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PERMUTATION STATISTICS ON MULTISETS

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CONTENTS

NOTATION

The following notation is used in this dissertation.

$\mathbf{m}$	the multiset of the form $1^{k_1} \cdots m^{k_m}$
$\cup$	the union of sets
$\cap$	the intersection of sets
$\uplus$	the union (merge) of multisets
$\lvert A \rvert$	the cardinality of set A
$\mathbf{P}_<$	the partially ordered set with relation $<$
$\sum$	the sum
$\prod$	the product
$\binom{n}{k}$	the binomial coefficient, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
$\binom{n}{k_1,\ldots,k_p}$	the multinomial coefficient, $\binom{n}{k_1,\ldots,k_p} = \frac{n!}{k_1!\cdots k_p!}$
$\mathbb{Z}$	the set of integer numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\zeta(s)$	the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$
B_k	the k-th Bernoulli number
∇f	the difference operator, $\nabla f(x) = f(x) - f(x-1)$
$a \equiv b \pmod{p}$	the congruence, $a - b$ is divisible by p
$\gcd$	the greatest common divisor
H_n	the n-th Harmonic number, $H_n = \sum_{i=1}^{n} \frac{1}{i}$
$c(n,k)$	the Stirling number of the first kind
$S(n,k)$	the Stirling number of the second kind

INTRODUCTION

Problems in computer science, algorithms analysis and many effective computational techniques operate with discrete combinatorial structures. For example, sets, graphs, finite sequences, generating functions and many others. Combinatorial analysis studies properties of these objects which is a wide and active area of research. Its applications arise in computer science, chemistry, biology, physics and other fields of science.

Many well-known properties of discrete objects that based on sets can naturally be generalized to *multisets*. Definition of sets referred to *multi* means that we allow repetitions of its elements.

This thesis presents an exploration of several computational, combinatorial, algebraic and number-theoretic properties which emerge in problems with multiset basis. All the results are of computational spirit and applicable from the aspect of computer science (e.g., in the analysis of algorithms, as discussed below).

Motivation of research

Let us consider one classical example of enumerative combinatorics. The number of ways to choose k objects from a set of n elements is equal to

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The same problem which allows repetitions of elements gives the answer

$$\binom{n+k-1}{k} = (-1)^k \binom{-n}{k}.$$

We see that changes to multisets derive an interesting result (very deep, in fact), known as *combinatorial reciprocity* [47, 48].

If we consider Worpitzky's identity [25, 50] (or, equivalently, the corresponding generating function) as the central object, observation gives that:

(1) The classical case [25, 50]:

$$n^m = \sum_i a_{m,i} \binom{n+i}{m}$$

relates the power function n^m with some integers $a_{m,i}$. The numbers $a_{m,i}$ are called *Eulerian* and they count the number of permutations $(p_1, \dots, p_m)$ of $(1, \dots, m)$ having i descent ('falling') positions $p_k > p_{k+1}$ $(1 \le k < m)$ or $k = m$.

(2) The generalized case for all permutations on multisets [14]:

$$\binom{n+k_1-1}{k_1} \cdots \binom{n+k_m-1}{k_m} = \sum_i a_{\mathbf{m},i} \binom{n+k_1+\cdots+k_m-i}{k_1+\cdots+k_m}$$

relates the product of binomial coefficients with generalized Eulerian numbers $a_{\mathbf{m},i}$. Notice that multiset $\mathbf{m} = \{1^{k_1}, \dots, m^{k_m}\}$ leads to a changes in the basis of algebraic statement. Namely, the power function is changed by the product of binomial coefficients:

$$n^m \to \binom{n}{k_1} \cdots \binom{n}{k_m}.$$

As the sum of powers of consecutive integers has many well-known properties, this case naturally moves us to study the *power sum of binomial coefficients:*

$$\sum_{n=1}^{N} n^m \to \sum_{n=1}^{N} \binom{n}{k_1} \cdots \binom{n}{k_m} \to \sum_{n=1}^{N} \binom{n}{k}^m.$$

For instance, we generalize a classical result of Faulhaber's theorem [21, 35].

We also introduce *Stirling numbers on multisets* which related to the product of binomial coefficients by the following polynomial identity:

$$\binom{n}{k_1} \cdots \binom{n}{k_m} = \sum_{i=0}^{k_1+\cdots+k_m} S(\mathbf{m}, i) \binom{n}{i}.$$

(3) The case of special multipermutations, called *Stirling permutations* [22], is defined on the multiset $\{1^2, \dots, m^2\}$ and yields the following identity:

$$S(n+m, m) = \sum_{i} A_{m,i} \binom{n+2m-1-i}{2m}.$$

It relates Stirling numbers of the second kind $S(n, m)$ with Eulerian numbers $A_{m,i}$ on Stirling permutations. We consider the *generalization of Stirling permutations* on any multiset $\{1^{k_1}, \dots, m^{k_m}\}$, which is the last part of thesis.

Related work

The usual finite power sum of consecutive integers $\sum_{i=1}^{N} i^n$ studied in many works. Classical results are due to Jacob Bernoulli (1654–1705), Johann Faulhaber (1580–1635) [21, 35], Carl G. J. Jacobi (1804–1851) [30], Leonhard Euler (1707-1783). Nevertherless, the research is still full of different approaches and attempts to find new properties, e.g., the survey of Knuth [35] on Faulhaber's results, overview and proofs with matrices by Edwards [18], Beardon [1] studied polynomial relations between power sums, Chen et al. [9] considered Faulhaber's theorem for arithmetic progressions, Guo and Zeng [26] presented a q-analog of Faulhaber's formula and its combinatorial interpretations (Guo et al., [27]), applications in the work of Gessel and Viennot [23] contain combinatorial properties for Bernoulli numbers and Faulhaber's coefficients, the Faulhaber's coefficients also arise in KdV equations [20]. Sums of powers of binomial coefficients whose types different to ours were considered, e.g., in [7, 12]. On the other hand, by viewing reciprocal case, the studies are related to important problems with the Riemann zeta function and its arithmetic properties. A special choice of functions and series with *reflectivity* property introduced in [35] to prove Faulhaber's theorem, also used in proofs by Rivoal [43] and Zudilin [52] on irrationality of zeta functions. Note that general question about irrationality of any of $\zeta(5), \zeta(7), \ldots$ is still open and challenging.

Broder [6] introduced Stirling numbers with special restrictions of elements in a set, called the r-Stirling numbers. In chapter 2, these numbers generalized in the notion of multiset covers and restricted partitions or permutations of sets. Such generalized Stirling numbers of the second kind occur (with another combinatorial interpretation) in the problem of boson normal ordering [38], which has the origin related to the composition of differential operators (or annihilation and creation in bosons terminology).

