Pisano period and permutations of $n \times n$ matrices

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Abstract

Repeated application of a particular permutation to an $n \times n$ matrix results in the original matrix. The number of iterations $I(n)$ for $n = 1, 2, \ldots$ is the same as the length of the period of the periodic sequence that results from the Fibonacci sequence modulus n known as the Pisano period.

Introduction

The well-known Fibonacci sequence of numbers, $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ shows periodicity modulus n [1]. For example, the Fibonacci sequence modulus 2 is $0, 1, 1, 0, 1, 1, \ldots$ which has the repeating pattern of length 3 $\{1, 1, 0\}$. Modulus 3 the pattern is of length 8, $\{0, 1, 1, 2, 0, 2, 2, 1, \ldots\}$. The length of the period modulus n, is called the Pisano period [2] after Fibonacci's real name Leonardo Pisano [3]. The Pisano sequence is the sequence of Pisano periods for $n = 1, 2, \ldots$. Table 1 gives the values of the Pisano sequence for $n = 1, 2, \ldots, 100$ [4].

Table 1. Pisano period values for $n = 1, 2, \ldots, 100$

1	3	8	6	20	24	16	12	24	60
10	24	28	48	40	24	36	24	18	60
16	30	48	24	100	84	72	48	14	120
30	48	40	36	80	24	76	18	56	60
40	48	88	30	120	48	32	24	112	300
72	84	108	72	20	48	72	42	58	120
60	30	48	96	140	120	136	36	48	240
70	24	148	228	200	18	80	168	78	120
216	120	168	48	180	264	56	60	44	120
112	48	120	96	180	48	196	336	120	300

Permutations of $n \times n$ matrices

There is a procedure for reordering the elements of an $n \times n$ matrix A which in this paper is called the diagrow procedure because it involves turning diagonals into rows. Start by taking two copies of A called $A1$ and $A2$ side by side as shown in Figure 1. The reordered matrix is formed from the elements along the diagonals $\{A1_{1,1}$ to $A1_{n,n}\}, \{A1_{1,2}$ to $A2_{n,1}\} \ldots \{A1_{1,n}$ to $A2_{n,n-1}\}$. In Figure 1, the reordered matrix is shown below the two copies of the original. If the process is repeated on the resultant reordered matrix, eventually the original matrix is obtained. The number iterations for $n = 1, 2, 3 \ldots$ required to get back to the original ordering is $1, 3, 8, 6, 20, 24, 16, \ldots$, respectively. This is exactly the same sequence as the Pisano sequence. Figures 2 and 3 illustrate the permutation cycles for 3×3 and 4×4 matrices.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 6 & 11 & 16 \\ 2 & 7 & 12 & 13 \\ 3 & 8 & 9 & 14 \\ 4 & 5 & 10 & 15 \end{vmatrix}$$

Figure 1. Example of a 4×4 matrix reordering

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow{1} \begin{vmatrix} 1 & 5 & 9 \\ 2 & 6 & 7 \\ 3 & 4 & 8 \end{vmatrix} \xrightarrow{2} \begin{vmatrix} 1 & 6 & 8 \\ 5 & 7 & 3 \\ 9 & 2 & 4 \end{vmatrix}$$

$$\xrightarrow{3} \begin{vmatrix} 1 & 7 & 4 \\ 6 & 3 & 9 \\ 8 & 5 & 2 \end{vmatrix} \xrightarrow{4} \begin{vmatrix} 1 & 3 & 2 \\ 7 & 9 & 8 \\ 4 & 6 & 5 \end{vmatrix} \xrightarrow{5} \begin{vmatrix} 1 & 9 & 5 \\ 3 & 8 & 4 \\ 2 & 7 & 6 \end{vmatrix}$$

$$\xrightarrow{6} \begin{vmatrix} 1 & 8 & 6 \\ 9 & 4 & 2 \\ 5 & 3 & 7 \end{vmatrix} \xrightarrow{7} \begin{vmatrix} 1 & 4 & 7 \\ 8 & 2 & 5 \\ 6 & 9 & 3 \end{vmatrix} \xrightarrow{8} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Figure 2. Example of 3×3 matrix permutations

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \xrightarrow{1} \begin{vmatrix} 1 & 6 & 11 & 16 \\ 2 & 7 & 12 & 13 \\ 3 & 8 & 9 & 14 \\ 4 & 5 & 10 & 15 \end{vmatrix} \xrightarrow{2} \begin{vmatrix} 1 & 7 & 9 & 15 \\ 6 & 12 & 14 & 4 \\ 11 & 13 & 3 & 5 \\ 16 & 2 & 8 & 10 \end{vmatrix}$$

$$\xrightarrow{3} \begin{vmatrix} 1 & 12 & 3 & 10 \\ 7 & 14 & 5 & 16 \\ 9 & 4 & 11 & 2 \\ 15 & 6 & 13 & 8 \end{vmatrix} \xrightarrow{4} \begin{vmatrix} 1 & 14 & 11 & 8 \\ 12 & 5 & 2 & 15 \\ 3 & 16 & 9 & 6 \\ 10 & 7 & 4 & 13 \end{vmatrix}$$

$$\xrightarrow{5} \begin{vmatrix} 1 & 5 & 9 & 13 \\ 14 & 2 & 6 & 10 \\ 11 & 15 & 3 & 7 \\ 8 & 12 & 16 & 4 \end{vmatrix} \xrightarrow{6} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$$

Figure 3. Example of 4×4 matrix permutations

Why is this so?

