

Moments, Narayana Numbers, and the Cut and Paste for Lattice Paths

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Abstract. Let $\mathcal{U}(n)$ denote the set of lattice paths that run from $(0, 0)$ to $(n, 0)$ with permitted steps $(1, 1)$, $(1, -1)$, and perhaps a horizontal step. Let $\mathcal{E}(n+2)$ denote the set of paths in $\mathcal{U}(n+2)$ that run strictly above the horizontal axis except initially and finally. First we review *the cut and paste bijection* which relates points under paths of $\mathcal{E}(n+2)$ to points on paths of $\mathcal{U}(n)$. We apply it to obtain area and enumeration results for paths, some involving the Narayana distribution. We extend the bijection to a formula relating factorial moments for the paths of $\mathcal{E}(n+2)$ to moments for the paths of $\mathcal{U}(n)$.

Key phases: lattice path moments, Catalan numbers, Narayana distribution, Schröder numbers.

1 Introduction.

Consider lattice paths in the integer plane which we represent as concatenations of the directed steps $U := (1, 1)$, $D := (1, -1)$, and, perhaps, $H := (h, 0)$ where h is a positive integer. For a given set of steps, let $\mathcal{U}(n)$ denote the set of all *unrestricted* paths running from $(0, 0)$ to $(n, 0)$. Let $\mathcal{C}(n)$ denote the set of paths in $\mathcal{U}(n)$ *constrained* never to pass beneath the horizontal axis. Let $\mathcal{E}(n)$ denote the set of paths in $\mathcal{C}(n)$ that are *elevated* strictly above the horizontal axis except at their initial and final points. For any set of paths \mathcal{L} with unit-weighted steps, $|\mathcal{L}|$ will denote its cardinality. For example, for the step set $\mathcal{S} = \{U, D\}$ and for $n \geq 0$, we have that $|\mathcal{U}(2n)|$ is the central binomial coefficient $\binom{2n}{n}$, $\mathcal{C}(2n)$ is the set of Dyck paths of length $2n$, and $|\mathcal{C}(2n)| = |\mathcal{E}(2n+2)|$ is a Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

For any step set \mathcal{S} and any path P running from $(0, 0)$ to $(n, 0)$, let

$$(0, p_0), (1, p_1), (2, p_2), \dots, (x, p_x), \dots, (n, p_n) \quad (1)$$

denote all lattice points traced by the path, not just the end points of steps.

Our principal result relates factorial moments in terms of binomial coefficients for constrained paths to those for restricted paths. Specifically, for the steps U , D , and H , and for any real r ,

$$\sum_{P \in \mathcal{C}(n)} \sum_{x=0}^n \binom{p_x + r + 2}{p_x} = \sum_{Q \in \mathcal{U}(n)} \sum_{x=0}^n \binom{q_x + r}{q_x}. \quad (2)$$

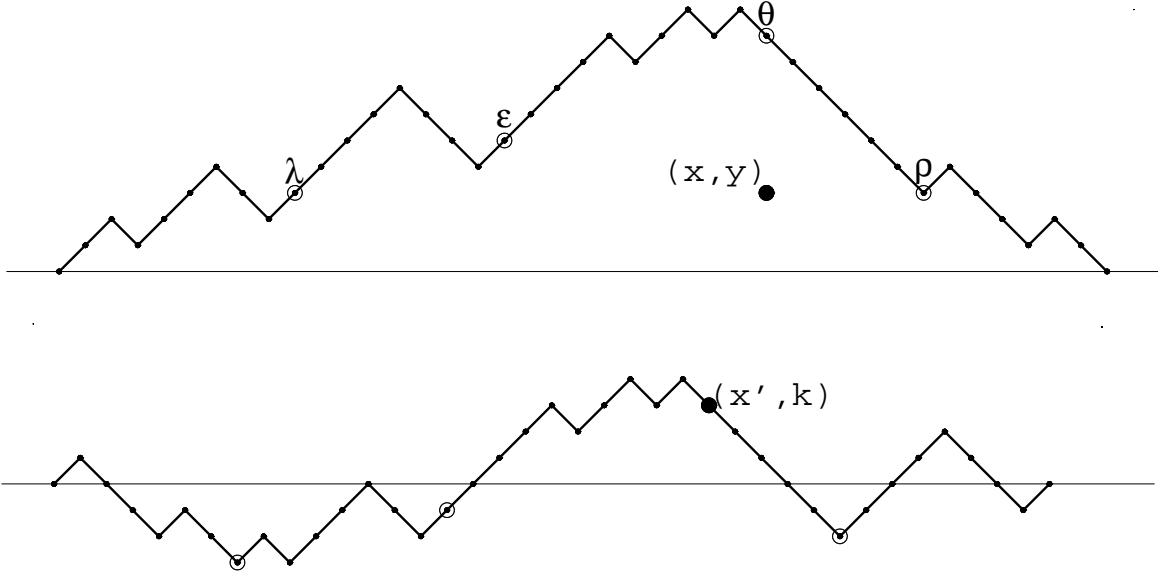


Figure 1: For $k = 3$, the dot $[P, (27, 3), 3]$ in $\text{DOTS}(40)$ and its image $(Q, (25, 3))$ in $\text{PPATHS}(38)$.

