ON RATIONAL RIGHT-ANGLED TRIANGLES

By Artemas Martin.

—"and are you such fools,
To Square for this?"—*Titus Andronicus*, ii, 1.

In a former paper the writer gave three methods of proof of the celebrated proposition that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the legs (sides including the right angle). Other proofs are presented here which may be of interest.

IX. Let ABC be any right-angled triangle (Fig. 1), B the right angle, O the centre of the inscribed circle.

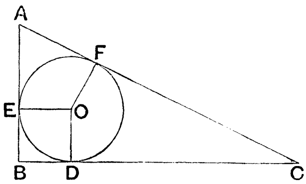

Fig. 1.

Put OD, OE, $OF = r =$ radius of inscribed circle, $AC = a$, $AB = b$, $BC = c$.

Then

area of triangle $ABC = \frac{1}{2}bc = \frac{1}{2}r(a + b + c)$...(1).

From the figure we have

$$BF + BD = b + c - a = 2r \text{..............(2)},$$

since $AE = AF$, $DC = FC$, and $BD = BE = r$.

Hence from (2) we have

$$r = \frac{1}{2}(b + c - a) \text{ ....(3).}$$

Substituting this value in (1) and reducing we get

$$a^2 = b^2 + c^2 \text{ ....(4).}$$

This proof was communicated in May, 1891, by Dr L. A. Bauer, then of the U.S. Coast and Geodetic Survey, now of the Carnegie Institution of Washington.

Another method of the same proof.

Let $CD = CF = m$, $AE = AF = n$, $OD = OE = OF = r$, in Fig. 1; then

$$AC = a = m + n, \ AB = b = n + r, \ BC = c = m + r.$$

Now $$(m + r)(n + r) = r(2m + 2n + 2r) \text{ ....(1),}$$

since each expression is double the area of the triangle ABC.

Therefore $$mn = mr + nr + r^2,$$

and $$2mn = 2mr + 2nr + 2r^2 \text{ ....(2).}$$

Adding $m^2 + n^2$ to each member of (2),

$$(m+n)^2 = (m+r)^2 + (n+r)^2,$$

or

$$a^2 = b^2 + c^2.$$

This method was communicated by Lucius Brown, Hudson, Mass.

The following is not new, but is of interest because of its simplicity.

Let ABC (Fig. 2) be any right-angled triangle (right-angled at B), $DEAC$ the square on the hypotenuse AC. Produce BC to H, making $CH = AB$; produce BA to F, making $AF = BC$; draw EH perpendicular to BH, and produce HE to G, making $EG = CH$; draw FG through D; then

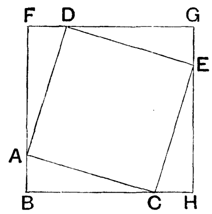

Fig. 2.

$$FD = AB = CH = GE,$$

and

$$DG = EH = BC = AF.$$

Let $AC = a$, $AB = b$, $BC = c$; then

$$BH = BF = FG = GH = b + c; \quad \square DEAC = c^2,$$

$$\triangle ABC = \triangle DFA = \triangle GED = \triangle CHE = \tfrac{1}{2}bc.$$

$$\square FGBH = \square DEAC + 4\triangle ABC;$$

therefore

$$(b+c)^2 = a^2 + 2bc,$$

or

$$b^2 + 2bc + c^2 = a^2 + 2bc.$$

Dropping $2bc$ from each side we have left

$$b^2 + c^2 = a^2.$$

X. Every prime number of the form $4m+1$ is the hypotenuse of a prime rational right-angled triangle.

The hypotenuse of a prime rational right-angled triangle is either a prime number of the form $4m+1$ or the product of two or more such primes.

A number that is the product of n different prime numbers of the form $4m+1$ is the common hypotenuse of 2^{n-1} different prime rational right-angled triangles.

Every number which contains a prime factor of the form $4m+1$ is the hypotenuse of a rational (but not prime) right-angled triangle.

The hypotenuse of any rational right-angled triangle is n times a prime number of the form $4m+1$, or n times the product of two or more such primes, where n may be any whole number. The triangle will be prime when n is a prime of the form $4m+1$.

A prime number of the form $4n+3$ can not be the hypotenuse of a rational right-angled triangle.

A number which does not contain a prime factor of the form $4m+1$ can not be the hypotenuse of a rational right-angled triangle.

XI. In *Miscellaneous Notes and Queries*, edited by the late S. C. Gould, Manchester, N.H., Vol. IX, No. 6 (June, 1892), p. 141, in a paper on "Rational Right-Angled Triangles," the late Benjamin F. Burleson of Oneida Castle, N.Y., in his solution of Problem 4, "Write out all the rational right-angled triangles there are whose hypotenuses do not exceed 100," stated that there are 55 such triangles; but he inadvertently included in his second series of triangles three that are in the first series, so that three triangles (10, 24, 26), (14, 48, 50), (54, 72, 90), were counted twice. Deducting 3 from 55 we have left 52, which is the number of rational right-angled triangles whose hypotenuses do not exceed 100.

See *Mathematical Magazine*, Vol. II, No. 12 (Sept. 1910), pp. 319—20; *Miscellaneous Notes and Queries*, Vol. x, No. 3 (Sept. 1892), p. 230; *Educational Times Reprint*, Vol. XXXIX (London, 1883), pp. 37—8, Quest. 6966.

The prime rational right-angled triangles whose hypotenuses are less than 3000 are given in the appended table for reference, and are arranged in the order of the magnitude of the hypotenuses for convenience of use in this paper. I have extended this table, in manuscript, so as to include all prime rational right-angled triangles whose hypotenuses are less than 10000. For triangles whose hypotenuses exceed 3000, consult the table on pp. 301—8 of No. 12, Vol. II, of the *Mathematical Magazine*, beginning with p. 304. I have extended this table in manuscript to $p = 101$, $q = 100$.

There are 158 prime rational right-angled triangles whose hypotenuses are less than 1000; 161 whose hypotenuses are between 1000 and 2000; 158 whose hypotenuses are between 2000 and 3000; or 477 whose hypotenuses are less than 3000.

I have checked the number of these triangles in the appended table with the table by Edward Sang (*Transactions of the Royal Society of Edinburgh*, Vol. XIII, 1864, pp. 757—58), and both agree to the extent of Sang's table, except that he omitted the following two triangles (576, 943, 1105), (744, 817, 1105), at the end of his table.

XII. To find rational right-angled triangles whose sides are whole numbers and hypotenuses consecutive whole numbers.

Examples. 1. To find pairs or couples of right-angled triangles whose hypotenuses are consecutive integers.

