; $k-2$ ways to color v that cannot get the colors of u and w; $k-3$ ways to color x that cannot get the colors of u, v, and w.
- k ways to color u and w with the same color; $k-1$ ways to color v that cannot get the color of u and w; $k-2$ ways to color x that cannot get the colors of u, v, and w.

4 Vertices and 5 Edges



$$
\begin{aligned}
P_G(k) &= k(k-1)(k-2)(k-3) + k(k-1)(k-2) \\
&= k(k-1)(k-2)^2 \\
&= k^4 - 5k^3 + 8k^2 - 4k
\end{aligned}
$$

$P_G(k)$ – Properties

- $P_G(k)$ is a polynomial in k:

$$P_G(k) = a_n k^n + a_{n-1} k^{n-1} + \cdots + a_1 k + a_0$$

- The degree of the polynomial is n: the number of vertices in the graph.
- All the coefficients are integers (could be 0).
- The coefficient of k^n is 1: $a_n = 1$.
- The coefficient of k^0 is 0: $a_0 = 0$.
- The coefficient of k^{n-1} is $-m$: $a_{n-1} = -m$.

$P_G(k)$ – More Properties

- ⋆ Signs of coefficients alternate between positive and negative.

$$P_G(k) = k^n - mk^{n-1} + b_{n-2}k^{n-2} \cdots +- \; b_1 k +- \; 0$$

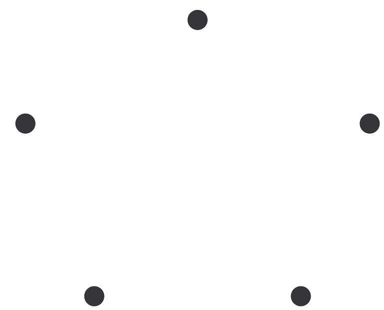
for non-negative coefficients $b_1, \ldots, b_{n-2}$.

- ⋆ For a graph with at least one edge, the sum of the coefficients is 0.

$$a_n + a_{n-1} + \cdots + a_1 = 0$$

for positive or negative or zero coefficients $a_1, \ldots, a_{n-2}$.

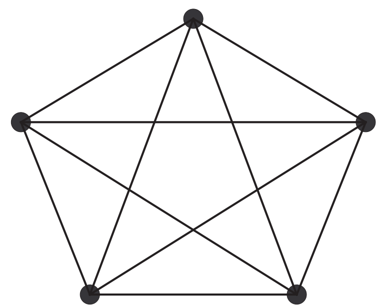
Null Graphs – N_n



- ⋆ The null graph N_n has n vertices and no edges.
- ⋆ Each vertex can be colored independently with k colors.

$$\begin{aligned} P_{N_n}(k) &= k^n \\ &= k^n - 0 \cdot k^{n-1} + 0 \cdot k^{n-2} - \cdots +- \; 0 \end{aligned}$$

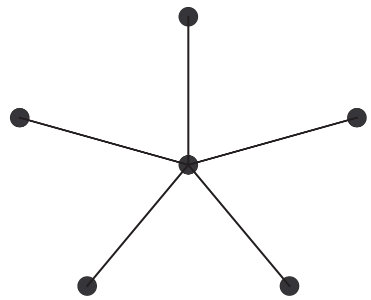
Complete Graphs – K_n



- The Complete graph K_n has n vertices and all possible edges: $m = \frac{n(n-1)}{2}$.
- The first vertex can be colored with k colors, the second with $k-1$ colors . . . and the last with $k-n+1$ colors.

$$
\begin{aligned}
P_{K_n}(k) &= k(k-1)(k-2)\cdots(k-n+1) \\
&= k^n - (1+2+\cdots+(n-1))k^{n-1} + \cdots +- 0
\end{aligned}
$$

Stars – S_n



- ⋆ The Star graph S_n has $n-1$ edges. A root vertex is connected to the rest of the $n-1$ vertices each connected only to the root.
- ⋆ The root can be colored with k colors and each of the other $n-1$ vertices can be colored with $k-1$ colors.