The codomain for the proposed bijection is a set of *pointed paths*, each path being marked by a distinguished lattice point on its trace. Hence the codomain is

$$\text{PPATHS}(n) := \{(Q, (x, q_x)) : Q \in \mathcal{U}(n), 0 \leq x \leq n\}.$$

We now define the bijection

$$\phi : \text{DOTS}(n+2) \rightarrow \text{PPATHS}(n) \quad (3)$$

First assume $k \geq 0$. Each $[P, (x, y), k] \in \text{DOTS}(n+2)$ determines four points on P (See Fig. 1):

- Let θ be the point on P directly above (x, y) ; i.e., $\theta := (x, p_x)$.
- Let $\epsilon = (\epsilon_1, \epsilon_2)$ be the nearest point on P to the left of (x, y) such that $\epsilon_2 = p_x - k - 1$.
(This indicates the role k plays in defining the bijection.)
- Let $\lambda = (\lambda_1, \lambda_2)$ be the nearest point on P to the left of (x, y) such that $\lambda_2 = y$.
- Let $\rho = (\rho_1, \rho_2)$ be the nearest point on P to right of (x, y) such that $\rho_2 = y$.

Let L_1 be that subpath of P running from $(0, 0)$ to λ ; let L_2 be that subpath of P running from λ to ϵ ; let R_1 be that subpath of P running from ϵ to ρ ; let R_2 be that subpath of P running from ρ to $(n+2, 0)$. Some of these subpaths may be empty. Here $P = L_1 L_2 R_1 R_2$. Let $\overline{L_1 R_1}$ be the path obtained from the concatenation $L_1 R_1$ by deleting its first and last steps. When $y = 0$, L_1 has zero length. Define

$$\phi([L_1 L_2 R_1 R_2, (x, y), k]) = (R_2 \overline{L_1 R_1} L_2, (x', k)) \quad (4)$$

where the point θ is moved along with $\overline{R_1}$ so that $x' = x + n + \lambda_1 - \epsilon_1 - \rho_1 - 1$.

If $k < 0$, let $\text{REFL}(Q)$ denote the reflection of the path Q about the x -axis and define

$$\phi([P, (x, y), k]) = (\text{REFL}(Q), (x', k))$$

where $\phi([P, (x, y), |k|]) = (Q, (x', |k|))$.

In a case where the steps are weighted, as defined in following subsection, one needs only to account for the weights of the deleted steps from $\mathcal{E}(n+2)$ to insure that ϕ is weight preserving.

2.2 Addition background and notation.

In this paper n will denote an arbitrary nonnegative integer. Usually, U , D , and $H := (h, 0)$ will denote unit-weighted steps, while U_t and H_s will denote steps weighted by the indeterminates t and s . When the steps are weighted, the weight of a path P , denoted by $|P|$, is the product of the weights of its steps. The weight of a path set \mathcal{L} , denoted $|\mathcal{L}|$, is the sum of the weights of its paths.

For notational brevity, one can allow h to be either a positive integer or ' ∞ '. Effectively, $\{U, D, (\infty, 0)\} = \{U, D\}$, and the power z^∞ will make no contribution to any power series. In the following, when we embellish ' \mathcal{S} ', denoting the allowed step set, we then embellish ' \mathcal{E} ', ' \mathcal{C} ', ' \mathcal{U} ', and generating functions accordingly.

For any step set $\mathcal{S} = \{U_t, D, H_s\}$, consider the following generating functions: $c(z) := \sum_{n \geq 0} |\mathcal{C}(n)|z^n$, $e(z) := tz^2c(z) = \sum_{n \geq 0} |\mathcal{E}(n+2)|z^{n+2}$, and $u(z) := \sum_{n \geq 0} |\mathcal{U}(n)|z^n$.

From the known decompositions of paths sets we have,

$$c(z) = 1 + sz^hc(z) + tz^2c(z)^2 \tag{5}$$

$$u(z) = 1 + sz^hu(z) + 2tz^2c(z)u(z) \tag{6}$$

To see (6) note that (i) every path in $\mathcal{U}(n)$ either has zero length; (ii) begins with H ; or (iii) begins with U followed by a constrained path or begins with D followed by the reflection about the x -axis of a constrained path, and then later returns to the horizontal axis for the first time. Identity (5) follows in a similar manner. Solving these yields

$$\begin{aligned} e(z) &= tz^2c(z) = (1 - sz^h - \sqrt{(1 - sz^h)^2 - 4tz^2})/2, \\ u(z) &= 1/\sqrt{(1 - sz^h)^2 - 4tz^2}. \end{aligned} \tag{7}$$

We remark that Example 6.3.8 of [9] extends easily to an alternative derivation of (7).