Let u, v, w; x, y, $w+1$ denote the sides of a pair of such triangles, and, to get a general solution, we must solve the equations

$$u^2 + v^2 = w^2 \quad \ldots\ldots\ldots\ldots(1),$$

$$x^2 + y^2 = (w+1)^2 \quad \ldots\ldots\ldots\ldots(2);$$

but it is much easier to find the triangles wanted by trial from the appended table with the aid of Table I in Barlow's Tables (London, 1814), when the hypotenuses are not greater than 3000; when the hypotenuses are greater than 3000, use the table of right-angled triangles in the *Mathematical Magazine* on pp. 301—8, beginning with p. 304.

If we multiply the sides of the first triangle in the table at the end of this paper by 5, we have a triangle whose sides are

15, 20, 25.

Multiplying the sides of the second triangle in the table by 2, we get the triangle whose sides are

10, 24, 26;

and we have two triangles whose hypotenuses differ by unity or are consecutive numbers.

In a similar manner we find from the first and fifth triangles in the table the pair

20, 21, 29;
18, 24, 30.

From the first and third triangles we get the pair

16, 30, 34;
21, 28, 35.

The following pairs, and many others, are easily found:

15, 36, 39;	14, 48, 50;	20, 48, 52;
24, 32, 40.	24, 45, 51.	28, 45, 53.
48, 55, 73;	60, 80, 100;	40, 96, 104;
24, 70, 74.	20, 99, 101.	63, 84, 105.
24, 143, 145;	119, 120, 169;	65, 156, 169;
96, 110, 146.	102, 136, 170.	80, 150, 170.
252, 844, 900;	696, 697, 985;	324, 945, 999;
451, 780, 901.	680, 714, 986.	352, 936, 1000.

2. To find groups of three rational right-angled triangles whose hypotenuses are consecutive numbers.

Multiplying the second triangle in the appended table by 3, the first triangle by 8, we have, with the seventh triangle,

15, 36, 39;
24, 32, 40;
9, 40, 41;

a group of three triangles whose hypotenuses are consecutive numbers.

In a similar manner the following groups of three such triangles have been found, and many other groups may be found with hypotenuses not exceeding 1000.

48, 55, 73;	60, 80, 100;	40, 96, 104;
24, 70, 74;	20, 99, 101;	63, 84, 105;
45, 60, 75.	48, 90, 102.	56, 90, 106.
69, 92, 115;	81, 108, 135;	48, 140, 148;
80, 84, 116;	64, 120, 136;	51, 140, 149;
45, 108, 117.	88, 105, 137.	90, 120, 150.
52, 165, 173;	95, 168, 193;	160, 168, 232;
120, 126, 174;	130, 144, 194;	105, 208, 233;
105, 140, 175.	117, 156, 195.	90, 216, 234.

44, 240, 244;	84, 245, 259;	128, 240, 272;
147, 196, 245;	100, 240, 260;	105, 252, 273;
54, 240, 246.	180, 189, 261.	176, 210, 274.
102, 280, 298;	140, 336, 364;	81, 360, 369;
115, 276, 299;	240, 275, 365;	222, 296, 370;
180, 240, 300.	66, 360, 366.	196, 315, 371.
155, 372, 403;	280, 294, 406;	120, 391, 409;
80, 396, 404;	132, 385, 407;	168, 374, 410;
243, 324, 405.	192, 360, 408.	264, 315, 411.

3. To find groups of four rational right-angled triangles whose hypotenuses are consecutive numbers less than 1000.

In the *School Messenger*, edited by G. H. Harvill, Vol. II, No. 6 (June, 1885), p. 218, the late B. F. Burleson, in a paper on "Complementary Squares" (right-angled triangles), said: "With hypotenuses less than 1000 I have found 21 sets of series, 4 in each, in which the hypotenuses differ by unity"; but he did not give the triangles.

The present writer has found 45 such sets or groups of four triangles which are exhibited below. It is to be regretted that Mr Burleson did not publish the groups he found so that it could be seen whether any of them are missing here.

30, 40, 50;	108, 144, 180;	40, 198, 202;
24, 45, 51;	19, 180, 181;	140, 147, 203;
20, 48, 52;	70, 168, 182;	96, 180, 204;
28, 45, 53.	33, 180, 183.	45, 200, 205.
120, 182, 218;	160, 168, 232;	44, 240, 244;
144, 165, 219;	105, 208, 233;	147, 196, 245;
132, 176, 220;	90, 206, 234;	54, 240, 246;
21, 220, 221.	120, 225, 235.	95, 228, 247.
128, 240, 272;	136, 255, 289;	120, 288, 312;
105, 252, 273;	200, 210, 290;	25, 312, 313;
176, 210, 274;	195, 216, 291;	170, 264, 314;
77, 264, 275.	192, 220, 292.	189, 252, 315.
75, 308, 317;	175, 288, 337;	240, 252, 348;
168, 270, 318;	130, 312, 338;	180, 299, 349;
220, 231, 319;	45, 336, 339;	210, 280, 350;
192, 256, 320.	160, 300, 340.	135, 324, 351.
260, 288, 388;	155, 372, 403;	132, 385, 407;
189, 340, 389;	80, 396, 404;	192, 360, 408;
150, 360, 390;	243, 324, 405;	120, 391, 409;
184, 345, 391.	280, 294, 406.	168, 374, 410.

280, 351, 449;	168, 425, 457;	108, 480, 492;
126, 432, 450;	120, 442, 458;	340, 357, 493;
99, 440, 451;	216, 405, 459;	190, 456, 494;
60, 448, 452.	276, 368, 460.	287, 396, 495.
168, 490, 518;	57, 540, 543;	352, 420, 548;
156, 495, 519;	256, 480, 544;	99, 540, 549;
312, 416, 520;	100, 455, 545;	154, 528, 550;
279, 440, 521.	210, 504, 546.	380, 399, 551.
215, 516, 559;	48, 575, 577;	390, 432, 582;
336, 448, 560;	272, 511, 578;	408, 495, 583;
264, 495, 561;	285, 504, 579;	384, 440, 584;
320, 462, 562.	400, 420, 580.	225, 540, 585.
354, 472, 590;	420, 441, 609;	273, 560, 623;
84, 585, 591;	110, 600, 610;	440, 576, 624;
192, 560, 592;	235, 564, 611;	175, 600, 625;
368, 465, 593.	288, 540, 612.	50, 624, 626.
150, 616, 634;	315, 572, 653;	385, 552, 673;
381, 508, 635;	360, 546, 654;	350, 576, 674;
336, 540, 636;	393, 524, 655;	189, 648, 675;
245, 588, 637.	144, 640, 656.	260, 624, 676.
52, 675, 677;	480, 504, 696;	196, 672, 700;
90, 672, 678;	153, 680, 697;	260, 651, 701;
455, 504, 679;	360, 598, 698;	270, 648, 702;
104, 672, 680.	315, 624, 699.	228, 665, 703.
480, 550, 730;	481, 600, 769;	217, 744, 775;
344, 645, 731;	462, 616, 770;	520, 576, 776;
132, 720, 732;	96, 765, 771;	252, 735, 777;
108, 725, 733.	380, 672, 772.	378, 640, 778.
324, 768, 800;	310, 744, 806;	315, 756, 819;
351, 720, 801;	207, 780, 807;	180, 800, 820;
80, 798, 802;	160, 792, 808;	421, 700, 821;
498, 605, 803.	280, 759, 809.	528, 630, 822.
540, 629, 829;	504, 672, 840;	600, 630, 870;
498, 664, 830;	580, 609, 841;	335, 804, 871;
345, 756, 831;	58, 840, 842;	480, 728, 872;
320, 768, 832.	480, 693, 843.	585, 648, 873.