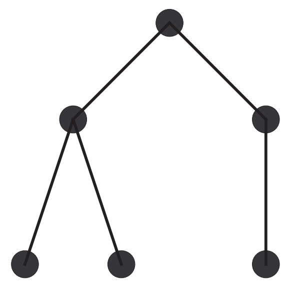
$$\begin{aligned} P_{S_n}(k) &= k(k-1)^{n-1} \\ &= k^n - (n-1)k^{n-1} + \cdots +- 0 \end{aligned}$$

Paths – P_n

- ⋆ The Path graph P_n has $n-1$ edges. The vertices are connected as a path of length $n-1$ edges.
- ⋆ The first vertex can be colored with k colors and each one of the other $n-1$ vertices, in order, can be colored with $k-1$ colors.

$$\begin{aligned} P_{P_n}(k) &= k(k-1)^{n-1} \\ &= k^n - (n-1)k^{n-1} + \cdots +- 0 \end{aligned}$$

Trees – T_n



- ⋆ A tree T_n is an acyclic (connected) graph with n vertices and $n-1$ edges.
- ⋆ The root can be colored with k colors and each of the other $n-1$ vertices can be colored with $k-1$ colors if it is colored after its parent and before all of its children.

$$\begin{aligned} P_{T_n}(k) &= k(k-1)^{n-1} \\ &= k^n - (n-1)k^{n-1} + \cdots +- 0 \end{aligned}$$

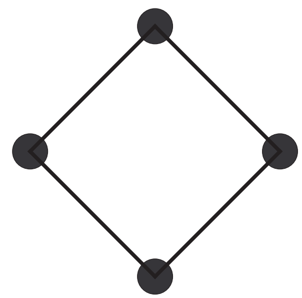
Finding the Chromatic Polynomial

- ⋆ Let G be a labelled graph with n vertices.
- ⋆ Suppose that there are $f(r)$ different ways to partition G into r independent sets.
- ⋆ Each color class in any coloring is an independent set.

⇒ A given partition into r independent sets can be colored in $k(k-1)\cdots(k-r+1)$ ways with k colors where each independent set gets a different color.

$$P_G(k) \quad = \quad \sum_{r=1}^{n} f(r) \cdot k(k-1)\cdots(k-r+1)$$

The Cycle C_4



⋆ $f(1) = 0 \quad f(2) = 1 \quad f(3) = 2 \quad f(4) = 1.$

$$\begin{aligned} P_{C_4}(k) &= f(1)k + f(2)k(k-1) + f(3)k(k-1)(k-2) \\ &\quad + f(4)k(k-1)(k-2)(k-3) \\ &= k^2 - k + 2k^3 - 6k^2 + 4k + k^4 - 6k^3 + 11k^2 - 6k \\ &= k^4 - 4k^3 + 6k^2 - 3k \\ &= (k-1)^4 + (k-1) \end{aligned}$$

The Coefficients in $P_G(k)$

$$P_G(k) = \sum_{r=1}^{n} f(r) \cdot k(k-1)\cdots(k-r+1)$$

- ⋆ $P_G(k)$ is a polynomial.
- ⋆ All the coefficients in $P_G(k)$ are integers.
- ⋆ The degree of $P_G(k)$ is n because $f(r) = 0$ for $r > n$.
- ⋆ The coefficient of k^n is 1 because $f(n) = 1$.
- ⋆ The coefficient of k^0 is $0.$ because $f(0) = 0$.

The Sum of All the Coefficients

Lemma: Let G be a graph with n vertices and at least 1 edge. Then the sum of all the coefficients in $P_G(k)$ is 0.

Proof:

- ⋆ It is impossible to color G with 1 color $\Rightarrow$ $P_G(1) = 0$.
- ⋆ By definition, $P_G(1) = a_n 1^n + a_{n-1} 1^{n-1} + \cdots + a_1 1^1$.
- ⋆ Therefore, $\sum_{i=1}^{n} a_i = 0$.

The Coefficient of k^{n-1}

Lemma: Let G be a graph with n vertices and m edges. Then the coefficient of k^{n-1} in $P_G(k)$ is $-m$.