Stirling permutations were originally introduced by Gessel and Stanley in [22]. They obtain relations between special permutations on multisets with Stirling numbers. Further research on the subject was done by Park [39, 40] who studied the multiset $\{1^r, \ldots, m^r\}$, in the same case Klingsberg and Schmalzried [33] gave combinatorial interpretations for barred permutations in terms of posets, Bóna [2] proved that corresponding Eulerian polynomial has only real zeros, Haglund and Visontai [28] consider Eulerian multivariate polynomial and discuss corresponding stability property, Janson (et al.) [32, 31] showed the connection with plane recursive trees and urn model, Egge [19] obtained similar theory of Legendre–Stirling permutations.

Application perspective

The results and methods of research presented in this thesis can be applied:

(1) In the analysis of algorithms and data structures. It is the case when solution of problem (e.g., NP-complete) is based on exhaustive search and requires a generation or enumeration of permutations, subsets, partitions, etc. Or the case that needs a probabilistic and asymptotic analysis. For example, the analysis of *skip lists* data structure is based on the *binomial transform* of sequences [36, 41]; in the analysis of optimum caching [34] we may observe the study of special Stirling numbers and permutations; the usual Stirling numbers arise in many applications (hash functions, bloom filters, see, e.g., [4, 29]); multisets are the input source for hash functions with application to memory integrity checking [10]. A complete and efficient way for computation of certain problems on partitions, decompositions, permutations, posets and other objects is therefore valuable.

(2) In other areas, such as combinatorial physics. For instance, generalized Stirling numbers which we study in chapter 2 occur in the problem of boson normal ordering [38].

Many interesting applications of multisets can be found in [8]. Modern research on subject of permutations which includes different approaches, statistics and applications can be found in the book of Bóna [3].

Main results and structure of thesis

The first and the most extensive part in chapter 1 studies the power sum of binomial coefficients. We establish algebraic, combinatorial and number-theoretic properties. They particularly include the generalization of Faulhaber's theorem, specialization on coefficients and power sums of triangular numbers with coming reciprocity, the problem of Wolstenholme's theorem in a class of binomial coefficients, reciprocal power sums of binomial coefficients and connections with the Riemann zeta function.

In chapter 2, Stirling numbers on multisets are introduced and studied. In fact, we show that these numbers are almost the same (but differ in order and distinguished elements) as those which defined on restricted partitions of sets or cycle decompositions of permutations.

Chapter 3 is the part concerning generalized Stirling permutations. It contains new combinatorial meanings for the B numbers. These interpretations are related to the barred permutations, weighted lattice paths, bipartite multigraphs, theory of P-partitions, systems of partitions of sets. The presented construction allows us to specify the B numbers and their combinatorial meanings for particular cases as, e.g., the sums of powers of consecutive integers, the binomial coefficient, the Stirling numbers, the central factorial numbers [42] and their generalizations.

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1 POWER SUM OF BINOMIAL COEFFICIENTS

While the sum of powers of consecutive nonnegative integers was studied by many mathematicians from ancient times, two names should be especially mentioned: Jacob Bernoulli (1654-1705) and Johann Faulhaber (1580-1635). It is well known that

$$\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{i=1}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left(\frac{n(n+1)}{2}\right)^2.$$

In general,

$$\sum_{i=1}^{N-1} i^m = \frac{1}{m+1} \sum_{i=0}^{m} \binom{m+1}{i} B_i N^{m+1-i},$$

where B_i are Bernoulli numbers.

Faulhaber [21, 35] noticed that odd power sums can be represented as a polynomial in $t = n(n+1)/2$. For example,

$$\sum_{i=1}^{n} i^3 = t^2,$$

$$\sum_{i=1}^{n} i^5 = \frac{4t^3 - t^2}{3},$$
$$\sum_{i=1}^{n} i^7 = \frac{12t^4 - 8t^3 + 2t^2}{6}.$$

He computed these sums up to degree 17. The first proof of Faulhaber's theorem was given by Jacobi [30]. The general formula for odd power sums can be written as

$$\sum_{i=1}^{n} i^{2p+1} = \frac{1}{2^{2p+2}(2p+2)} \sum_{i=0}^{p} \binom{2p+2}{2i} (2 - 2^{2i}) B_{2i} ((8t+1)^{p+1-i} - 1).$$

Faulhaber knew that odd power sums are divisible by t^2 and even power sums can be expressed in terms of odd power sums. If

$$\sum_{i=1}^{n} i^{2p+1} = c_1 t^2 + c_2 t^3 + \cdots + c_p t^{p+1},$$

then

$$\sum_{i=1}^{n} i^{2p} = \frac{2n+1}{2(2p+1)}(2c_1 t + 3c_2 t^2 + \cdots + (p+1)c_p t^p).$$

In this chapter we consider power sum of binomial coefficients

$$f_{k,m}(N) = \sum_{i=0}^{N-1} \binom{i+k-1}{k}^m,$$

and for $k = 1$ we obtain the usual power sum

$$f_{1,m}(N) = \sum_{i=1}^{N-1} i^m.$$

We establish an analog of Faulhaber's theorem for any positive integer k. Namely, we show that:

- $f_{k,m}(N)$ is a polynomial in N and it can therefore be considered as a polynomial $f_{k,m}(x)$ in any variable x;
- $f_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}^2$ if m, k are odd and $m > 1$;
- $f_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}(2x+k-2)$ if k is odd and m is even;
- $f_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}$, otherwise;
- and in all cases the quotients are polynomials in $(2x+k-2)^2$ with rational coefficients.

For example,

$$f_{3,3}(N) = \sum_{i=0}^{N-1} \binom{i+2}{3}^3 = \binom{N+2}{4}^2 \frac{(2N+1)^2 - 10}{15},$$

$$f_{3,2}(N) = \sum_{i=0}^{N-1} \binom{i+2}{3}^2 = \binom{N+2}{4}(2N+1)\frac{5(2N+1)^2 - 41}{420},$$

$$f_{2,2}(N) = \sum_{i=0}^{N-1} \binom{i+1}{2}^2 = \binom{N+1}{3}\frac{3N^2 - 2}{10}.$$

By Faulhaber's theorem, any odd power sum can be expressed as a combination of powers of triangular numbers

$$\sum_{i=0}^{N} i^{2m-1} = \frac{1}{2m}\sum_{i=0}^{m-1} A_i^{(m)} (N(N+1))^{m-i},$$

and any even power sum can be expressed as

$$\sum_{i=0}^{N-1} i^{2m} = (N - \frac{1}{2}) \sum_{i=0}^{m} \widetilde{A}_i^{(m)} (N(N-1))^{m-i},$$

where $\widetilde{A}_i^{(m)} = \frac{m+1-i}{(2m+1)(m+1)} A_i^{(m+1)}$. [1]

Knuth [35] showed that the coefficients $A_i^{(m)}$ have many interesting properties. Our generalization of Faulhaber's theorem tends to consider the inverse problem: expressing the power sum of triangular numbers $f_{2,m}(N)$ in terms of powers of N. We show that this expression can be presented as a combination of odd powers of N,

$$f_{2,m}(N) = \sum_{i=1}^{N-1} \left(\frac{i(i+1)}{2} \right)^m = \frac{1}{2^m} \sum_{i=0}^{m} B_i^{(m)} N^{2m-2i+1}.$$

We find the following duality relations between coefficients $B_i^{(m)}$ and $A_i^{(m)}, \widetilde{A}_i^{(m)}$:

$$B_i^{(m+i)} = (-1)^{i-1} \frac{m+i}{m(2m+1)} A_i^{(-m)},$$

$$B_i^{(m+i-1)} = (-1)^i \widetilde{A}_i^{(-m)}.$$

Properties of $B_i^{(m)}$ which similar to the properties of $A_i^{(m)}$ are also established.