345, 828, 897;	75, 936, 939;	365, 876, 949;
560, 702, 898;	564, 752, 940;	266, 912, 950;
620, 651, 899;	580, 741, 941;	225, 924, 951;
252, 864, 900.	510, 792, 942.	448, 840, 952.

These groups of four triangles are distinct and independent; no group contains a triangle found in any other group.

4. To find groups of five triangles whose hypotenuses are consecutive whole numbers less than 1000.

I have found the following 15 groups of five such triangles whose hypotenuses are less than 1000.

120, 182, 218;	136, 255, 289;	155, 372, 403;
144, 165, 219;	200, 210, 290;	80, 396, 404;
132, 176, 220;	195, 216, 291;	243, 324, 405;
21, 220, 221;	192, 220, 292;	280, 294, 406;
72, 210, 222.	68, 285, 293.	132, 385, 407.
168, 490, 518;	390, 432, 582;	385, 552, 673;
156, 495, 519;	408, 495, 583;	350, 576, 674;
312, 416, 520;	384, 440, 584;	189, 648, 675;
279, 440, 521;	225, 540, 585;	260, 624, 676;
360, 378, 522.	136, 570, 586.	52, 675, 677.
480, 504, 696;	481, 600, 769;	217, 744, 775;
153, 680, 697;	462, 616, 770;	520, 576, 776;
360, 598, 698;	96, 765, 771;	252, 735, 777;
315, 624, 699;	380, 672, 772;	378, 640, 778;
196, 672, 700.	195, 748, 773.	171, 760, 779.
483, 644, 805;	240, 782, 818;	540, 629, 829;
310, 744, 806;	315, 756, 819;	498, 664, 830;
207, 780, 807;	180, 800, 820;	345, 756, 831;
160, 792, 808;	421, 700, 821;	320, 768, 832;
280, 759, 809.	528, 630, 822.	392, 735, 833.
345, 828, 897;	75, 936, 939;	365, 876, 949;
560, 702, 898;	564, 752, 940;	266, 912, 950;
620, 651, 899;	580, 741, 941;	225, 924, 951;
252, 864, 900;	510, 792, 942;	448, 840, 952;
451, 780, 901.	207, 920, 943.	615, 728, 953.

5. To find groups of six triangles whose hypotenuses are consecutive whole numbers less than 1000.

I have found the following seven groups of six such triangles:

155, 372, 403;	385, 552, 673;	417, 556, 695;
80, 396, 404;	350, 576, 674;	480, 504, 696;
243, 324, 405;	189, 648, 675;	153, 680, 697;
280, 294, 406;	260, 624, 676;	360, 598, 698;
132, 385, 407;	52, 675, 677;	315, 624, 699;
192, 360, 408.	90, 672, 678.	196, 672, 700.
217, 744, 775;	483, 644, 805;	345, 828, 897;
520, 576, 776;	310, 744, 806;	560, 702, 898;
252, 735, 777;	207, 780, 807;	620, 651, 899;
378, 640, 778;	160, 792, 808;	252, 864, 900;
171, 760, 779;	280, 759, 809;	451, 780, 901;
396, 672, 780.	486, 648, 810.	198, 880, 902.
365, 876, 949;	225, 924, 951;	615, 728, 953;
266, 912, 950;	448, 840, 952;	504, 810, 954.

6. To find groups of seven right-angled triangles whose hypotenuses are consecutive whole numbers less than 1000.

I have found the following four groups of seven such right-angled triangles:

155, 372, 403;	385, 552, 673;
80, 396, 404;	350, 576, 674;
243, 324, 405;	189, 648, 675;
280, 294, 406;	260, 624, 676;
132, 385, 407;	52, 675, 677;
192, 360, 408;	90, 672, 678;
120, 391, 409.	455, 504, 679.
417, 556, 695;	365, 876, 949;
480, 504, 696;	266, 912, 950;
153, 680, 697;	225, 924, 951;
360, 598, 698;	448, 840, 952;
315, 624, 699;	615, 728, 953;
196, 672, 700;	504, 810, 954;
260, 651, 701.	573, 764, 955.

7. To find groups of eight right-angled triangles whose hypotenuses are consecutive whole numbers.

I have found the following seven groups of eight such triangles. The hypotenuses of the first three are less than 1000.

155, 372, 403;	385, 552, 673;	417, 556, 695;
80, 396, 404;	350, 576, 674;	480, 504, 696;

243, 324, 405;	189, 648, 675;	153, 680, 697;
280, 294, 406;	260, 624, 676;	360, 598, 698;
132, 385, 407;	52, 675, 677;	315, 624, 699;
192, 360, 408;	90, 672, 678;	196, 672, 700;
120, 391, 409;	455, 504, 679;	260, 651, 701;
168, 374, 410.	104, 672, 680.	270, 648, 702.

528, 1025, 1153;	715, 1428, 1597;	1008, 1344, 1680;
96, 1150, 1154;	752, 1410, 1598;	720, 1519, 1681;
693, 924, 1155;	351, 1560, 1599;	82, 1680, 1682;
644, 960, 1156;	448, 1536, 1600;	792, 1485, 1683;
765, 868, 1157;	80, 1599, 1601;	116, 1680, 1684;
570, 1008, 1158;	702, 1440, 1602;	164, 1677, 1685;
209, 1140, 1159;	420, 1547, 1603;	960, 1386, 1686;
800, 840, 1160.	160, 1596, 1604.	840, 1463, 1687.