Recall the rising factorial, $z^{\bar{k}}$, defined so $z^{\bar{k}} = z(z+1) \cdots (z+k-1)$ for positive integer k , $z^{\bar{0}} = 1$, and $z^{\bar{k}} = 0$ for negative integer k . We define $\binom{r+\bar{k}}{k} = (r+1)^{\bar{k}}/k!$ for nonnegative integer k and $\binom{r+\bar{k}}{k} = 0$ otherwise. For any statement A , we define its truth value by $\chi(A)$ so that $\chi(A) = 1$ if A is true, and $= 0$ otherwise.

3 Some examples of the cut and paste.

Here we illustrate the cut and paste method by restricting its domain and codomain to prove some known results, mainly concerning the Narayana distribution and the large Schröder numbers. Other such examples appear in [6].

3.1 Cardinality results analogous to the cycle lemma.

For $\mathcal{S} = \{U_t, D, H_s\}$ let $\mathcal{E}(n, m)$ and $\mathcal{U}(n, m)$ denote those subsets of $\mathcal{E}(n)$ and $\mathcal{U}(n)$, respectively, where each path has m U -steps.

Proposition 1 For $m \geq 0$,

$$(m + 1)|\mathcal{E}(n + 2, m + 1)| = t|\mathcal{U}(n, m)|.$$

When $\mathcal{S} = \{U, D\}$, this formula reduces to $(n + 1)|\mathcal{E}(2n + 2)| = |\mathcal{U}(2n)|$, and thus the cut and paste explains the factor $(n + 1)$. The paper [6] compares this explanation with that given by the classical cycle lemma of [5].

To obtain the proposition, place $m + 1$ dots on the x -axis under each path P in $\mathcal{E}(n + 2, m + 1)$ so that one dot is located directly below the final point of each U step. Specifically, apply the restricted bijection

$$\begin{aligned} \phi : \{[P, (x, 0), 0] : P \in \mathcal{E}(n + 2, m + 1) \text{ and } x \text{ below a final point of a } U \text{ step}\} \\ \rightarrow \{(Q, (0, 0)) : Q \in \mathcal{E}(n, m)\}. \end{aligned}$$

We see that the weight of the domain is $(m + 1)|\mathcal{E}(n + 2, m + 1)|$ while the weight of the codomain is $t|\mathcal{U}(n, m)|$, where the factor t corresponds to the deletion of a U in the cut and paste.

We define the elevated large Schröder paths to be the paths in $\bar{\mathcal{E}}(n + 2)$ having $\bar{\mathcal{S}} = \{U, D, (2, 0)\}$. We take the large Schröder numbers to be defined in terms of the cardinality of these path sets. Specifically, $(|\bar{\mathcal{E}}(2n + 2)|)_{n \geq 0} = (1, 2, 6, 22, 90, 394, \dots)$. (Sequence A006318 of [8].)

Proposition 1 shows that large Schröder numbers can be formulated as

$$\begin{aligned} |\bar{\mathcal{E}}(2n + 2)| &= \sum_{m \geq 0} |\bar{\mathcal{E}}(2n + 2, m + 1)| \\ &= \sum_{m \geq 0} \frac{1}{(m + 1)} |\bar{\mathcal{U}}(2n, m)| = \sum_{m \geq 0} \frac{(m + n)!}{(m + 1)! m! (n - m)!}. \end{aligned} \tag{8}$$

3.2 The Narayana distribution in terms of oddly positioned up steps.

Consider the step set $\tilde{\mathcal{S}} = \{U_t, U, D\}$, where in this example, U_t is an up step of weight t that must be oddly positioned on any path, U is a unit-weighted up step that must be evenly positioned on any path, and D is a unit weighted down step. As we will see, for $t = 2$, $(|\tilde{\mathcal{E}}(2n+2)|)_{n \geq 1} = (2, 6, 22, 90, 394, \dots)$, the large Schröder numbers with missing initial term.

Consider also the step set $\mathcal{S}^* = \{U_t, D, (1, 0)_{t+1}\}$. It is known that for $t = 1$, $(|\mathcal{E}^*(n+1)|)_{n \geq 1} = (1, 2, 5, 14, 42, \dots)$, the Catalan numbers missing the first term. For $t = 2$, $(|\mathcal{E}^*(n+1)|)_{n \geq 1} = (2, 6, 22, 90, 394, \dots)$, the large Schröder numbers missing the first term. In order to see that the cardinalities of $\mathcal{E}^*(n+1)$ and $\tilde{\mathcal{E}}(2n+2)$ agree, one can establish a straightforward isomorphism $\alpha : \mathcal{U}^*(n) \rightarrow \tilde{\mathcal{U}}(2n)$ using the step replacement rules: $\alpha(U_t) = U_t U$, $\alpha(D) = DD$, and $\alpha(H_{t+1}) = \{U_t D, DU\}$.