We study integer divisibility properties of $f_{k,m}(N)$ for integer N. We consider an analog of the Riemann zeta function

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}$$

for binomial coefficients. Let

$$\zeta_k(m) = \sum_{i=1}^{\infty} \binom{i+k-1}{k}^{-m}.$$

We prove that (for positive integers k, m and $l = \lceil m/2 \rceil$)

$$\zeta_k(m) \in \begin{cases} \mathbb{Q} + \mathbb{Q}\zeta(2) + \cdots + \mathbb{Q}\zeta(2l), & \text{if } km \text{ is even;} \\ \mathbb{Q} + \mathbb{Q}\zeta(3) + \cdots + \mathbb{Q}\zeta(2l-1), & \text{if } km \text{ is odd and } km > 1. \end{cases}$$

In case of $k = 2$ we obtain the formula for $\zeta_2(m)$,

$$\frac{\zeta_2(m)}{2^m} = (-1)^{m-1} \binom{2m-1}{m} + (-1)^m 2 \sum_{i=1}^{\lfloor m/2 \rfloor} \binom{2m-2i-1}{m-1} \zeta(2i).$$

[1] $A_i^{(m)}$ is referred to the sequences A093556, A093557 and $\widetilde{A}_i^{(m)}$ is referred to the sequences A093558, A093559 in [45].

The structure of this chapter is as follows. In section 1.1, we consider the sum of products of binomial coefficients, which has combinatorial origin and is used in establishing polynomial properties of $f_{k,m}(x)$. In section 1.2, an analog of Faulhaber's theorem for the powers of binomial coefficients is proved. In section 1.3, we study integer properties for $f_{k,m}(x)$ and for $f_{k,-1}$. In section 1.4, we consider the property of Wolstenholme's theorem in a class of binomial coefficients [15]. In section 1.5, the properties of infinite sum $\zeta_k(m)$ are derived. In section 1.6, we focus on the partial case $k = 2$ and express the power sum of triangular numbers $f_{2,m}(N)$ as a sum of powers of N.

1.1 Sum of products of binomial coefficients

1.1.1 Generalized Worpitzky's identity

Let

$$f_{k_1,\dots,k_m}(N) = \sum_{i=0}^{N-1} \binom{i+k_1-1}{k_1} \cdots \binom{i+k_m-1}{k_m}, \tag{1.1}$$

where $k_1 \le \cdots \le k_m$.

To study $f_{k_1,\dots,k_m}$, we need a generalization of Worpitzky's identity for multisets. To formulate this identity let us introduce Eulerian numbers for multisets.

Let $\mathbf{m} = 1^{k_1} \cdots m^{k_m}$ be a multiset where l repeats k_l times, $l = 1, \dots, m$. Let $S_\mathbf{m}$ be the set of permutations of $\mathbf{m}$ and set $K = k_1 + \cdots + k_m$.

For a permutation $\sigma = \sigma(1) \dots \sigma(K) \in S_\mathbf{m}$, let i be a *descent* index if $i = K$ or $\sigma(i) > \sigma(i+1), i < K$. A descent number $\operatorname{des}(\sigma)$ is defined as a number of descent indices of σ and Eulerian number $a_{\mathbf{m},p}$ is defined as a number of permutations with p descents,

$$a_{\mathbf{m},p} = |\{\sigma \in S_\mathbf{m} | \operatorname{des}(\sigma) = p\}|.$$

For $\mathbf{m} = 1^1 2^1 \cdots m^1$, we obtain the usual Eulerian numbers $a_{m,p}$ (A008292 in [45]) and the well-known Worpitzky identity

$$x^m = \sum_{p>0} \binom{x+m-p}{m} a_{m,p}.$$

Example 1.1.1. Let $\mathbf{m} = 1^2 2^2$. Then

$$S_{1^2 2^2} = \{1122, 1212, 2112, 2121, 2211, 1221\},$$

and

$$\operatorname{des}(1122) = 1, \operatorname{des}(1212) = 2, \operatorname{des}(2112) = 2, \operatorname{des}(2121) = 3,$$
$$\operatorname{des}(2211) = 2, \operatorname{des}(1221) = 2.$$

Therefore,

$$a_{1^2 2^2,1} = 1,\ a_{1^2 2^2,2} = 4,\ a_{1^2 2^2,3} = 1,\ \text{and}\ a_{1^2 2^2,i} = 0,\ \text{if}\ i \ne 1, 2, 3.$$

Theorem 1.1.2. *[14] For any nonnegative integers* $k_1, \dots, k_m$,

$$\prod_{i=1}^{m} \binom{x+k_i-1}{k_i} = \sum_{p>0} \binom{x+K-p}{K} a_{\mathbf{m},p},$$

where $a_{\mathbf{m},p}$ *are Eulerian numbers of permutations of the multiset* $\mathbf{m} = 1^{k_1} \cdots m^{k_m}$.

Example 1.1.3. If $\mathbf{m} = 1^2 2^2$, then

$$\binom{x+1}{2}^2 = \binom{x+3}{4} + 4\binom{x+2}{4} + \binom{x+1}{4}.$$

For a more detailed overview and other properties of Eulerian numbers on multisets and generalized Worpitzky's identity, see [14].

1.1.2 Sum of products of binomial coefficients as a polynomial

Theorem 1.1.4. *Let* $0 \le k_1 \le \cdots \le k_m$. *Then the sum (1.1) induces a polynomial* $f_{k_1,\dots,k_m}(x)$ *of degree* $K+1$ *with rational coefficients. As a polynomial with rational coefficients, the polynomial* $f_{k_1,\dots,k_m}(x)$ *is divisible by* $\binom{x+k_m-1}{k_m+1}$.