1240, 1302, 1798;
224, 1785, 1799;
504, 1728, 1800;
649, 1680, 1801;
952, 1530, 1802;
720, 1653, 1803;
396, 1760, 1804;
1083, 1444, 1805.

The late B. F. Burleson gave the first and second of the foregoing groups of eight right-angled triangles whose hypotenuses are consecutive numbers in the late S. C. Gould's *Notes and Queries*, Vol. III, No. 12 (December, 1886), p. 201, and remarked: "These are the only 2 sets of 8 triangles in a series in which the hypotenuses are in regular sequence ever obtained. They were found by Charles Kriele, of Pennsylvania, when aged, infirm, and nearly blind, and sent by him to the writer of this article."

Mr Burleson gave these two groups in Harvill's *School Messenger*, Vol. II, No. 6 (June, 1885), p. 219, saying he had found "2 sets of series, 8 in each, in which the hypotenuses differ only by unity," and made no mention there of Kriele.

I have added five other groups of 8 triangles, so *seven* such groups are now known. I have also added another triangle to the first, third and fifth groups, making three groups of *nine* triangles whose hypotenuses are consecutive whole numbers, which are exhibited below:

155, 372, 403;	417, 556, 695;	715, 1428, 1597;
80, 396, 404;	480, 504, 696;	752, 1410, 1598;
243, 324, 405;	153, 680, 697;	351, 1560, 1599;
280, 294, 406;	360, 598, 698;	448, 1536, 1600;

132, 385, 407;	315, 624, 699;	80, 1599, 1601;
192, 360, 408;	196, 672, 700;	702, 1440, 1602;
120, 391, 409;	260, 651, 701;	420, 1547, 1603;
168, 374, 410;	270, 648, 702;	160, 1596, 1604;
264, 315, 411.	420, 665, 703.	963, 1284, 1605.

I have added another triangle to the last of the groups above and thus have *one* group of 10 rational right-angled triangles whose hypotenuses are consecutive whole numbers which is given below:

715, 1428, 1597;	702, 1440, 1602;
752, 1410, 1598;	420, 1547, 1603;
351, 1560, 1599;	160, 1596, 1604;
448, 1536, 1600;	963, 1284, 1605;
80, 1599, 1601;	1056, 1210, 1606.

Since what precedes was written, Mr B. O. M. De Beck of Cincinnati, Ohio, has sent me, from notes he made from Barlow's Tables more than 60 years ago, eight groups of 10 triangles whose hypotenuses are consecutive numbers; eight groups of 11 such triangles; six groups of 12 triangles; two groups of 13 triangles, and one group of 14 triangles, the hypotenuses of all the groups being between 1000 and 10000.

He also sent me later the following groups with hypotenuses between 10000 and 20000: thirteen groups of 10 triangles; nine groups of 11 triangles; ten groups of 12 triangles; three groups of 13 triangles; one group of 14 triangles, and one group of 15 triangles.

Mr De Beck also contributed one group of 15 triangles with hypotenuses from 85478 to 85492, inclusive.

Space considerations prevent me from giving here more than one of each of the different groups whose hypotenuses are between 1000 and 10000. I give also the group of 15 triangles whose hypotenuses are between 10000 and 20000.

1247, 2140, 3103;	1440, 1650, 2190;	1692, 2256, 2820;
2080, 2304, 3104;	175, 2184, 2191;	1085, 2604, 2821;
1863, 2484, 3105;	1408, 1680, 2192;	1328, 2490, 2822;
990, 2944, 3106;	1032, 1935, 2193;	1740, 2223, 2823;
1195, 2868, 3107;	1170, 1856, 2194;	1800, 2176, 2824;
1008, 2940, 3108;	1317, 1756, 2195;	1695, 2260, 2825;
1309, 2820, 3109;	396, 2160, 2196;	1530, 2376, 2826;
1866, 2488, 3110;	845, 2028, 2197;	352, 2805, 2827;
561, 3060, 3111;	1190, 1848, 2198;	560, 2772, 2828;
1512, 2720, 3112.	324, 2175, 2199;	621, 2760, 2829;
	1320, 1760, 2200.	1698, 2264, 2830;
		969, 2660, 2831.

3630, 4840, 6050;	5535, 6552, 8577;	2460, 12177, 12423;
2376, 5565, 6051;	2080, 8322, 8578;	3960, 11776, 12424;
2848, 5340, 6052;	5796, 6325, 8579;	3479, 11928, 12425;
1635, 5828, 6053;	5148, 6864, 8580;	6840, 10374, 12426;
3354, 5040, 6054;	131, 8580, 8581;	5848, 10965, 12427;
3633, 4844, 6055;	490, 8568, 8582;	4780, 11472, 12428;
3744, 4760, 6056;	5700, 6417, 8583;	8379, 9180, 12429;
3465, 4968, 6057;	3584, 7800, 8584;	1650, 12320, 12430;
3142, 4600, 6058;	5151, 6868, 8585;	1240, 12369, 12431;
3984, 4565, 6059;	4536, 7290, 8586;	4032, 11760, 12432;
3636, 4848, 6060;	3565, 7812, 8587;	4495, 11592, 12433;
4180, 4389, 6061;	1140, 8512, 8588;	6384, 10670, 12434;
2030, 5712, 6062.	2520, 8211, 8589;	8100, 9435, 12435;
	5154, 6872, 8590.	5236, 11280, 12436;
		2355, 12212, 12437.

XIII. In each of the following nine groups of three right-angled triangles, the three *hypotenuses* of each group are the sides of a right-angled triangle.

21, 72, 75;	40, 96, 104;	56, 105, 119;
60, 80, 100;	72, 135, 153;	72, 96, 120;
35, 120, 125.	57, 176, 185.	65, 156, 169.
64, 120, 136;	60, 63, 87;	60, 144, 156;
60, 252, 273;	160, 384, 416;	460, 483, 667;
55, 300, 305.	243, 324, 425.	440, 525, 685.
240, 320, 400;	480, 504, 696;	387, 516, 645;
264, 495, 561;	455, 528, 697;	560, 588, 812;
111, 680, 689.	473, 864, 985.	315, 988, 1037.

Such groups of right-angled triangles could be obtained by solving the following simultaneous equations, but it is easier to find them from the table of right-angled triangles, with the aid of Barlow's Table I, as I have done.

Let p, q, r be the legs and hypotenuse of the first triangle; s, t, u the legs and hypotenuse of the second triangle; v, w, x the legs and hypotenuse of the third triangle; then we shall have

$$p^2 + q^2 = r^2,$$
$$s^2 + t^2 = u^2,$$
$$v^2 + w^2 = x^2,$$
$$r^2 + u^2 = x^2.$$

XIV. In the following six groups of three rational right-angled triangles, the *hypotenuses* of each group form a rational scalene triangle.