Proposition 2 For $n \geq 1$,

$$|\tilde{\mathcal{E}}(2n+2)| = |\mathcal{E}^*(n+1)| = \sum_{i=1}^n \frac{1}{n} \binom{n}{i} \binom{n}{i-1} t^i, \quad (9)$$

where $\frac{1}{n} \binom{n}{i} \binom{n}{i-1}$ is a Narayana number. (Sequence A001263 in [8].)

The first equality is clear. By Proposition 1,

$$\begin{aligned} |\mathcal{E}^*(n+2)| &= \sum_m |\mathcal{E}^*(n+2, m)| \\ &= \sum_m t |\mathcal{U}^*(n, m)| / (m+1) = t \sum_m \frac{t^m (t+1)^{n-2m} n!}{(m+1)! m! (n-2m)!}. \end{aligned}$$

Applying the binomial theorem and then the Chu-Vandermonde convolution yields the proposition. We remark that, for $t = 2$, the right side of (9) is a known formula for the large Schröder numbers.

3.3 The Narayana distribution in terms of peaks

For this application we count elevated paths with respect to the number of bicolored peaks. Again we derive the formula for the Narayana numbers and the large Schröder numbers. On any path, a ‘right-hand turn’ or a ‘peak’ is the intermediate vertex of a consecutive UD pair. For $\hat{\mathcal{S}} = \{U, D\}$, let $\hat{\mathcal{E}}(n, b, r)$ denote the set of elevated paths using the steps U and D and having b blue peaks, r red peaks, and no others.

Proposition 3 For $1 \leq b+r \leq n$,

$$|\hat{\mathcal{E}}(2n+2, b, r)| = \frac{1}{b+r} \binom{n-1}{b+r-1} \binom{n}{b+r-1} \binom{b+r}{b}.$$

For each path P in $\widehat{\mathcal{E}}(2n+2, 0, i)$, place i dots on the x -axis below the peaks (all being red) of the path. With $k = 0$ each dot is mapped by ϕ to a point $\phi([P, (x, 0), 0]) = [\overline{L_1 R_1} L_2, (0, 0)]$ (L_1 has no length) where the image path begins with a D step and has $i - 1$ right-hand turns. If we tilt each image path counterclockwise by 45 degrees, one can check that in the tilted path there would be $\binom{n-1}{i-1}$ ways to choose the abscissae and $\binom{n}{i-1}$ ways to choose the ordinates for the intermediate vertices of the right-hand turns, where these turns uniquely determine the path. Thus, $i|\widehat{\mathcal{E}}(2n+2, 0, i)| = \binom{n-1}{i-1} \binom{n}{i-1}$. Now, allowing b of the peaks to be independently colored blue, while the remainder are red, yields the factor $\binom{b+r}{b}$.

When we disallow blue peaks, the proposition shows that $\widehat{\mathcal{E}}(2n+2, 0, r)$ is a Narayana number. When we do not limit the coloring or the number of peaks, we see that the number of paths in the $\widehat{\mathcal{E}}(2n+2)$ with independently bicolored peaks is the large Schröder number:

$$\begin{aligned} \sum_b \sum_r |\widehat{\mathcal{E}}(2n+2, b, r)| &= \sum_b \sum_i |\widehat{\mathcal{E}}(2n+2, b, i-b)| \\ &= \sum_i \sum_b |\widehat{\mathcal{E}}(2n+2, b, i-b)| = \sum_{i=1}^n \frac{1}{i} \binom{n-1}{i-1} \binom{n}{i-1} 2^i \end{aligned} \quad (10)$$

Now return to the large Schröder paths, considered in (8), which used $\overline{\mathcal{S}} = \{U, D, (2, 0)\}$. In that notation, $\overline{\mathcal{E}}(2n, n+1-j)$ will be the set of elevated paths having j of the $(2, 0)$ steps. There is a simple matching between $\cup_r \widehat{\mathcal{E}}(2n+2, b, r)$ and $\overline{\mathcal{E}}(2n+2, n+1-b)$ that is obtained by transforming each UD pair with a blue intermediate vertex into a $(2, 0)$ step and by removing the color red. Hence the number of paths in $\overline{\mathcal{E}}(2n+2)$ is also counted by large Schröder numbers of (10).