Proof:

- ⋆ The coefficient of k^{n-1} in $k(k-1)\cdots(k-n+1)$ is $-\frac{1}{2}n(n-1)$ and $f(n) = 1$.
- ⋆ The coefficient of k^{n-1} in $k(k-1)\cdots(k-n+2)$ is 1 and $f(n-1)$ is equal to the number of non-adjacent pairs of vertices: $f(n-1) = \frac{1}{2}n(n-1) - m$.
- ⋆ The coefficient of k^{n-1} in $k(k-1)\cdots(k-r+1)$ for $r < n-1$ is 0.
- ⋆ The coefficient of k^{n-1} in $P_G(k)$ is
$$1 \cdot \left(-\tfrac{1}{2}n(n-1)\right) + \left(\tfrac{1}{2}n(n-1) - m\right) \cdot 1 = -m.$$

Disconnected Graphs

Lemma: Let $G_1, G_2, \ldots, G_h$ be the h connected components of G. Then $P_G(k) = P_{G_1}(k) \cdot P_{G_2}(k) \cdots P_{G_h}(k)$.

Proof: The colorings of the h connected components are independent.

Example:

$$\begin{aligned} P_G(k) &= P_{K_2}(k) \cdot P_{K_1}(k) \\ &= k(k-1)k \\ &= k^3 - k^2 \end{aligned}$$

Disconnected Graphs

Corollary: If G is composed of h connected components, then the coefficient of k^ℓ for $\ell < h$ is 0.

Proof: The coefficient of k^0 is 0 in $P_{G_i}(k)$ for all $1 \leq i \leq h$ $\Rightarrow$ in the product of the h polynomials the smallest degree with a positive coefficient is k^h.

Example: The null graph N_n has n connected components $\Rightarrow$ all the coefficients are 0 except the coefficient of k^n which is 1 $\Rightarrow$ $P_{N_n}(k) = k^n$.

Trees

Lemma: Assume that the chromatic polynomial of a graph G is $P_G(k) = k(k-1)^{n-1}$. Then G is a tree with n vertices.

Proof:

⋆ The degree of $P_G(k)$ is $n \Rightarrow G$ has n vertices.

⋆ The coefficient of k^{n-1} is $-(n-1) \Rightarrow G$ has $n-1$ edges.

⋆ The coefficient of k in a disconnected graph is 0 and the coefficient of k in $k(k-1)^{n-1}$ is greater than $0 \Rightarrow G$ is connected.

⋆ A connected graph with n vertices and $n-1$ edges is a tree.

Three Transformations

Delete an edge: $G - (u, v)$ is G without the old edge (u, v).

Add an edge: $G + (u, v)$ is G with the new edge (u, v).

Contract 2 vertices: $G/(u, v)$ is G

- ⋆ without the old vertices u and v and all the edges that are connected to them,
- ⋆ with a new vertex uv that is connected to all the neighbors of u and v.

First Recursive Formula for $P_G(k)$

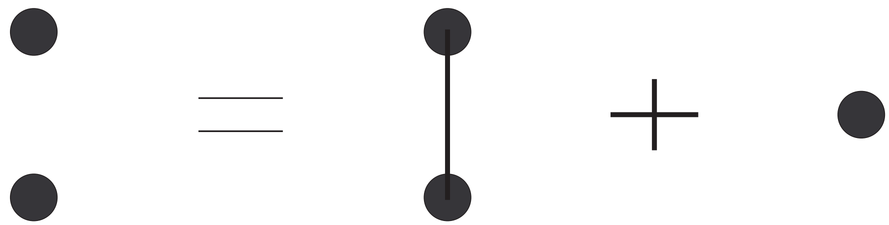
Theorem: For any non-adjacent vertices u and v,

$$P_G(k) \quad = \quad P_{G+(u,v)}(k) + P_{G/(u,v)}(k)$$

Proof:

- ⋆ $P_{G+(u,v)}(k)$ covers all the colorings in which the color of u is different than the color of v.
- ⋆ $P_{G/(u,v)}(k)$ covers all the colorings in which the color of u is the same as the color of v.