5, 12, 13;	15, 36, 39;	8, 15, 17;
24, 32, 40;	9, 40, 41;	16, 63, 65;
27, 36, 45.	14, 48, 50.	48, 64, 80.
15, 112, 113;	27, 120, 123;	88, 105, 137;
24, 143, 145;	48, 140, 148;	96, 110, 146;
130, 144, 194.	21, 220, 221.	140, 225, 265.

In each of the groups below the three *hypotenuses* of each group are the sides of a rational scalene triangle whose sides are consecutive numbers.

24, 45, 51;	95, 168, 193;	360, 627, 723;
20, 48, 52;	130, 144, 194;	76, 720, 724;
28, 45, 53.	117, 156, 195.	120, 715, 725.

1776, 2035, 2701;	5040, 8733, 10083;
1330, 2352, 2702;	280, 10080, 10084;
1428, 2295, 2703.	3960, 9275, 10085.

XV. As stated on a preceding page, a number which is the product of n different prime factors of the form $4m+1$ is the hypotenuse of 2^{n-1} (incorrectly given 2^n in Barlow's *Theory of Numbers*, p. 177) different prime right-angled triangles.

See Matteson's *Diophantine Problems with Solutions* (Washington, 1888), p. 10.

Examples. 1. The least two primes of the form $4m+1$ are 5 and 13, and their product $= 65 = 8^2 + 1^2 = 7^2 + 4^2$, is the hypotenuse of each of the two prime triangles

16, 63, 65; 33, 56, 65.

There are also two other triangles, not prime triangles, with hypotenuses = 65, obtained as follows:

$$(3, \ 4, \ 5) \times 13 = 39, \ 52, \ 65;$$
$$(5, \ 12, \ 13) \times 5 = 25, \ 60, \ 65.$$

The product of 5 and $17 = 85 = 9^2 + 2^2 = 7^2 + 6^2$, and therefore 85 is the common hypotenuse of the two prime triangles

13, 84, 85; 36, 77, 85.

The other two triangles in this case are

$$(3, \ 4, \ 5) \times 17 = 51, \ 68, \ 85;$$
$$(8, \ 15, \ 17) \times 5 = 40, \ 75, \ 85.$$

The product of 13 and $17 = 221 = 14^2 + 5^2 = 11^2 + 10^2$, and 221 is the common hypotenuse of the two prime triangles

21, 220, 221; 140, 171, 221.

The other two triangles with hypotenuses = 221 are

$$(5, \ 12, \ 13) \times 17 = 85, \ 204, \ 221;$$
$$(8, \ 15, \ 17) \times 13 = 104, \ 195, \ 221.$$

And, generally,

$$(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ad+bc)^2+(ac-bd)^2.$$

There are 36 couples or pairs of prime right-angled triangles having a common hypotenuse less than 1000, 41 such pairs of triangles whose hypotenuses are between 1000 and 2000, 39 pairs whose hypotenuses are between 2000 and 3000; or, 116 pairs with hypotenuses less than 3000. See table of prime rational right-angled triangles at end of paper.

2. The product of three prime numbers of the form $4m+1$ is the common hypotenuse of $2^2=4$ prime right-angled triangles. The product of the least three primes of this form is

$$5\times 13\times 17=1105=33^2+4^2=32^2+9^2=31^2+12^2=24^2+23^2,$$

which is, therefore, the hypotenuse of each of the following four prime right-angled triangles, viz.:

47, 1104, 1105;
264, 1073, 1105;
576, 943, 1105;
744, 817, 1105.

The last two of these triangles are omitted at the end of Edward Sang's table of right-angled triangles referred to on a preceding page.

There are nine other triangles having 1105 for hypotenuse which are not prime triangles, but are multiples of prime triangles. They are given below with the method of obtaining them.

$$(3,\ 4,\ 5)\times 13\times 17=663,\ 884,\ 1105;$$
$$(5,\ 12,\ 13)\times 5\times 17=425,\ 1020,\ 1105;$$
$$(8,\ 15,\ 17)\times 5\times 13=520,\ 975,\ 1105;$$
$$(16,\ 63,\ 65)\times 17=272,\ 1071,\ 1105;$$
$$(33,\ 56,\ 65)\times 17=561,\ 952,\ 1105;$$
$$(36,\ 77,\ 85)\times 13=468,\ 1001,\ 1105;$$
$$(13,\ 84,\ 85)\times 13=169,\ 1092,\ 1105;$$
$$(21,\ 220,\ 221)\times 5=105,\ 1100,\ 1105;$$
$$(140,\ 171,\ 221)\times 5=700,\ 885,\ 1105.$$

In general,

$$\begin{aligned}(a^2+b^2)(c^2+d^2)(e^2+f^2)&=[e(ad+bc)+f(ac-bd)]^2+[e(ac-bd)-f(ad+bc)]^2,\\ &=[e(ac-bd)+f(ad+bc)]^2+[e(ad+bc)-f(ac-bd)]^2,\\ &=[e(ad-bc)+f(ac+bd)]^2+[e(ac+bd)-f(ad-bc)]^2,\\ &=[e(ac+bd)+f(ad-bc)]^2+[e(ad-bc)-f(ac+bd)]^2.\end{aligned}$$

Take $a=2,\ b=1;\ c=3,\ d=2;\ e=4,\ f=1;$ then

$$5\times 13\times 17=1105=32^2+9^2=24^2+23^2=31^2+12^2=33^2+4^2,$$

as found before.

$$5\times 13\times 29=1885=43^2+6^2=42^2+11^2=38^2+21^2=34^2+27^2;$$

hence 1885 is the common hypotenuse of the four prime right-angled triangles

427, 1836, 1885;
516, 1813, 1885;
924, 1643, 1885;
1003, 1596, 1885.

There are also nine other right-angled triangles having 1885 for a common hypotenuse that are not prime triangles, which are given below:

221, 1872, 1885;	675, 1760, 1885;	957, 1624, 1885;
312, 1859, 1885;	725, 1740, 1885;	1131, 1508, 1885;
464, 1787, 1885;	760, 1725, 1885;	1300, 1365, 1885.