Proof. By Theorem 1.1.2,

$$f_{k_1,\dots,k_m}(N) = \sum_{p>0} \sum_{i=0}^{N-1} \binom{i+K-p}{K} a_{\mathbf{m},p} = \sum_{p>0} \binom{N+K-p}{K+1} a_{\mathbf{m},p},$$

for any positive integer N. Therefore, we can substitute any variable x in N and see that

$$f_{k_1,\dots,k_m}(x) \in \mathbb{Q}[x], \qquad \deg\, f_{k_1,\dots,k_m}(x) = K+1.$$

If $k_m = 0$, then $k_1 = \cdots = k_m = 0$, and

$$f_{0,\dots,0}(N) = N - 1.$$

Therefore, $f_{0,\dots,0}(x) = x-1$. So, in this case divisibility of $f_{k_1,\dots,k_m}(x)$ by $\binom{x+k_m-1}{k_m+1}$ is evident.

Suppose now that $k_m > 0$. Let us consider a difference polynomial

$$\Delta f(x) = f_{k_1,\dots,k_m}(x+1) - f_{k_1,\dots,k_m}(x).$$

By (1.1),

$$\Delta f(x) = \binom{x+k_1-1}{k_1} \cdots \binom{x+k_m-1}{k_m}.$$

Therefore, $\Delta f(x)$ has k_m zeros: $0, -1, \ldots, -(k_m - 1)$. Hence,

$$\begin{gathered}
f_{k_1,\ldots,k_m}(1) - f_{k_1,\ldots,k_m}(0) = \Delta f(0) = 0, \\
f_{k_1,\ldots,k_m}(0) - f_{k_1,\ldots,k_m}(-1) = \Delta f(-1) = 0, \\
\ldots \\
f_{k_1,\ldots,k_m}(-(k_m - 2)) - f_{k_1,\ldots,k_m}(-(k_m - 1)) = \Delta f(-(k_m - 1)) = 0.
\end{gathered}$$

If $k_m > 0$, then $\binom{i+k_m-1}{k_m} = 0$ for $i = -k_m + 1$. Therefore,

$$f_{k_1,\ldots,k_m}(-k_m + 1) = 0.$$

We thus obtain a polynomial $f_{k_1,\ldots,k_m}(x)$ with $k_m + 1$ zeros

$$1, 0, -1, \ldots, -(k_m - 1).$$

This means that $f_{k_1,\ldots,k_m}(x)$ is divisible by $\binom{x+k_m-1}{k_m+1}$. □

Set $f_{k,m}(x) = f_{k,k,\ldots,k}(x)$. In other words, $f_{k,m}(x)$ is a polynomial defined by the following relations

$$f_{k,m}(N) = \sum_{i=0}^{N-1} \binom{i+k-1}{k}^m. \tag{1.2}$$

Corollary 1.1.5. *The polynomial $f_{k,m}(x)$ has the following properties*

- $f_{k,m}(x) \in \mathbb{Q}[x]$,
- $\deg f_{k,m}(x) = km + 1$,
- $f_{k,m}(x)$ *is divisible by* $\binom{x+k-1}{k+1}$.

A more detailed version of this result is given in the next section.

1.2 Faulhaber's theorem for powers of binomial coefficients

1.2.1 Formulation of the main result

We know that $f_{1,m}(N) = \sum_{i=1}^{N-1} i^m$ is a polynomial in N of degree $m+1$. By Faulhaber's theorem [21, 35], the polynomial $f_{1,m}(x)$ is divisible by the polynomial $f_{1,1}(x) = x(x-1)/2$. For odd m, the polynomial $f_{1,m}(x)$ is divisible by $f_{1,1}(x)^2$ and the quotient is a polynomial in $f_{1,1}(x)$. For even m, the polynomial $f_{1,m}(x)$ can be presented as product of $f_{1,1}(x)(2x-1)$ and a polynomial in $f_{1,1}(x)$.

The following theorem is an analog of Faulhaber's theorem for the sum of powers of binomial coefficients.

Theorem 1.2.1. *There exist polynomials* $Q_{k,m}(x) \in \mathbb{Q}[x]$, *such that*

$$f_{k,m}(x) = \begin{cases} \binom{x+k-1}{k+1}^2 Q_{k,m}((2x+k-2)^2), & \text{if } m, k \text{ are odd, } m > 1; \\ \\ \binom{x+k-1}{k+1}(2x+k-2)Q_{k,m}((2x+k-2)^2), & \text{if } k \text{ is odd, } m \text{ is even;} \\ \\ \binom{x+k-1}{k+1}Q_{k,m}((2x+k-2)^2), & \text{otherwise.} \end{cases}$$

Note that $f_{k,1}(x) = \binom{x+k-1}{k+1}$ and our theorem states that the polynomial $f_{k,m}(x)$ is divisible by $f_{k,1}(x)$. Moreover, if m and k are odd $(m > 1)$, then $f_{k,m}(x)$ is divisible by $f_{k,1}(x)^2$.

Let us show Theorem 1.2.1 for some small values of k.

If $k = 1$, then there exist polynomials $Q_{1,m}(x) \in \mathbb{Q}[x]$, such that the polynomial $f_{1,m}(N) = \sum_{i=1}^{N-1} i^m$ can be expressed as:

$$f_{1,m}(x) = \begin{cases} \binom{x}{2}^2 Q_{1,m}((2x-1)^2), & \text{if } m \text{ is odd and } m > 1; \\ \\ \binom{x}{2}(2x-1)Q_{1,m}((2x-1)^2), & \text{if } m \text{ is even.} \end{cases}$$

Note that the polynomial $Q_{1,m}((2x-1)^2) = Q_{1,m}(8\frac{x(x-1)}{2} + 1)$ can be written as a polynomial in $\binom{x}{2} = \frac{x(x-1)}{2}$. Hence, Faulhaber's theorem is a particular case of Theorem 1.2.1.

If $k = 2$, then for any $m > 0$, there exists a polynomial $Q_{2,m}(x) \in \mathbb{Q}[x]$, such that

$$f_{2,m}(x) = \binom{x+1}{3} Q_{2,m}(x^2).$$

Since

$$\binom{x+1}{3} = \frac{x(x^2-1)}{6},$$

this implies that $f_{2,m}(x)$ is an odd polynomial.

If $k = 3$, then $f_{3,3}(x) = \binom{x+2}{4}^2 Q_{3,3}(x)$, where

$$Q_{3,3}(x) = \frac{1}{15}(4x^2 + 4x - 9).$$

Then for each positive integer m, $(m > 1)$, there exists a polynomial $Q_{3,m}(x) \in \mathbb{Q}[x]$, such that

$$f_{3,m}(x) = \binom{x+2}{4}^2 Q_{3,m}(Q_{3,3}(x)), \text{ if } m \text{ is odd, and}$$

$$f_{3,m}(x) = \binom{x+2}{4}(2x+1)Q_{3,m}(Q_{3,3}(x)), \text{ if } m \text{ is even.}$$

1.2.2 Reflective functions

The proof of Theorem 1.2.1 is based on the notion of reflective functions introduced by Knuth [35]. The function $f(x)$ is called r*-reflective* if for all $x,$ we have

$$f(x) = f(-x-r);$$

and $f(x)$ is called *anti-*r*-reflective* if for all $x,$ we have

$$f(x) = -f(-x-r).$$

In other words, reflective functions are even or odd functions shifted by $r/2$.