The Null Graph N_2



$$
\begin{aligned}
P_{N_2}(k) &= P_{K_2}(k) + P_{K_1}(k) \\
&= k(k-1) + k \\
&= k^2
\end{aligned}
$$

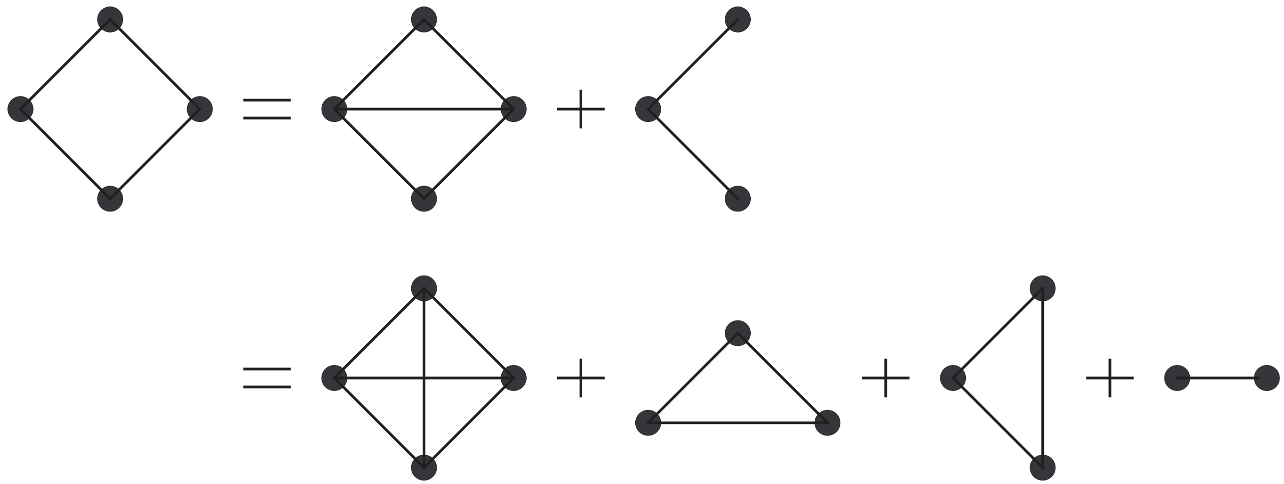
First Recursive Formula for $P_G(k)$

Corollary: The chromatic polynomial of G is a linear combination of chromatic polynomials of complete graphs with at most n vertices,

$$P_G(k) \;=\; P_{K_n}(k) + b_{n-1}P_{K_{n-1}}(k) + \cdots + b_1 P_{K_1}(k)$$

for some non-negative integers $b_{n-1}, \ldots, b_1$.

The Cycle C_4



$$
\begin{aligned}
P_{C_4}(k) &= P_{K_4}(k) + 2P_{K_3}(k) + P_{K_2}(k) \\
&= k(k-1)(k-2)(k-3) + 2k(k-1)(k-2) + k(k-1) \\
&= k^4 - 4k^3 + 6k^2 - 3k \\
&= (k-1)^4 + (k-1)
\end{aligned}
$$

Second Recursive Formula for $P_G(k)$

Theorem: For any edge (u, v),

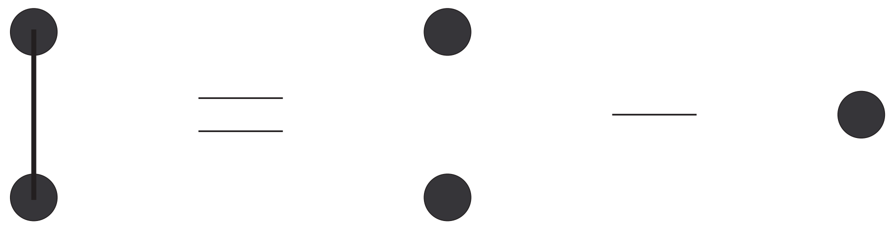
$$P_G(k) \;=\; P_{G-(u,v)}(k) - P_{G/(u,v)}(k)$$

Proof:

⋆ $P_{G-(u,v)}(k)$ covers all the colorings in which the color of u is the same as the color of v and all the colorings in which the color of u is different than the color of v.

⋆ $P_{G/(u,v)}(k)$ covers all the colorings in which the color of u is the same as the color of v.