There are only two groups of four prime rational right-angled triangles having a common hypotenuse less than 2000, and only three groups of such triangles with hypotenuses between 2000 and 3000, viz.:

$$5 \times 13 \times 37 = 2405, \; 5 \times 17 \times 29 = 2465, \; 5 \times 13 \times 41 = 2665;$$

$$2405 = 49^2 + 2^2 = 47^2 + 14^2 = 46^2 + 17^2 = 38^2 + 31^2;$$

$$2465 = 49^2 + 8^2 = 47^2 + 16^2 = 44^2 + 23^2 = 41^2 + 28^2;$$

$$2665 = 51^2 + 8^2 = 48^2 + 19^2 = 44^2 + 27^2 = 37^2 + 36^2.$$

Hence the three groups of 4 prime triangles are

196, 2397, 2405;	784, 2337, 2465;	73, 2664, 2665;
483, 2356, 2405;	897, 2296, 2465;	816, 2537, 2665;
1316, 2013, 2405;	1407, 2024, 2465;	1207, 2376, 2665;
1564, 1827, 2405.	1504, 1953, 2465.	1824, 1943, 2665.

The following curious theorem is given without proof in Matteson's *Diophantine Problems with Solutions*, p. 11.

If a number N is the product of 1, 2, 3, 4, 5, ... or n different prime factors of the form $4m+1$, the terms of the following series,

$$1_1, \; 4_2, \; 13_3, \; 40_4, \; 121_5, \; 364_6, \; 1093_7, \; 3280_8, \text{ etc.},$$

will represent the number of ways N^2 can be the sum of two squares, where the subscripts, 1, 2, 3, 4, ... n, denote the number of prime factors composing N. Any term of this series is equal to three times the preceding term, + 1. Thus the square of a prime number of the form $4m+1$ is the sum of two squares in *one* way only; the square of a number which is the product of two different primes of the form $4m+1$ is the sum of two squares in $1 \times 3 + 1 = 4$ different ways; the square of a number which is the product of three different primes of the form $4m+1$ is the sum of two squares in $4 \times 3 + 1 = 13$ different ways; the square of a number which is the product of four different primes of the form $4m+1$ is the sum of two squares in $13 \times 3 + 1 = 40$ different ways, etc.

If we put u_n for the nth term in the above series we have

$$u_{n+1} = 3u_n + 1.$$

The solution of this Finite-Difference equation gives

$$u_n = \tfrac{1}{2}(3^n - 1).$$

See Boole's *Calculus of Finite Differences*, second edition (London, 1872), p. 165.

Hence if a number is the product of n different prime numbers of the form $4m+1$, it is the common hypotenuse of $\tfrac{1}{2}(3^n - 1)$ different rational integral-sided right-angled triangles, 2^{n-1} of which are *prime* triangles and $\tfrac{1}{2}(3^n - 2^n - 1)$ are *not* prime triangles.

Examples. 1. If $n = 1$, then $\tfrac{1}{2}(3^1 - 1) = 1$, and a prime number of the form $4m + 1$ can be the hypotenuse of but *one* rational right-angled triangle.

2. If $n = 2$, then $\tfrac{1}{2}(3^2 - 1) = 4$, and any number which is the product of two different primes of the form $4m + 1$ is the common hypotenuse of four different rational right-angled triangles.

3. If $n = 3$, then $\tfrac{1}{2}(3^3 - 1) = 13$, and any number which is the product of three different primes of the form $4m + 1$ is the common hypotenuse of 13 different rational right-angled triangles.

4. If $n = 4$, then $\tfrac{1}{2}(3^4 - 1) = 40$, and any number which is the product of four different primes of the form $4m + 1$ is the common hypotenuse of 40 different rational right-angled triangles.

5. If $n = 5$, then $\tfrac{1}{2}(3^5 - 1) = 121$, and any number which is the product of five different primes of the form $4m+1$ is the common hypotenuse of 121 different rational right-angled triangles.

And so on, to any value of n.

Find all the rational right-angled triangles whose hypotenuses are 32045.

Solution.

$$32045 = 5 \times 13 \times 17 \times 29$$
$$= 179^2 + 2^2 = 178^2 + 19^2 = 173^2 + 46^2 = 166^2 + 67^2$$
$$= 163^2 + 74^2 = 157^2 + 86^2 = 144^2 + 109^2 = 131^2 + 122^2.$$

Hence the eight *prime* right-angled triangles whose hypotenuses are 32045 are as follows:

716, 32037, 32045;	15916, 27813, 32045;
2277, 31964, 32045;	17253, 27004, 32045;
6764, 31323, 32045;	21093, 24124, 32045;
8283, 30956, 32045;	22244, 23067, 32045.

The other thirty-two triangles are found as follows:

$$(3,\ 4,\ 5) \times 13 \times 17 \times 29 = 19227,\ 25636,\ 32045;$$
$$(5,\ 12,\ 13) \times 5 \times 17 \times 29 = 12325,\ 29580,\ 32045;$$
$$(8,\ 15,\ 17) \times 5 \times 13 \times 29 = 15080,\ 28275,\ 32045;$$
$$(20,\ 21,\ 29) \times 5 \times 13 \times 17 = 22100,\ 23205,\ 32045;$$
$$(16,\ 63,\ 65) \times 17 \times 29 = 7888,\ 31059,\ 32045;$$
$$(33,\ 56,\ 65) \times 17 \times 29 = 16269,\ 27608,\ 32045;$$
$$(13,\ 84,\ 85) \times 13 \times 29 = 4901,\ 31668,\ 32045;$$