Consider referencing the rows and columns of a matrix starting with row 0 and column 0 as is common with some programming languages. Let $\mathbb{N}_0$ be the set of non-negative integers and consider a mapping f from $\mathbb{N}_0^2$ to $\mathbb{N}_0^2$. In general the diagrow procedure can be characterised as the mapping f on the indices (i, j), $i, j \in \{0, 1, \ldots, n-1\}$, of an $n \times n$ matrix where:

$$f(i, j) = (k, i) \quad \text{and} \quad k = (j - i) \bmod (n).$$

In relation to the diagrow procedure this means that the element indexed by (i, j) is relocated to the index (k, i).

For example, the application of the diagrow procedure to each index of a 3×3 matrix results in:

$$\begin{array}{lll} (0,0) \to (0,0) & (0,1) \to (1,0) & (0,2) \to (2,0) \\ (1,0) \to (2,1) & (1,1) \to (0,1) & (1,2) \to (1,1) \\ (2,0) \to (1,2) & (2,1) \to (2,2) & (2,2) \to (0,2). \end{array}$$

In the 4×4 matrix case, the diagrow procedure results in the following translations:

$$\begin{array}{llll} (0,0) \to (0,0) & (0,1) \to (1,0) & (0,2) \to (2,0) & (0,3) \to (3,0) \\ (1,0) \to (3,1) & (1,1) \to (0,1) & (1,2) \to (1,1) & (1,3) \to (2,1) \\ (2,0) \to (2,2) & (2,1) \to (3,2) & (2,2) \to (0,2) & (2,3) \to (1,2) \\ (3,0) \to (1,3) & (3,1) \to (2,3) & (3,2) \to (3,3) & (3,3) \to (0,3). \end{array}$$

The Fibonacci sequence is generated by successive addition of the previous two terms. The translation of indices using the diagrow procedure is performed by the successive difference modulo n of the previous row and column numbers. Given any two numbers in order from the Fibonacci sequence all the smaller Fibonacci numbers can be generated by a similar process of successive differences. More clearly, the inverse of the diagrow mapping is

$$f^{-1}(i, j) = (j, k) \quad \text{where } k = (i + j) \bmod (n)$$

which uses the same generating process as the Fibonacci sequence.

The generalised Fibonacci sequence has the same recursive formula as the Fibonacci sequence $G(a, b) = (b, a + b)$ but with different starting values. With $a = 0$ and $b = 1$ or $a = 1$ and $b = 0$, the usual Fibonacci sequence is generated, starting with 0 with the former and 1 with the latter initial values. The generalised Fibonacci sequence is shown to be related to the Fibonacci sequence as follows: Start with

$$a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, 5a + 8b, \ldots$$

and separate terms into two groups with terms in each group having the same factor;

$$a, 0a, 1a, 1a, 2a, 3a, 5a, \ldots + 0b, 1b, 1b, 2b, 3b, 5b, 8b, \ldots$$

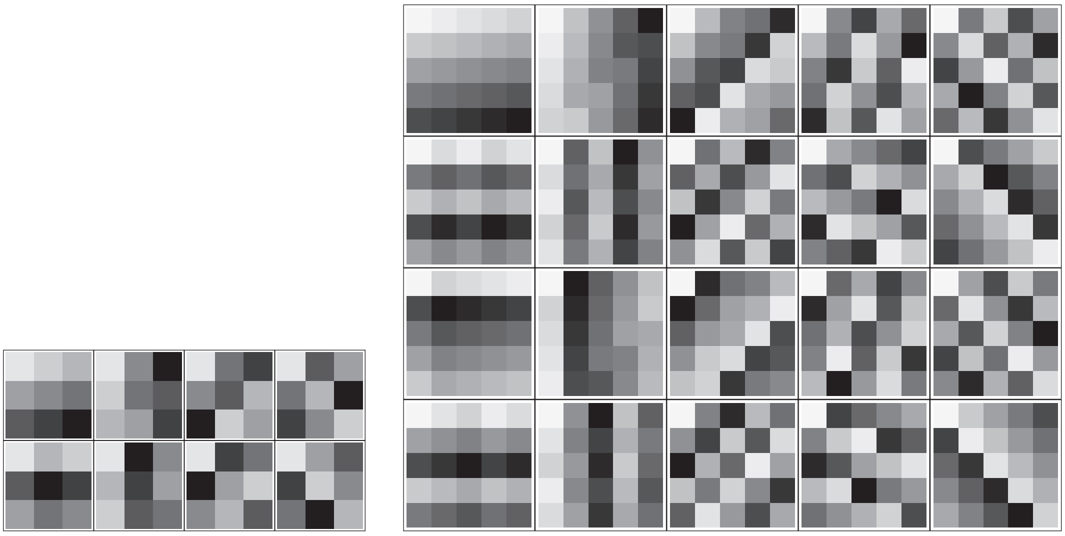
then factor the common factors from each group