The Complete Graph K_2



$$
\begin{aligned}
P_{K_2}(k) &= P_{N_2}(k) - P_{N_1}(k) \\
&= k^2 - k \\
&= k(k-1)
\end{aligned}
$$

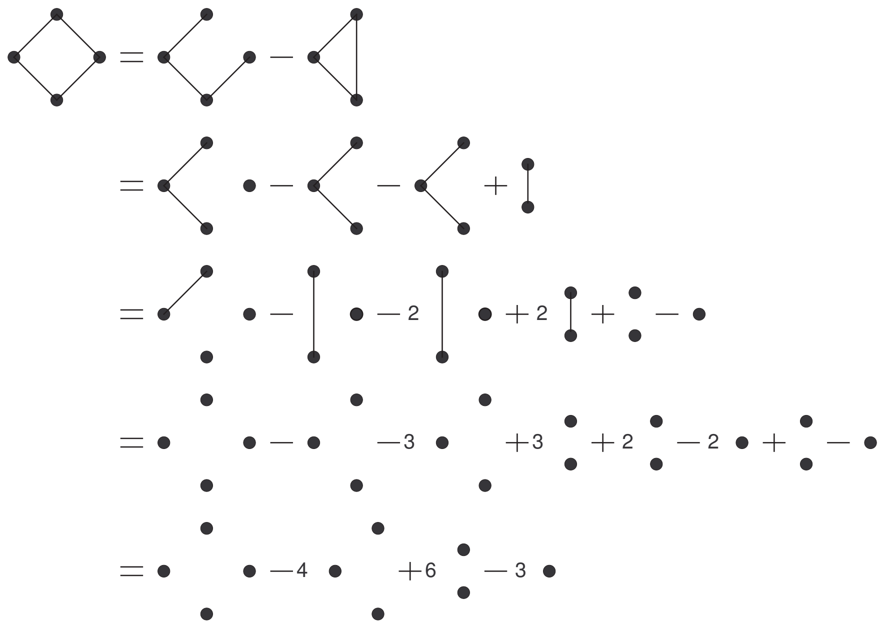
Second Recursive Formula for $P_G(k)$

Corollary: The chromatic polynomial of G is a linear combination of chromatic polynomials of null graphs with at most n vertices,

$$P_G(k) \;=\; P_{N_n}(k) + c_{n-1} P_{N_{n-1}}(k) + \cdots + c_1 P_{N_1}(k)$$

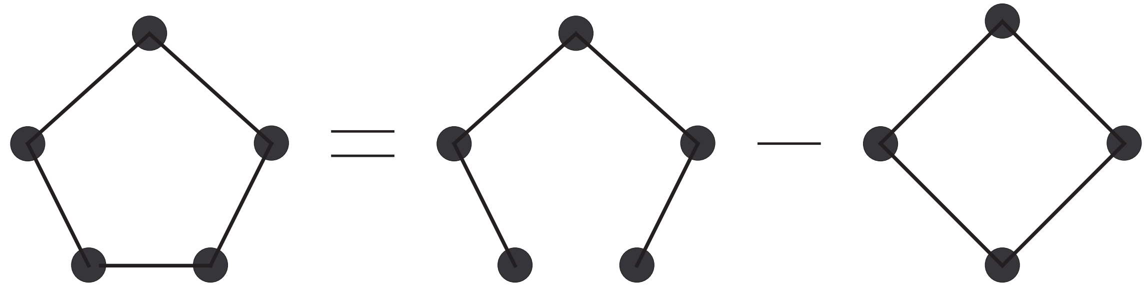
for integers (positive, negative, or 0) $c_{n-1}, \ldots, c_1$.

The Cycle C_4



$$
\begin{aligned}
P_{C_4}(k) &= P_{N_4}(k) - 4P_{N_3}(k) + 6P_{N_2}(k) - 3P_{N_1}(k) \\
&= k^4 - 4k^3 + 6k^2 - 3k \\
&= (k-1)^4 + (k-1)
\end{aligned}
$$

The Chromatic Polynomial of the Cycle C_n



$$
\begin{aligned}
P_{C_n}(k) &= P_{P_n}(k) - P_{C_{n-1}}(k) \\
&= P_{P_n}(k) - P_{P_{n-1}}(k) + P_{C_{n-2}}(k) \\
&\vdots \\
&= P_{P_n}(k) - P_{P_{n-1}}(k) + \cdots +- P_{P_2}(k) \\
&= k(k-1)^{n-1} - k(k-1)^{n-2} + \cdots +- k(k-1)
\end{aligned}
$$