$$(36,\ 77,\ 85) \times 13 \times 29 = 13572,\ 29029,\ 32045;$$
$$(17,\ 144,\ 145) \times 13 \times 17 = 3757,\ 31824,\ 32045;$$
$$(24,\ 143,\ 145) \times 13 \times 17 = 5304,\ 31603,\ 32045;$$
$$(21,\ 220,\ 221) \times 5 \times 29 = 3045,\ 31900,\ 32045;$$
$$(140,\ 171,\ 221) \times 5 \times 29 = 20300,\ 24795,\ 32045;$$
$$(135,\ 352,\ 377) \times 5 \times 17 = 11475,\ 29920,\ 32045;$$
$$(152,\ 345,\ 377) \times 5 \times 17 = 12920,\ 29325,\ 32045;$$
$$(132,\ 425,\ 493) \times 5 \times 13 = 8580,\ 30875,\ 32045;$$
$$(155,\ 468,\ 493) \times 5 \times 13 = 10075,\ 30420,\ 32045;$$
$$(47,\ 1104,\ 1105) \times 29 = 1363,\ 32016,\ 32045;$$
$$(264,\ 1073,\ 1105) \times 29 = 7656,\ 31117,\ 32045;$$
$$(576,\ 943,\ 1105) \times 29 = 16704,\ 27347,\ 32045;$$
$$(744,\ 817,\ 1105) \times 29 = 21576,\ 23693,\ 32045;$$
$$(427,\ 1836,\ 1885) \times 17 = 7259,\ 31212,\ 32045;$$
$$(516,\ 1813,\ 1885) \times 17 = 8772,\ 30821,\ 32045;$$
$$(924,\ 1643,\ 1885) \times 17 = 15708,\ 27931,\ 32045;$$
$$(1003,\ 1596,\ 1885) \times 17 = 17051,\ 27132,\ 32045;$$
$$(784,\ 2337,\ 2465) \times 13 = 10192,\ 30381,\ 32045;$$
$$(897,\ 2296,\ 2465) \times 13 = 11661,\ 29848,\ 32045;$$
$$(1407,\ 2024,\ 2465) \times 13 = 18291,\ 26312,\ 32045;$$
$$(1504,\ 1953,\ 2465) \times 13 = 19552,\ 25389,\ 32045;$$
$$(480,\ 6391,\ 6409) \times 5 = 2400,\ 31955,\ 32045;$$
$$(791,\ 6360,\ 6409) \times 5 = 3955,\ 31800,\ 32045;$$
$$(3959,\ 5040,\ 6409) \times 5 = 19795,\ 25200,\ 32045;$$
$$(4200,\ 4841,\ 6409) \times 5 = 21000,\ 24205,\ 32045.$$

See Matteson's *Diophantine Problems with Solutions*, p. 10; also, *Miscellaneous Notes and Queries*, Vol. x, No. 3 (Sept., 1892), p. 225.

XVI. As stated elsewhere, a prime number of the form $4m+1$ is the sum of two squares in one way only, and therefore is the hypotenuse of only one rational right-angled triangle; but the *square* of such a prime is the sum of two squares in two ways, and therefore is the common hypotenuse of two different right-angled triangles; the *cube* of such a prime is the sum of two squares in three ways, and so is the common hypotenuse of three different right-angled triangles; and, generally, the *n*th *power* of a prime number of the form $4m+1$ is the sum of two squares in n ways, and is the common hypotenuse of n right-angled triangles, but only *one* of the triangles is prime.

Examples. 1. $$5^2 = 3^2 + 4^2,$$

and there is *one* triangle whose hypotenuse is 5.

$$25^2 = 7^2 + 24^2 = 15^2 + 20^2,$$

and 25 is the common hypotenuse of the two triangles

7, 24, 25;

15, 20, 25.

$$125^2 = (5^3)^2 = 44^2 + 117^2 = 35^2 + 120^2 = 75^2 + 100^2,$$

and we have the three triangles

44, 117, 125;

35, 120, 125;

75, 100, 125.

$$(5^4)^2 = 625^2 = 336^2 + 527^2 = 220^2 + 585^2 = 175^2 + 600^2 = 375^2 + 500^2;$$

hence we have the four triangles

175, 600, 625;

220, 585, 625;

336, 527, 625;

375, 500, 625.

2. The next prime of the form $4m + 1$ being 13, we have

$$13^2 = 5^2 + 12^2,$$

and there is only *one* triangle whose hypotenuse is 13.

$$169^2 = 119^2 + 120^2 = 65^2 + 156^2,$$

and the two triangles with hypotenuse 169 are

65, 156, 169;

119, 120, 169.

$$13^3 = 2197, \text{ and } 2197^2 = 828^2 + 2035^2 = 1547^2 + 1560^2 = 845^2 + 2028^2;$$

therefore the three triangles having hypotenuse = 2197 are

828, 2035, 2197;

845, 2028, 2197;

1547, 1560, 2197.

$13^4 = 28561$ is beyond the limit of the table of right-angled triangles given at the end of this paper, but the triangles can be computed by the aid of Barlow's Tables, before-mentioned, Table 1, p. 3. I find

$$28561^2 = 239^2 + 28560^2 = 10764^2 + 26455^2 = 20111^2 + 20280^2 = 10895^2 + 26364^2,$$

and therefore the four triangles are

239, 28560, 28561;

10764, 26455, 28561;

10895, 26364, 28561;

20111, 20280, 28561.

Table of Prime Rational Right-Angled Triangles whose Hypotenuses are less than 3000.

3	4	5	152	345	377	39	760	761	423	1064	1145
5	12	13	189	340	389	481	600	769	704	903	1145
8	15	17	228	325	397	195	748	773	528	1025	1153
7	24	25	40	399	401	56	783	785	68	1155	1157
20	21	29	120	391	409	273	736	785	765	868	1157
12	35	37	29	420	421	168	775	793	204	1147	1165
9	40	41	87	416	425	432	665	793	517	1044	1165
28	45	53	297	304	425	555	572	797	340	1131	1181
11	60	61	145	408	433	280	759	809	611	1020	1189
16	63	65	84	437	445	429	700	821	660	989	1189
33	56	65	203	396	445	540	629	829	832	855	1193
48	55	73	280	351	449	41	840	841	49	1200	1201
13	84	85	168	425	457	116	837	845	147	1196	1205
36	77	85	261	380	461	123	836	845	476	1107	1205
39	80	89	31	480	481	205	828	853	245	1188	1213
65	72	97	319	360	481	232	825	857	705	992	1217
20	99	101	44	483	485	287	816	865	140	1221	1229
60	91	109	93	476	485	504	703	865	612	1075	1237
15	112	113	132	475	493	348	805	877	280	1209	1241
44	117	125	155	468	493	369	800	881	441	1160	1241
88	105	137	217	456	505	60	899	901	799	960	1249
17	144	145	336	377	505	451	780	901	420	1189	1261
24	143	145	220	459	509	464	777	905	539	1140	1261
51	140	149	279	440	521	616	663	905	748	1035	1277
85	132	157	92	525	533	43	924	925	637	1116	1285
119	120	169	308	435	533	533	756	925	893	924	1285
52	165	173	341	420	541	129	920	929	560	1161	1289
19	180	181	33	544	545	215	912	937	72	1295	1297
57	176	185	184	513	545	580	741	941	51	1300	1301
104	153	185	165	532	557	301	900	949	255	1288	1313
95	168	193	276	493	565	420	851	949	735	1088	1313
28	195	197	396	403	565	615	728	953	360	1271	1321
84	187	205	231	520	569	124	957	965	357	1276	1325
133	156	205	48	575	577	387	884	965	884	987	1325
21	220	221	368	465	593	248	945	977	504	1247	1345
140	171	221	240	551	601	473	864	985	833	1056	1345
60	221	229	35	612	613	696	697	985	561	1240	1361
105	208	233	105	608	617	372	925	997	840	1081	1369
120	209	241	336	527	625	559	840	1009	148	1365	1373
32	255	257	100	621	629	45	1012	1013	931	1020	1381
23	264	265	429	460	629	660	779	1021	296	1353	1385
96	247	265	200	609	641	64	1023	1025	663	1216	1385
69	260	269	315	572	653	496	897	1025	53	1404	1405
115	252	277	300	589	661	192	1015	1033	444	1333	1405
160	231	281	385	552	673	315	988	1037	159	1400	1409
161	240	289	52	675	677	645	812	1037	265	1392	1417
68	285	293	37	684	685	320	999	1049	792	1175	1417
136	273	305	156	667	685	620	861	1061	371	1380	1429
207	224	305	111	680	689	731	780	1069	592	1305	1433
25	312	313	400	561	689	448	975	1073	76	1443	1445
75	308	317	185	672	697	495	952	1073	477	1364	1445
36	323	325	455	528	697	132	1085	1093	228	1435	1453
204	253	325	260	651	701	585	928	1097	583	1344	1465
175	288	337	259	660	709	47	1104	1105	936	1127	1465
180	299	349	333	644	725	264	1073	1105	380	1419	1469
225	272	353	364	627	725	576	943	1105	740	1269	1469
27	364	365	108	725	733	744	817	1105	969	1120	1481
76	357	365	216	713	745	141	1100	1109	689	1320	1489
252	275	373	407	624	745	235	1092	1117	532	1395	1493
135	352	377	468	595	757	329	1080	1129	55	1512	1513