$$a(1, 0, 1, 1, 2, 3, 5, \ldots) + b(0, 1, 1, 2, 3, 5, 8, \ldots).$$

The first sequence is a times the Fibonacci sequence starting with $1, 0$ and the second sequence is b times the Fibonacci sequence starting with $0, 1$. Clearly, the lengths of the periods modulus n of the first sequence will be the same as the lengths of the periods modulus n of the second sequence. Both of these lengths are the Pisano sequence. The lengths of the periods modulus n of the sum of the two sequences will therefore also be the same as the Pisano sequence. It is clear that the Pisano sequence will also apply to the generalised Fibonacci sequence. Since the diagrow procedure follows the same generating process as the generalised Fibonacci sequence, except in reverse, the repeated diagrow procedure applied to an $n \times n$ matrix has a period with the same length as the length of the period modulus n of the Fibonacci sequence, that is the Pisano period.

Extra bits

An $n \times n$ matrix can be represented graphically in the manner of a chessboard with each square filled in with a shade of grey ranging from white for the lowest number to black for the highest number. The elements of the matrices used in the graphical representations shown in Figures 4, 5, 6 and 7 have the same numbering convention as shown in the examples of Figures 2 and 3 except that each number has been divided by n^2 so that the smallest number is $1/n^2$ and the largest number is 1. This is most clearly seen by comparing Figure 2 with Figure 4(a).



(a) 8 permutations of a 3×3 matrix

(b) 20 permutations of a 5×5 matrix

Figure 4. Graphical representation of the permutations resulting from a diagrow reordering of a 3×3 matrix Figure 4(a) and a 5×5 matrix Figure 4(b). Each matrix has elements with values $1/n^2, 2/n^2, \ldots, 1$ represented by squares filled with shades of grey.

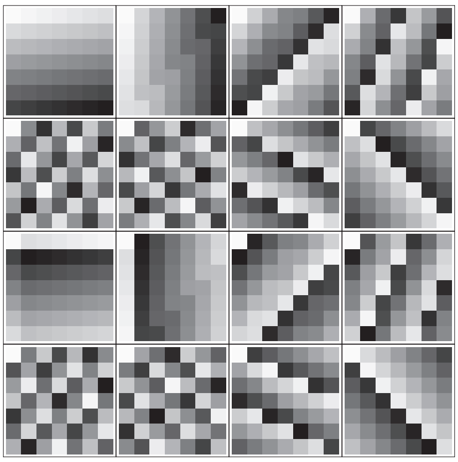


Figure 5. Permutations resulting from a diagrow reordering of a 7×7 matrix. Each matrix has elements with values $1/7^2, 2/7^2, \ldots, 1$ represented by squares filled with shades of grey.

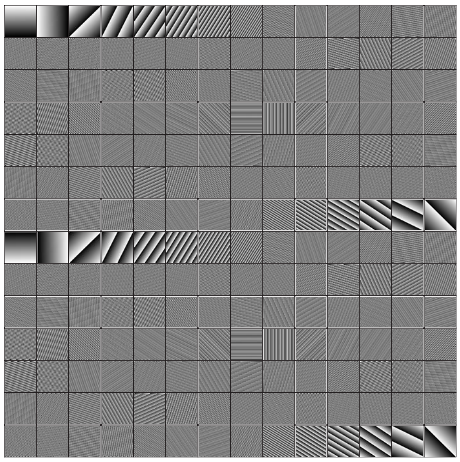


Figure 6. The 196 permutations resulting from a diagrow reordering of a 97×97 matrix with elements $1/97^2, 2/97^2, \ldots, 1$ represented by squares filled with shades of grey.

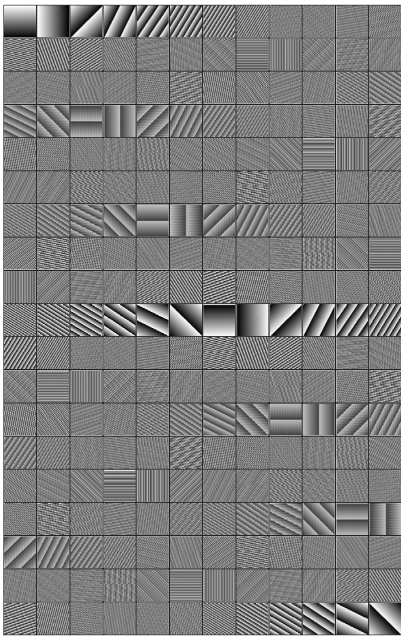


Figure 7. The 228 permutations resulting from a diagrow reordering of a 74×74 matrix with elements $1/74^2, 2/74^2, \ldots, 1$ represented by squares filled with shades of grey.

Interesting patterns appear in these graphical representations especially for larger values of n as seen in Figures 6 and 7. Some representations have the appearance of woven fabric. Diagrow permutations of the $n \times n$ matrices with $n = 81, 82, \ldots, 160$ shown as simultaneous animations can be found at [5].

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