3.4 The total heights of the peaks

We will consider the total area under the paths of $\mathcal{E}(n+2)$ more extensively in Section 6. Here, for $\mathcal{S} = \{U_t, D, H_s\}$, we will sum the heights of the peaks over all paths in the set of constrained paths $\mathcal{C}(n)$. Equivalently, by the manner in which dots are arrayed under each lattice point on a path of $\mathcal{E}(n+2)$, we will find the weighted cardinality of the dots $[P, (x, y), 1]$ (Note, $k = 1$) under the peaks of the paths of $\mathcal{E}(n+2)$. By the cut and paste one can check that the restricted bijection is

$$\begin{aligned} \phi : \{[P, (x, y), 1] \in \text{DOTS}(n+2) : (x, p_x) \text{ is a peak}\} \\ \rightarrow \{(P, (x, 1)) \in \text{PPATHS}(n) : p_{x-1} = p_{x+1} = 0\}. \end{aligned}$$

Here, each dot in the restricted domain is mapped to a path in $\mathcal{U}(n)$ having a marked UD with intermediate vertex of ordinate 1. Each such marked path results from the concatenation of three paths, namely, an unrestricted path from $(0, 0)$ to the marked UD followed by an unrestricted path to $(n, 0)$. Hence, with tz^2 corresponding to the marked UD , we have

Proposition 4

$$\sum_{n \geq 0} \sum_{P \in \mathcal{C}(n)} \sum_{0 < x < n} \chi((x, p_x) \text{ is a peak}) p_x z^n = tz^2 u(z)^2.$$

Consequently, for $\mathcal{S} = \{U, D\}$, the power series for the sum of heights of the peaks on constrained paths is $tz^2u(z)^2 = z^2(1 - 4z^2)^{-1}$, whose coefficients are powers of 4.

4 Relating moments for constrained paths to those for unrestricted paths.

Our principal consequence of the the cut and paste bijection is

Proposition 5 *For step set $\mathcal{S} = \{U, D, H\}$ and for real r ,*

$$\sum_{P \in \mathcal{C}(n)} \sum_{x=0}^n \binom{p_x + r + 2}{p_x} = \sum_{P \in \mathcal{E}(n+2)} \sum_{x=1}^{n+1} \binom{p_x + r + 1}{p_x - 1} = \sum_{Q \in \mathcal{U}(n)} \sum_{x=0}^n \binom{q_x + r}{q_x}. \quad (11)$$

To prove the second equality, we assign the value $\binom{k+r}{k}$ to each dot $[P, (x, y), k]$ and also to its image. The cut and paste yields trivially

$$\sum_{\substack{[P, (x, y), k] \in \\ \text{DOTS}(n+2)}} \binom{k+r}{k} = \sum_{\substack{(Q, (x, k)) \in \\ \text{PPATHS}(n)}} \binom{k+r}{k} = \sum_{\substack{(Q, (x, y)) \in \\ \text{PPATHS}(n)}} \binom{y+r}{y}. \quad (12)$$

The left side of (12) becomes

$$\begin{aligned} \sum_{P \in \mathcal{E}(n+2)} \sum_{x=1}^{n+1} \sum_{y=0}^{p_x-1} \sum_{k=0}^{p_x-y-1} \binom{k+r}{k} &= \sum_{P \in \mathcal{E}(n+2)} \sum_{x=1}^{n+1} \sum_{y=0}^{p_x-1} \binom{p_x - y + r}{p_x - y - 1} \\ &= \sum_{P \in \mathcal{E}(n+2)} \sum_{x=1}^{n+1} \binom{p_x - 1 + r + 2}{p_x - 1} \\ &= \sum_{P \in \mathcal{C}(n)} \sum_{x=0}^n \binom{p_x + r + 2}{p_x}, \end{aligned}$$

where the last identity follows from a simple shift. This identity with (12) yields (11).
the result.

More generally, for step set $\mathcal{S} = \{U_t, D, H_s\}$ and for any nonnegative integer m , a similar computation shows

$$\begin{aligned} \sum_{P \in \mathcal{C}(n)} |P| \sum_{x=0}^n \binom{p_x + r + 2 - m}{p_x - m} &= \sum_{P \in \mathcal{E}(n+2)} t^{-1} |P| \sum_{x=1}^{n+1} \binom{p_x + r - m + 1}{p_x - m - 1} \\ &= \sum_{Q \in \mathcal{U}(n)} |Q| \sum_{x=0}^n \binom{q_x + r - m}{q_x - m}. \end{aligned} \quad (13)$$