Table of Prime Rational Right-Angled Triangles whose Hypotenuses are less than 3000 (*cont.*)

888	1225	1513	1003	1596	1885	469	2220	2269	1065	2408	2633
165	1508	1517	1311	1360	1889	752	2145	2273	1568	2145	2657
795	1292	1517	549	1820	1901	1440	1769	2281	73	2664	2665
156	1517	1525	688	1785	1913	603	2204	2285	816	2537	2665
684	1363	1525	671	1800	1921	1196	1947	2285	1207	2376	2665
312	1505	1537	1121	1560	1921	1235	1932	2293	1824	1943	2665
385	1488	1537	1092	1595	1933	1575	1672	2297	219	2660	2669
901	1260	1549	88	1935	1937	96	2303	2305	1300	2331	2669
495	1472	1553	1312	1425	1937	737	2184	2305	365	2652	2677
836	1323	1565	264	1927	1945	940	2109	2309	511	2640	2689
1036	1173	1565	793	1776	1945	480	2279	2329	1725	2068	2693
624	1457	1585	860	1749	1949	871	2160	2329	1020	2501	2701
1007	1224	1585	440	1911	1961	1365	1892	2333	1349	2340	2701
715	1428	1597	1239	1520	1961	1380	1891	2341	104	2703	2705
80	1599	1601	915	1748	1973	672	2255	2353	657	2624	2705
240	1591	1609	63	1984	1985	1128	2065	2353	312	2695	2713
988	1275	1613	616	1887	1985	1005	2132	2357	803	2604	2725
780	1421	1621	1032	1705	1993	1495	1848	2377	1764	2077	2725
57	1624	1625	315	1972	1997	69	2380	2381	520	2679	2729
1113	1184	1625	1037	1716	2005	1139	2100	2389	1491	2300	2741
285	1612	1637	1357	1476	2005	345	2368	2393	949	2580	2749
399	1600	1649	792	1855	2017	196	2397	2405	728	2655	2753
560	1551	1649	180	2021	2029	483	2356	2405	1095	2552	2777
935	1368	1657	360	2009	2041	1316	2013	2405	936	2623	2785
1140	1219	1669	1159	1680	2041	1564	1827	2405	1633	2256	2785
720	1519	1681	693	1924	2045	392	2385	2417	1700	2211	2789
164	1677	1685	1204	1653	2045	1056	2183	2425	1428	2405	2797
627	1564	1685	1428	1475	2053	1273	2064	2425	1960	2001	2801
1045	1332	1693	819	1900	2069	588	2365	2437	1241	2520	2809
328	1665	1697	1281	1640	2081	759	2320	2441	75	2812	2813
741	1540	1709	720	1961	2089	784	2337	2465	212	2805	2813
492	1645	1717	1144	1767	2105	897	2296	2465	424	2793	2825
1092	1325	1717	1376	1593	2105	1407	2024	2465	1144	2583	2825
880	1749	1721	65	2112	2113	1504	1953	2465	1775	2208	2833
1155	1292	1733	92	2115	2117	1248	2135	2473	525	2788	2837
59	1740	1741	195	2188	2117	1748	1755	2477	636	2773	2845
177	1736	1745	276	2107	2125	100	2499	2501	1387	2484	2845
656	1617	1745	1403	1596	2125	980	2301	2501	1632	2345	2857
295	1728	1753	1071	1840	2129	300	2491	2509	1900	2139	2861
84	1763	1765	455	2088	2137	1541	1980	2509	825	2752	2873
413	1716	1765	460	2091	2141	71	2520	2521	848	2745	2873
969	1480	1769	585	2072	2153	213	2516	2525	1533	2444	2885
1040	1431	1769	1320	1711	2161	1173	2236	2525	1917	2156	2885
1248	1265	1777	644	2067	2165	355	2508	2533	975	2728	2897
531	1700	1781	1197	1804	2165	1692	1885	2533	1060	2709	2909
820	1581	1781	715	2052	2173	497	2496	2545	108	2915	2917
420	1739	1789	1525	1548	2173	1176	2257	2545	1560	2479	2929
649	1680	1801	828	2035	2197	700	2451	2549	1679	2400	2929
767	1656	1825	188	2205	2213	1675	1932	2557	540	2891	2941
984	1537	1825	1260	1829	2221	639	2480	2561	2059	2100	2941
172	1845	1853	376	2193	2225	1311	2200	2561	1272	2665	2953
885	1628	1853	1496	1647	2225	781	2460	2581	1275	2668	2957
61	1860	1861	1012	1995	2237	900	2419	2581	77	2964	2965
183	1856	1865	67	2244	2245	1632	2015	2593	756	2867	2965
344	1833	1865	564	2173	2245	204	2597	2605	231	2960	2969
305	1848	1873	201	2240	2249	923	2436	2605	385	2952	2977
1148	1485	1877	1449	1720	2249	1809	1880	2609	1825	2352	2977
427	1836	1885	335	2232	2257	408	2585	2617	1425	2632	2993
516	1813	1885	1105	1968	2257	1100	2379	2621	1768	2415	2993
924	1643	1885									