The Chromatic Polynomial of the Cycle C_n

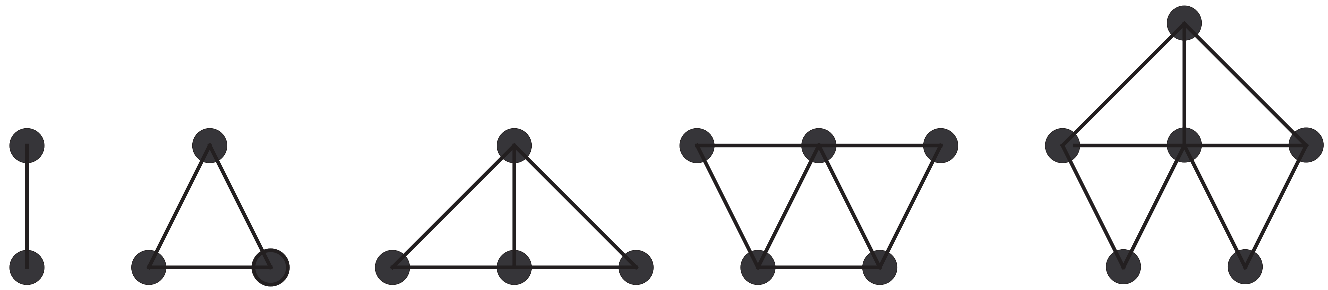
Proposition: For $n \geq 3$, $P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$.

Proof:

- $\star$ $P_{C_3} = k(k-1)(k-2) = k^3 - 3k^2 + 2k = (k-1)^3 - (k-1)$.
- $\star$ $P_{C_4} = k^4 - 4k^3 + 6k^2 - 3k = (k-1)^4 + (k-1)$.

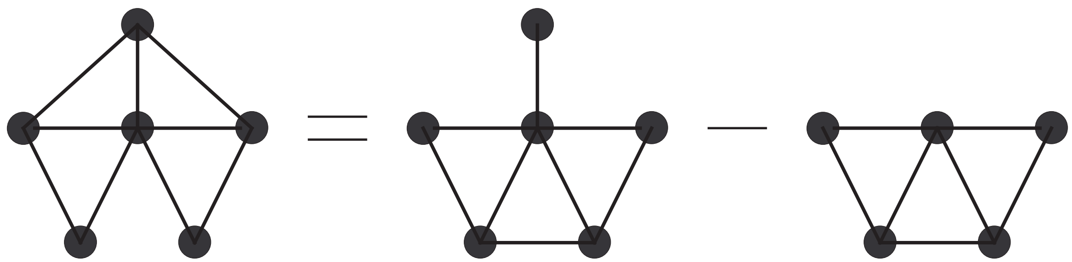
$$\begin{aligned} P_{C_n}(k) &= P_{P_n}(k) - P_{C_{n-1}}(k) \\ &= k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) \\ &= (k-1)^n + (-1)^n(k-1) \end{aligned}$$

The Chromatic Polynomial of the Broken Wheel B_n



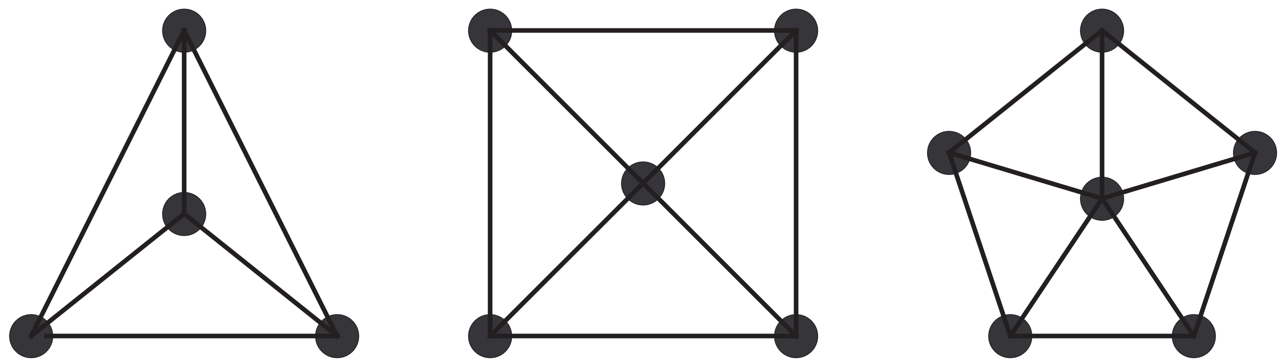
- $P_{B_2} = k(k-1)$.
- $P_{B_3} = k(k-1)(k-2)$.
- $P_{B_4} = k(k-1)(k-2)^2$.