Example. An initial example of (13) is that the total second moments for the paths of $\mathcal{E}(n+2)$ is

$$\begin{aligned}
\sum_{P \in \mathcal{E}(n+2)} |P| \sum_{x=1}^{n+1} p_x^2 &= \sum_{P \in \mathcal{E}(n+2)} |P| \sum_{x=1}^{n+1} \binom{p_x+1}{p_x-1} + \binom{p_x}{p_x-2} \\
&= \sum_{Q \in \mathcal{U}(n)} t|Q| \sum_{x=0}^n \binom{q_x}{q_x} + \binom{q_x-1}{q_x-1} \\
&= \sum_{Q \in \mathcal{U}(n)} t|Q| \sum_{x=0}^n \chi(q_x \geq 0) + \chi(q_x \geq 1) \\
&= \sum_{Q \in \mathcal{U}(n)} t|Q| \sum_{x=0}^n \chi(q_x \geq 0) + \chi(q_x < 0) \\
&= \sum_{Q \in \mathcal{U}(n)} t|Q| \sum_{x=0}^n 1 = (n+1)t|\mathcal{U}(n)|.
\end{aligned}$$

Of course, referring directly to the cut and paste ϕ in (3), if we count the domain, which consists of arrays of y_x^2 dots for each point y_x on the trace of a path, and we count the codomain, we find the same result.

Example. Here we count two ‘Catalan structures’ not in the original catalog of Exercise 6.19 of [9].

(i) For the steps U and D and for $n \geq 0$, the total number of intercepts of the horizontal axis by the Dyck paths running from $(0, 0)$ to $(2n, 0)$ is $\frac{1}{n+2} \binom{2n+2}{n+1}$.

(ii) As a consequence of (11), if the values 1, -2 , and 1 are assigned respectively to the points of ordinate 0, 1, and 2 on the unrestricted paths running from $(0, 0)$ to $(2n, 0)$, then the sum of these values over all these paths is the Catalan number of (i).

These follow from

$$\begin{aligned}
\sum_{n \geq 0} \sum_{Q \in \mathcal{U}(n)} \sum_x (-1)^{q_x} \binom{2}{q_x} z^n &= \sum_{n \geq 0} \sum_{Q \in \mathcal{U}(n)} \sum_x \binom{q_x-3}{q_x} z^n = \\
\sum_{n \geq 0} \sum_{P \in \mathcal{C}(n)} \sum_x \binom{p_x-1}{p_x} z^n &= \sum_{n \geq 0} \sum_{P \in \mathcal{C}(n)} \sum_x \chi(p_x = 0) z^n = c(z)^2 = z^{-2}(c(z) - 1).
\end{aligned}$$

To see the next to the last identity, notice that $\sum_{P \in \mathcal{C}(n)} \sum_x \chi(p_x = 0)$ counts the set of intercept-marked paths formed from constrained paths of $\mathcal{C}(n)$. Since each intercept is realized as the concatenation of a constrained path from $(0, 0)$ to the intercept with a constrained path from the intercept to $(n, 0)$, $\sum_{P \in \mathcal{C}(n)} \sum_x \chi(p_x = 0) = \sum_i |C(i)||C(n-i)|$.

5 Recurrences for moments.

We will give a recurrence for binomial moments for constrained paths generalizing one given by Chapman [3] for Dyck paths. (See also [7] and [12]). We will then use Proposition 5 to convert that recurrence into one for moments of unrestricted paths.

For path set $\mathcal{S} = \{U_t, D, H_s\}$, we define the generating functions

$$c_r(z) := \sum_{n \geq 0} \sum_{P \in \mathcal{C}(n)} |P| \sum_{x=0}^n \binom{p_x + r - m + 2}{p_x - m} z^n$$

$$u_r(z) := \sum_{n \geq 0} \sum_{Q \in \mathcal{U}(n)} |Q| \sum_{x=0}^n \binom{q_x + r - m}{q_x - m} z^n.$$

Proposition 6 *For real r and nonnegative m ,*

$$(1 - tz^2c(z)^2)c_r(z) = c_{r-1}(z) \quad (14)$$

$$(1 - tz^2c(z)^2)u_r(z) = u_{r-1}(z) \quad (15)$$

$$c_r(z) = \frac{1}{2}(1 + (1 - sz^h)u(z))c_{r-1}(z) \quad (16)$$

We prove (14) in the form

$$c_r(z) = c_{r-1}(z) + tz^2c(z)^2c_r(z). \quad (17)$$

and only for $m = 0$ and $s = t = 1$. The binomial coefficients satisfy

$$\sum_{P \in \mathcal{C}(n)} \sum_{x=0}^n \binom{p_x + r + 2}{p_x} = \sum_{P \in \mathcal{C}(n)} \sum_{x=0}^n \binom{p_x + r + 1}{p_x} + \sum_{P \in \mathcal{C}(n)} \sum_{x=0}^n \binom{p_x + r + 1}{p_x - 1}. \quad (18)$$