Note that the

- sum of two (anti)-r-reflective functions is (anti)-r-reflective;
- product of two r-reflective functions is r-reflective;
- product of anti-r-reflective and r-reflective is anti-r-reflective function and
- product of two anti-r-reflective functions is r-reflective.

Lemma 1.2.2. *Let* $\nabla f(x) = f(x) - f(x-1)$. *Suppose that* $f(0) = f(-r) = 0$ *and the function* f *is defined on the set of integers. Then the following is true:*

- *if the function* ∇f *is* $(r-1)$*-reflective, then* f *is anti-*r*-reflective and*
- *if the function* ∇f *is anti-*$(r-1)$*-reflective, then* f *is* r*-reflective.*

Proof. Suppose that ∇f is $(r-1)$-reflective. Then we have

$$f(N) - f(0) = \sum_{i=1}^{N} \nabla f(i) = \sum_{i=1}^{N} \nabla f(-i-r+1) = f(-r) - f(-N-r),$$

which gives $f(N) = -f(-N-r)$ and this implies that f is anti-r-reflective.

Now if ∇f is anti-$(r-1)$-reflective, then

$$f(N) - f(0) = \sum_{i=1}^{N} \nabla f(i) = -\sum_{i=1}^{N} \nabla f(-i-r+1) = -f(-r) + f(-N-r).$$

So, $f(N) = f(-N-r)$ and f is r-reflective. □

Lemma 1.2.3 ([35], Lemma 4)**.** *A polynomial* $f(x)$ *is* r*-reflective if and only if it can be presented as a polynomial in* $x(x+r)$ *(or* $(2x+r)^2$*); it is anti-*r*-reflective if and only if it can be presented as* $2x+r$ *times a polynomial in* $x(x+r)$ *(or* $(2x+r)^2$*).*

Lemma 1.2.4. *The polynomial* $\binom{x+k-1}{k}$ *is* $(k-1)$*-reflective if* k *is even and anti-*$(k-1)$*-reflective if* k *is odd.*

Proof. The proof follows from the identity $\binom{x+k-1}{k} = (-1)^k\binom{-x}{k}$. $\square$

By Theorem 1.1.4, there exist polynomials $g_{k,m}(x) \in \mathbb{Q}[x]$, such that

$$f_{k,m}(x) = \binom{x+k-1}{k+1} g_{k,m}(x).$$

Lemma 1.2.5. *Reflective properties of functions* $f_{k,m}$ *and* $g_{k,m}$ *are*

- $f_{k,m}(x)$ *is* $(k-2)$*-reflective if* $km+1$ *is even and anti-*$(k-2)$*-reflective if* $km+1$ *is odd.*

- $g_{k,m}(x)$ *is* $(k-2)$*-reflective if* $(m-1)k$ *is even and anti-*$(k-2)$*-reflective if* $(m-1)k$ *is odd.*

Proof. Let $\nabla f = f(x) - f(x-1)$. Since $\nabla f_{k,m}(x) = \binom{x+k-2}{k}^m$, we see that $\nabla f_{k,m}(x)$ is $(k-3)$-reflective if km is even and anti-$(k-3)$-reflective, otherwise. By Theorem 1.1.4, $f(0) = f(-(k-2)) = 0$. Thus, by Lemma 1.2.2, $f_{k,m}(x)$ is anti-$(k-2)$-reflective if km is even and $(k-2)$-reflective, otherwise.

Note that $g_{k,m} = f_{k,m}/\binom{x+k-1}{k+1}$. Therefore by Lemma 1.2.4, $g_{k,m}(x)$ is $(k-2)$-reflective if $km-k$ is even and anti-$(k-2)$-reflective, otherwise. $\square$

Lemma 1.2.6. *Let* k *be an odd number. Then* $f_{k,m}(x)$ *is divisible by* $\binom{x+k-1}{k+1}(2x+k-2)$ *if* m *is even and is divisible by* $\binom{x+k-1}{k+1}^2$ *if* m *is odd and* $m>1$.

Proof. Suppose that m is even. By Lemma 1.2.5, the function $g_{k,m}$ is anti-$(k-2)$-reflective. The anti-$(k-2)$-reflectivity condition for $x = \frac{2-k}{2}$ gives us

$$g_{k,m}(-\frac{2-k}{2} - k + 2) = g_{k,m}(\frac{2-k}{2}) = -g_{k,m}(\frac{2-k}{2}).$$

Hence, $g_{k,m}(\frac{2-k}{2}) = 0$ and $g(x)$ is divisible by $(2x+k-2)$.

Now consider the case m is odd $(m>1)$. By Lemma 1.2.5, the function $g_{k,m}$ is $(k-2)$-reflective. We have

$$\begin{aligned} f_{k,m}(x+1) - f_{k,m}(x) &= \binom{x+k}{k+1} g_{k,m}(x+1) - \binom{x+k-1}{k+1} g_{k,m}(x) \\ &= \binom{x+k-1}{k}^m. \end{aligned}$$

Hence,

$$(x+k)g_{k,m}(x+1) - (x-1)g_{k,m}(x) = (k+1)\binom{x+k-1}{k}^{m-1}.$$

Therefore, for $i = 0, -1, \dots, -(k-1),$ we obtain

$$(k+i)g_{k,m}(i+1) - (i-1)g_{k,m}(i) = 0.$$

In other words,

$$\begin{aligned} kg_{k,m}(1) &= -g_{k,m}(0), \\ (k-1)g_{k,m}(0) &= -2g_{k,m}(-1), \\ (k-2)g_{k,m}(-1) &= -3g_{k,m}(-2), \\ &\dots \\ g_{k,m}(-(k-2)) &= -kg_{k,m}(-(k-1)). \end{aligned}$$

Hence,

$$g_{k,m}(1) = (-1)^k g_{k,m}(-(k-1)) = -g_{k,m}(-(k-1)).$$

Since $g_{k,m}(x)$ is $(k-2)$-reflective, the reflectivity condition for $x = 1$ gives us

$$g_{k,m}(1) = g_{k,m}(-(k-1)).$$

So, $g_{k,m}(1) = 0$ and

$$g_{k,m}(-(k-1)) = \dots = g_{k,m}(-1) = g_{k,m}(0) = g_{k,m}(1) = 0.$$

Thus, $g_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}$ and $f_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}^2$. $\square$