The Chromatic Polynomial of the Broken Wheel B_n



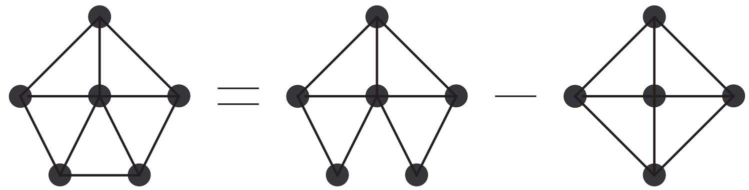
$$\begin{aligned}
P_{B_n}(k) &= P_{B'_{n-1}}(k) - P_{B_{n-1}}(k) \\
&= (k-1)P_{B_{n-1}}(k) - P_{B_{n-1}}(k) \\
&= (k-2)P_{B_{n-1}}(k) \\
&\vdots \\
&= (k-2)^{n-2}P_{B_2}(k) \\
&= k(k-1)(k-2)^{n-2}
\end{aligned}$$

The Chromatic Polynomial of the Wheel W_n



- ⋆ $P_{W_4} = k(k-1)(k-2)(k-3).$
- ⋆ $P_{W_5} = k(k-1)(k-2)(k^2-5k+7).$
- ⋆ $P_{W_6} = k(k-1)(k-2)(k-3)(k^2-4k+5).$

The Chromatic Polynomial of the Wheel W_n



$$
\begin{aligned}
P_{W_n}(k) &= P_{B_n}(k) - P_{W_{n-1}}(k) \\
&= k(k-1)(k-2)^{n-2} - P_{W_{n-1}}(k) \\
&= k(k-1)\left[(k-2)^{n-2} - (k-2)^{n-3}\right] + P_{W_{n-2}}(k) \\
&\vdots \\
&= k(k-1)\left[(k-2)^{n-2} - (k-2)^{n-3} \cdots +-(k-2)\right] \\
&= k(k-2)^{n-1} + (-1)^{n-1}k(k-2) \\
&= k(k-2)\left[(k-2)^{n-2} + (-1)^{n-1}\right]
\end{aligned}
$$

The Signs of the Coefficients of $P_G(k)$

Lemma: Let G be a graph with n vertices and m edges. Then the coefficients of $P_G(k)$ alternate between positive and negative.

Proof:

- ⋆ By induction on m.
- ⋆ If $m = 0$ then $P_G(k) = k^n$ and 0 can be $+0$ or -0.
- ⋆ Assume correctness for graphs with $m - 1$ edges or less.
- ⋆ Let (u, v) be an edge in G.

Proof Continue

- ⋆ Both $G-(u,v)$ and $G/(u,v)$ have at most $m-1$ edges. $G-(u,v)$ has n vertices and $G/(u,v)$ has $n-1$ vertices.
- ⋆ By induction, $P_{G-(u,v)} = k^n - b_{n-1}k^{n-1} + b_{n-2}k^{n-2} - \cdots$ for non-negative integers $b_1, \ldots, b_{n-1}$.
- ⋆ By induction, $P_{G/(u,v)} = k^{n-1} - c_{n-2}k^{n-2} + c_{n-3}k^{n-3} - \cdots$ for non-negative integers $c_1, \ldots, c_{n-2}$.
- ⋆ Recall that $P_G(k) = P_{G-(u,v)}(k) - P_{G/(u,v)}(k)$.
- ⋆ $P_G(k) = k^n - (b_{n-1}+1)k^{n-1} + (b_{n-2}+c_{n-2})k^{n-2} - \cdots$
- ⋆ The signs alternate since all b_i and c_i are not negative.