Consider the rightmost double sum. Each $P \in \mathcal{C}(n)$ either lies entirely on the x -axis and makes no contribution to that sum, or has a factor which is an elevated path. We will consider the contribution to the sum by the horizontal translations of all $Q = UQ'D$ belonging to $\cup_{0 \leq n' \leq n-2} \mathcal{E}(n'+2)$. Each time a translation of some Q , $Q \in \mathcal{E}(n'+2)$, appears as a factor in some concatenation forming a path of $\mathcal{C}(n)$, it is preceded by a (perhaps void) constrained path and followed by a (perhaps void) constrained path; thus translations of this Q makes $\sum_{i+i'=n-n'-2} |\mathcal{C}(i)||\mathcal{C}(i')|$ appearances in the paths of $\mathcal{C}(n)$. The moment contribution of Q to $\sum_{P \in \mathcal{C}(n)} \sum_{x=0}^n \binom{p_x+r+1}{p_x-1}$ is

$$\sum_{x=1}^{n'+1} \binom{q_x + r + 1}{q_x - 1} = \sum_{x=0}^{n'} \binom{q'_x + r + 2}{q'_x}$$

times the frequency of its appearance. Hence,

$$\sum_{P \in \mathcal{C}(n)} \sum_{x=0}^n \binom{p_x + r + 1}{p_x - 1} = \sum_{n'=0}^{n-2} \sum_{Q' \in \mathcal{C}(n')} \sum_{i+i'=n-n'-2} |\mathcal{C}(i)||\mathcal{C}(i')| \sum_{x=0}^{n'} \binom{q'_x + r + 2}{q'_x} \quad (19)$$

and the corresponding generating function is $z^2 c(z)^2 c_r(z)$. The identities (18) and (19) yield (17) for $m = 0$ and $s = t = 1$. The proof for cases other than $m = 0$ and $s = t = 1$ are similar. An application of Proposition 5 to (14) yields (15). A straightforward computation yields (16).

6 Further examples.

6.1 Areas and intercepts

Take $\mathcal{S} = \{U, D, H\}$. By the trapezoid rule, the total area bounded between the horizontal axis and the paths of $\mathcal{E}(n+2)$ is the middle formula of (11) in the form of $\sum_{P \in \mathcal{E}(n+2)} \sum_x p_x = \sum_{P \in \mathcal{E}(n+2)} \sum_x \binom{p_x}{p_x-1}$. Since $\binom{p_x-1}{p_x} = \chi(p_x = 0)$, the right side of (11) becomes $\sum_{Q \in \mathcal{U}(n)} \sum_x \chi(q_x = 0)$ which is the total number of intercepts of the horizontal axis by the paths of $\mathcal{U}(n)$. More generally, using (13), then

Proposition 7 *For $\mathcal{S} = \{U, D, H\}$ and for $m \geq 0$, the total area of the regions under the paths of $\mathcal{E}(n+2)$ and above the horizontal line $y = m$ is equal to the number of intercepts of that line by the paths of $\mathcal{U}(n)$.*

Next we give further results concerning area, intercepts, and the generating function $u(z)^2$, whose formula is obtained from (7).

Proposition 8 *For $\mathcal{S} = \{U, D, H\}$, the generating function for the number of intercepts of the horizontal axis by the step end points on the traces of the paths of $\mathcal{U}(n)$ satisfies*

$$\sum_{n \geq 0} \sum_{Q \in \mathcal{U}(n)} \sum_{0 \leq x \leq n} \chi((x, 0) \text{ is a step end point on } Q) z^n = u(z)^2.$$

This proposition follows by observing that each intercept contributing to the inner summations results from the concatenation of two paths, an unrestricted path from $(0, 0)$ to the intercept followed by an unrestricted path from the intercept to $(n, 0)$.

Proposition 9 *For $\mathcal{S} = \{U, D, H\}$, the generating function for the number of intercepts of the horizontal axis by lattice points on the traces of paths of $\mathcal{U}(n)$ satisfies*

$$\sum_{n \geq 0} \sum_{Q \in \mathcal{U}(n)} \sum_{0 \leq x \leq n} \chi(q_x = 0) z^n = (1 + (h-1)z^h) u(z)^2. \quad (20)$$

Equivalently, the generating function for the total area under the paths of $\mathcal{E}(n+2)$ satisfies

$$\sum_{n \geq 0} \sum_{P \in \mathcal{E}(n+2)} \sum_{1 \leq x \leq n+1} p_x z^{n+2} = z^2 (1 + (h-1)z^h) u(z)^2.$$

Intercepts, being end points of steps, make a contribution to the generating function as in Proposition 8. When a step lies on the horizontal axis, the $h - 1$ intercepts which are interior to a step can be collapsed along with the step to become the intercept of a step end point on a path of belonging to $\mathcal{U}(n - h)$ and thus make an adjusted contribution to the right side of (20). The second identity follows by the initial remarks of this section.