1.2.3 Proof of Theorem 1.2.1.

Let m and k be two odd positive integers and $m > 1$. By Lemma 1.2.6, there exist polynomials $R_{k,m}(x)$ such that $g_{k,m}(x) = \binom{x+k-1}{k+1} R_{k,m}(x)$. The function $g_{k,m}(x)$ is $(k-2)$-reflective and therefore, $R_{k,m}(x)$ (according to parity of k) is also $(k-2)$-reflective. So, by Lemma 1.2.3, there exist polynomials $Q_{k,m}(x) \in \mathbb{Q}[x]$ such that $R_{k,m}(x) = Q_{k,m}((2x+k-2)^2)$. In this case

$$f_{k,m}(x) = \binom{x+k-1}{k+1}^2 Q_{k,m}((2x+k-2)^2).$$

Now assume that k is odd and m is even. Then the function $g_{k,m}$ is anti-$(k-2)$-reflective. By Lemma 1.2.3, there exist polynomials $Q_{k,m}(x) \in \mathbb{Q}[x]$, such that $g_{k,m}(x) = (2x+k-2)Q_{k,m}((2x+k-2)^2)$. Therefore,

$$f_{k,m}(x) = \binom{x+k-1}{k+1}(2x+k-2)Q_{k,m}((2x+k-2)^2).$$

In all other cases $g_{k,m}$ is $(k-2)$-reflective. By Lemma 1.2.3, there exist polynomials $Q_{k,m}(x) \in \mathbb{Q}[x]$ such that $g_{k,m}(x) = Q_{k,m}((2x+k-2)^2)$. We have

$$f_{k,m}(x) = \binom{x+k-1}{k+1} Q_{k,m}((2x+k-2)^2).$$

$\square$

1.3 Integer divisibility for $f_{k,m}(x)$

1.3.1 Formulation of the main result

By Theorem 1.2.1, the polynomial $f_{k,m}(x)$ is divisible by $\binom{x+k-1}{k+1}^2$ if m and k are odd $(m>1)$. In particular, $f_{k,m}(x)$ is divisible by x^2. By Theorem 1.2.1, $f_{k,m}(x)$ is divisible by x for any k, m. Here divisibility refers to the divisibility of polynomials with rational coefficients. Now in this section, we study divisibility properties of $f_{k,m}(x)$ for integer x.

The divisibility properties of Theorem 1.2.1 do not hold for integers. Let us give counter-examples for all three cases:

$$\frac{f_{3,5}(7)}{7^2} = \frac{1}{7^2}\sum_{i=1}^{6}\binom{i+2}{3}^5 = \frac{86650668}{7} \notin \mathbb{Z},$$

$$\frac{f_{3,2}(5)}{5} = \frac{1}{5}\sum_{i=1}^{4}\binom{i+2}{3}^2 = \frac{517}{5} \notin \mathbb{Z},$$

$$\frac{f_{4,4}(11)}{11} = \frac{1}{11}\sum_{i=1}^{10}\binom{i+3}{4}^4 = \frac{335469880502}{11} \notin \mathbb{Z}.$$

Since $f_{k,m}(x) \in \mathbb{Q}[x]$, for any given k and m, there is a sufficiently large prime p, such that $f_{k,m}(p)$ is divisible by p and by p^2 if m, k are odd and $m>1$. The following theorem is a more detailed version of these statements.

Theorem 1.3.1. *Let N, k, m be positive integers. Let M be an odd positive integer such that $\gcd(M, k!) = 1$ and $M-k+1 \le N \le M+1$. Then*

$$f_{k,m}(N) = \sum_{i=0}^{N-1}\binom{i+k-1}{k}^m \equiv 0 \pmod{M}$$

in the following cases

- *m, k are odd numbers, or*
- *$\gcd(M, (km+1)!) = 1$ (k, m may be even or odd).*

If m, k are odd numbers and $m>1$, then

$$f_{k,m}(N) = \sum_{i=0}^{N-1}\binom{i+k-1}{k}^m \equiv 0 \pmod{M^2}$$

in the following cases

- *m is divisible by M;*
- *$\gcd(M, (km+1)!) = 1$.*

1.3.2 Proof

To prove Theorem 1.3.1 we need some preliminary facts.

Lemma 1.3.2. *Let* M *be an odd positive integer number such that* $\gcd(M, k!) = 1$. *Then for odd numbers* k, m *and for all integers* i *such that* $1 \leq i \leq M - k$, *the following relation holds*

$$\binom{M-i+k-1}{k}^m \equiv -\binom{i+k-1}{k}^m + Mm\binom{i+k-1}{k}^m \sum_{j=0}^{k-1} \frac{1}{i+j} \pmod{M^2}.$$

Proof. Let us consider an expression $\binom{M-i+k-1}{k}$ as a polynomial in M,

$$\begin{aligned}\binom{M-i+k-1}{k} &= \frac{(M-i+k-1)\cdots(M-i)}{k!}\\ &= a_k M^k + \cdots + a_1 M + a_0\\ &= \frac{M^k}{k!} + \cdots + M(-1)^{k-1}\binom{i+k-1}{k}\sum_{j=0}^{k-1}\frac{1}{i+j}\\ &+ (-1)^k\binom{i+k-1}{k}.\end{aligned}$$

Note that for $0 \leq j \leq k-1$, all numbers $i+j$ are relatively prime with M. Therefore,

$$\binom{M-i+k-1}{k} \equiv (-1)^k\binom{i+k-1}{k} + (-1)^{k-1}M\binom{i+k-1}{k}\sum_{j=0}^{k-1}\frac{1}{i+j} \pmod{M^2}.$$

Hence,

$$\binom{M-i+k-1}{k}^m \equiv \left(-\binom{i+k-1}{k} + M\binom{i+k-1}{k}\sum_{j=0}^{k-1}\frac{1}{i+j}\right)^m \pmod{M^2}$$

On expanding the right side of this congruence, we obtain the result. □

Lemma 1.3.3. $(km+1)!f_{k,m}(x) \in \mathbb{Z}[x]$.

Proof. By Theorem 1.1.2, there are some integers a_j for which we can present $f_{k,m}(x)$ in the form $\sum_{j>0} a_j \binom{x+km-j}{km+1}$. Note that $(km+1)!\binom{x+km-j}{km+1} \in \mathbb{Z}[x]$. □

Proof of Theorem 1.3.1. For $0 \le i \le M-k$ with odd k, by Lemma 1.3.2,

$$\binom{M-i+k-1}{k}^m + \binom{i+k-1}{k}^m \equiv 0 \pmod{M}.$$

Therefore,

$$\begin{aligned}\sum_{i=0}^{N-1} \binom{i+k-1}{k}^m &\equiv \sum_{i=1}^{M-k} \binom{i+k-1}{k}^m \\ &= \frac{1}{2}\sum_{i=1}^{M-k}\left(\binom{M-i+k-1}{k}^m + \binom{i+k-1}{k}^m\right) \\ &\equiv 0 \pmod{M}.\end{aligned}$$

Now suppose that m is divisible by N. Hence, by Lemma 1.3.2, we have

$$\begin{aligned}\binom{M-i+k-1}{k}^m &\equiv -\binom{i+k-1}{k}^m + Mm\binom{i+k-1}{k}^m \sum_{j=0}^{k-1}\frac{1}{i+j} \\ &\equiv -\binom{i+k-1}{k}^m \pmod{M^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=0}^{N-1} \binom{i+k-1}{k}^m &\equiv \sum_{i=1}^{M-k} \binom{i+k-1}{k}^m \\ &= \frac{1}{2}\sum_{i=1}^{M-k}\left(\binom{M-i+k-1}{k}^m + \binom{i+k-1}{k}^m\right) \\ &\equiv 0 \pmod{M^2}.\end{aligned}$$

Since $\gcd(M,(km+1)!) = 1$, by Lemma 1.3.3 and Theorem 1.2.1, $(km+1)!f_{k,m}(x) \in \mathbb{Z}[x]$ is divisible by $x(x+1)\cdots(x+k-1)$ and by $(x(x+1)\cdots(x+k-1))^2$ if m,k are odd numbers and $m>1$. Here divisibility refers to the divisibility of polynomials with integer coefficients. □

Remark 1.3.4. In Theorem 1.3.1 we can change M to a power of some prime number p. The property $\gcd(M,(km+1)!) = 1$ can be changed by an inequality $p > km+1$.

1.3.3 Integer divisibility in case $k = 2$

Theorem 1.3.5. *Assume that p is an odd prime number and $1 \le m \le p-1$. Then*

$$f_{2,m}(p) = \sum_{i=1}^{p-1} \binom{i+1}{2}^m \equiv \begin{cases} -2 \pmod p, & \text{if } m = p-1; \\ -\frac{1}{2^m}\binom{m}{p-1-m} \pmod p, & \text{otherwise.} \end{cases}$$

Proof. The following fact is known:

$$\sum_{i=1}^{p-1} i^t \equiv \begin{cases} 0 \pmod p, & \text{if } t \text{ is not divisible by } p-1; \\ -1 \pmod p, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} f_{2,m}(p) = \sum_{i=1}^{p-1} \binom{i+1}{2}^m &= \frac{1}{2^m} \sum_{i=1}^{p-1} \sum_{j=0}^{m} \binom{m}{j} i^{m+j} \\ &= \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} \sum_{i=1}^{p-1} i^{m+j} \end{aligned}$$

If $m = p-1$, then $m+j$ is divisible by $p-1$ only in two cases: $j = 0, m$. Hence,

$$f_{2,p-1}(p) \equiv -\frac{1}{2^{p-1}} \left(\binom{p-1}{0} + \binom{p-1}{p-1} \right) \equiv -2 \pmod p.$$

If $1 \le m < p-1$, there is only one integer $j \in [0, m]$ such that $m + j$ is divisible by $p-1$. Namely, $j = p-1-m$. In this case,

$$f_{2,m}(p) \equiv -\frac{1}{2^m} \binom{m}{p-1-m} \pmod p.$$

□

By Theorem 1.3.5, if $p-1-m > m$, then $f_{2,m}(p) \equiv 0 \pmod p$. This fact is compatible with Theorem 1.3.1, because $p > 2m+1$.

1.3.4 The case $f_{k,-1}(N)$

Theorem 1.3.6. *Let N, k be positive integer numbers and M be a positive integer such that $\gcd(M, k!) = 1$. Then, the rational number q defined as*

$$q = f_{(k,-1)}(N) = \sum_{i=1}^{N-1} \frac{1}{\binom{i+k-1}{k}},$$

is divisible by M (its denominator is relatively prime with M) in the following cases:

(i) $N \equiv 1 \pmod M$;

(ii) $N \equiv 1-k \pmod M$ and k is odd.

To prove this result we need one

Lemma 1.3.7. *Suppose that* $k > 1$. *Then*

$$f_{k,-1}(N) = \sum_{i=1}^{N-1} \frac{1}{\binom{i+k-1}{k}} = \frac{k}{k-1}\left(1 - \frac{1}{\binom{N+k-2}{k-1}}\right).$$

Proof. Let $\phi(x) = \frac{1}{\binom{x+k-1}{k-1}}$. Then it is easy to verify that

$$\frac{1}{\binom{i+k-1}{k}} = \frac{k}{k-1}\left(\frac{1}{\binom{i+k-2}{k-1}} - \frac{1}{\binom{i+k-1}{k-1}}\right) = -\frac{k}{k-1}\nabla\phi(i).$$

Therefore,

$$\begin{aligned} f_{(k,-1)}(N) &= \sum_{i=1}^{N-1} \frac{1}{\binom{i+k-1}{k}} = -\frac{k}{k-1}\sum_{i=1}^{N-1} \nabla\phi(i) \\ &= \frac{k}{k-1}(\phi(0) - \phi(N-1)) = \frac{k}{k-1}\left(1 - \frac{1}{\binom{N+k-2}{k-1}}\right). \end{aligned}$$

□

Proof of Theorem 1.3.6. By Lemma 1.3.7,

$$f_{k,-1}(N) = \frac{k!}{k-1}\left(\frac{1}{(k-1)!} - \frac{1}{(N+k-2)\cdots N}\right).$$

Notice that for both cases (i) and (ii), the numbers $N, \ldots, N+k-2$ are relatively prime with M. So, if $N \equiv 1 \pmod M$, then

$$(N+k-2)\cdots N \equiv (k-1)! \pmod M.$$

If $N \equiv 1-k \pmod M$ and k is odd, then

$$(N+k-2)\cdots N \equiv (-1)^{k-1}(k-1)! \equiv (k-1)! \pmod M.$$

□

1.4 Wolstenholme's theorem for binomial coefficients

We prove that the numerator of $\sum_{i=k}^{p-1}\binom{i}{k}^{-1}$ is divisible by p^2 for infinitely many primes p if and only if $k = 1$.

For rational numbers $\alpha = a/b$ and $\beta = c/d$ write $\alpha \equiv \beta \pmod n$, if $ad - bc$ is divisible by n and denominators b, d are relatively prime with n.

The following result was proved by Wolstenholme in [49]. For any prime number $p > 3$,

$$\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \pmod{p^2}.$$

In this section we consider the extension of Wolstenholme theorem to the class of binomial coefficients. Does there exist a number k, $k > 1$ such that

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}$$

for any $p > k$? We show that it is impossible.

Theorem 1.4.1. *For a given $k \geq 1$ the congruence*

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}, \quad 1 \leq k < p,$$

takes place for infinitely many primes p if and only if $k = 1$.

Before the proof of our main result let us introduce some notations

$$F_k(n) = \sum_{i=k}^{n} \frac{1}{\binom{i}{k}},$$

$$H_{a,b} = \sum_{1 \leq i_1 < \cdots < i_b \leq a} \frac{1}{i_1 \cdots i_b}, \qquad b > 0,$$

$$H_{a,0} = 1.$$

We establish the following slightly more general result.