6.2 Two comparable models. Here we examine some comparable, yet different, models whose area results are derivable from the the cut and paste, through Proposition 9. Notice the appearance of powers of 4, which also appeared in Section 3.3. In both cases below we will consider instances of a recurrence: If $a(n)$ is defined so that $\sum_n a(n)z^n = u(z)^2$, then

$$a(n) = 4ta(n - 2) + 2sa(n - h) - s^2a(n - 2h) \quad (21)$$

which follows the comparison of coefficients and (7).

Case for $s = t + 1$ and $h = 1$: Here the step set is $\mathcal{S}^* = \{U_t, D, (1, 0)_{t+1}\}$, as in Section 3.2. For $t = 1$ and $t = 2$, $u^*(z)$ is the generating function for the central binomial coefficients and the central Delannoy numbers, respectively. (See Sequence A001850 of [8] and Section 6.5 of [9].)

The weighted area under the elevated paths of $\mathcal{E}^*(n + 2)$, satisfies

$$\sum_{n \geq 0} \sum_{P \in \mathcal{E}^*(n+2)} \sum_x p_x z^{n+2} = tz^2 u^*(z)^2 = \frac{tz^2}{((t+1)z - 1)^2 - 4tz^2}$$

For $t = 1$, $z^2 u^*(z) = \frac{z^2}{1-4z}$, a generating function whose coefficients are powers of 4. We remark that there is a simple bijection from $\mathcal{E}^*(n + 2)$ to a set of parallelogram polyominoes of perimeter $2n + 4$ which is area preserving as discussed in [11]. Further attention to this result appears in [4].

For $t = 2$, $2z^2 u^*(z)^2 = \frac{2z^2}{1-6z+z^2} = 2z^2 + 12z^3 + 70z^4 + 408z^5 \dots$, where the coefficients are double the square-triangular numbers, or, equivalently, every other Pell number. (See sequences A000129, A001109, and A001542 in [8].) By (21), for $t = 2$, $u^*(z)^2 = \sum_n a^*(n)z^n$ satisfies

$$a^*(n) = 6a^*(n - 1) - a^*(n - 2) \quad (22)$$

subject to $a^*(2) = 2$ and $a^*(3) = 12$. This recurrence in terms of the total the areas of zebras (i.e., column-bicolored parallelogram polyominoes) was a principal topic of [11].

Case for $t = 1$: Here $\overline{\mathcal{S}} = \{U, D, (0, h)_s\}$. For $h = \infty$, $\overline{e}(z)$ is a generating function for the Catalan numbers. For $s = 1$ and $h = 1$, $\overline{e}(z)$ is the generating function for the Motzkin numbers. For $s = 1$ and $h = 2$, $\overline{\mathcal{C}}(n)$ are the usual large Schröder paths with $\overline{e}(z)$ being the generating function for the large Schröder numbers, as noted after formula (10).

For $h = \infty$, $\overline{u}(z)$ gives the central binomial coefficients. For $s = h = 1$, $\overline{u}(z)$ gives the central trinomial numbers. For $s = 1$ and $h = 2$, $\overline{u}(z)$ gives the central Delannoy numbers.

By Proposition 9 the area under the elevated paths of $\overline{\mathcal{E}}(n+2)$, satisfies

$$\sum_{n \geq 0} \sum_{P \in \overline{\mathcal{E}}(n+2)} \sum_x p_x z^{n+2} = \frac{1 + (h-1)z^h}{(sz^h - 1)^2 - 4z^2}.$$

For $h = \infty$, the coefficients of this power series are powers of 4. (See [13].) For $s = 1$ and $h = 2$, this power series becomes $\frac{z^2(1+z^2)}{1-6z^2+z^4} = 1z^2 + 7z^4 + 41z^6 + 239z^8 + 1393z^{10} \dots$, where the coefficients correspond to pairwise sums of consecutive square-triangular numbers, or equivalently to every other pairwise sum of consecutive Pell numbers, as noted in [2]. (See sequence A002315 in [8].) For $\overline{\mathcal{E}}(2n+2)$, we remark that the square-triangular numbers give both the sums of the ordinates of the points of the path restricted to be end points of steps and the sums of the ordinates of the points which are the mid points of steps. By (21), for $s = 1$ and $h = 2$, $\overline{u}(z)^2 = \sum_n \overline{a}(n)z^n$ satisfies

$$\overline{a}(n) = 6\overline{a}(n-2) - \overline{a}(n-4) \tag{23}$$

subject to $\overline{a}(2) = 1$ and $\overline{a}(4) = 7$, as noted in [2]. Compare this recurrence to (22). For recent considerations of (23) and other references, see [1]. For a bijective approach to the recurrences for the cardinality, area, and the second moments for large Schröder paths see [10].

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