Lemma 1.4.2. *For any integer p (not necessary prime) and $1 < k < p$,*

$$F_k(p-1) = \frac{(1+(-1)^k)k}{k-1} + \frac{k}{(k-1)\binom{p-1}{k-1}} \sum_{i=1}^{k-1} (-p)^i H_{k-1,i}. \tag{1.3}$$

Proof. Let ∇ be a difference operator

$$\nabla f(x) = f(x) - f(x-1).$$

Then for $\phi(x) = \frac{1}{\binom{x}{k-1}}$ we have

$$\frac{1}{\binom{i}{k}} = \frac{k}{k-1} \left(\frac{1}{\binom{i-1}{k-1}} - \frac{1}{\binom{i}{k-1}} \right) = -\frac{k}{k-1} \nabla \phi(i).$$

Since

$$F_k(p-1) = -\frac{k}{k-1}\sum_{i=k}^{p-1}\nabla\phi(i) = \frac{k}{k-1}(\phi(k-1)-\phi(p-1)) = \frac{k}{k-1}\left(1-\frac{1}{\binom{p-1}{k-1}}\right),$$

we obtain

$$F_k(p-1) = \frac{k}{(k-1)\binom{p-1}{k-1}}\left(\binom{p-1}{k-1}-1\right). \tag{1.4}$$

Let $c(n,i)$ be Stirling numbers of the first kind. The following relations are well-known

$$x(x-1)\cdots(x-k+1) = \sum_{i=1}^{k} c(k,i)x^i, \tag{1.5}$$

$$c(n,i) = (-1)^{n+i}(n-1)!H_{n-1,i-1}. \tag{1.6}$$

By (1.5) and (1.6)

$$\binom{p-1}{k-1} = \frac{\sum_{i=1}^{k} s_{k,i}p^{i-1}}{(k-1)!} = \sum_{i=1}^{k}(-1)^{k+i}H_{k-1,i-1}p^{i-1}.$$

Hence,

$$\binom{p-1}{k-1} = (-1)^{k+1} - \sum_{i=1}^{k-1}(-1)^{k+i}H_{k-1,i}p^i. \tag{1.7}$$

Thus, we obtain

$$F_k(p-1) = -\frac{k}{(k-1)\binom{p-1}{k-1}}\{((-1)^k+1)+\sum_{i=1}^{k-1}(-1)^{k+i}p^iH_{k-1,i}\} \tag{1.8}$$

It is not difficult to see that (1.8) is equivalent to (1.3). If k is odd, it is obvious. If k is even, it follows from (1.7).

Lemma 1.4.3. *Let p be a prime number. If $1 < k < p$ then*

$$\sum_{i=k}^{p-1}\frac{1}{\binom{i}{k}} \equiv \frac{(1+(-1)^k)k}{k-1} \pmod{p}, \tag{1.9}$$

$$\sum_{i=k}^{p-1}\frac{1}{\binom{i}{k}} \equiv \frac{(1+(-1)^k)k}{k-1} + \frac{(-1)^k pk}{k-1}\sum_{i=1}^{k-1}\frac{1}{i} \pmod{p^2}. \tag{1.10}$$

Proof. The equalities (1.9) and (1.10) are easy consequences of Lemma 1.4.2. In case of (1.10) we use the congruence $\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod{p}$.

Lemma 1.4.4. *Let p be a prime number and $1 < k < p$. Then the following conditions are equivalent*

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}, \tag{1.11}$$

$$k \textit{ is odd and } \sum_{i=1}^{k-1} \frac{1}{i} \equiv 0 \pmod{p}. \tag{1.12}$$

Proof. Suppose that (1.11) is true, $F_k(p-1) \equiv 0 \pmod{p^2}$. Then $F_k(p-1) \equiv 0 \pmod{p}$. Therefore, by (1.9) k is odd. Hence by Lemma 1.4.3, relation (1.10), we have $H_{k-1,1} \equiv 0 \pmod{p}$. So, (1.11) implies (1.12).

Conversely, if (1.12) is given then by (1.10) we have $F_k(p-1) \equiv 0 \pmod{p^2}$. From Lemma 1.4.3 we get the sufficient part of the lemma.

Proof of Theorem 1.4.1. For given $k > 1$ the numerator of the sum $\sum_{i=1}^{k-1} \frac{1}{i}$ has a finite number of prime divisors. Therefore, the congruence

$$\sum_{i=1}^{k-1} \frac{1}{i} \equiv 0 \pmod{p}$$

holds only for a finite number of primes $p \geq k$. So, if $k > 1$, then by Lemma 1.4.4 the congruence

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}$$

might be true only for a finite number of primes $p > k$.

If $k = 1$, by Wolstenholme's theorem our statement is true

$$\sum_{i=k}^{p-1} \frac{1}{\binom{i}{k}} \equiv 0 \pmod{p^2}$$

for any $p > 3$. Theorem 1.4.1 is proved.

1.5 Power sum of reciprocals of binomial coefficients

1.5.1 Formulation of the main result

In this section, we consider the case of power sums of binomial coefficients with negative powers,

$$\zeta_k(m) = \sum_{i=1}^{\infty} \binom{i+k-1}{k}^{-m}.$$

For $k = 1$ we have

$$\zeta_1(m) = \zeta(m) = \sum_{i=1}^{\infty} \frac{1}{i^m},$$

where $\zeta(m)$ is a Riemann zeta function.

In particular, for $m = 1$, by Lemma 1.3.7, one can obtain the exact value

$$\zeta_k(1) = \sum_{i=1}^{\infty} \binom{i+k-1}{k}^{-1} = \frac{k}{k-1}.$$

It is known that for any positive integer m,

$$\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}, \tag{1.13}$$

where B_{2m} is a Bernoulli number.

We prove similar results for binomial coefficients. Some examples that follow from our results:

$$\zeta_2(2) = \sum_{i=1}^{\infty} \binom{i+1}{2}^{-2} = \frac{4}{3}\pi^2 - 12,$$

$$\zeta_2(3) = \sum_{i=1}^{\infty} \binom{i+1}{2}^{-3} = -8\pi^2 + 80,$$

$$\zeta_3(2) = \sum_{i=1}^{\infty} \binom{i+2}{3}^{-2} = 9\pi^2 - \frac{351}{4}.$$

Similarly,

$$\zeta_3(3) = \sum_{i=1}^{\infty} \binom{i+2}{3}^{-3} = \frac{783}{4} - 162\zeta(3),$$

$$\zeta_5(3) = \sum_{i=1}^{\infty} \binom{i+4}{5}^{-3} = -\frac{1298125}{96} + 11250\zeta(3),$$

$$\zeta_3(5) = \sum_{i=1}^{\infty} \binom{i+2}{3}^{-5} = \frac{576639}{16} - \frac{47385}{2}\zeta(3) - 7290\zeta(5).$$

Below we show that these relations are based on reflectivity properties of binomial coefficients, and that $\zeta_k(m)$ can be expressed as a linear combination of certain values of the Riemann zeta function.

Theorem 1.5.1. *For any positive integers k and m, there exist rational numbers $\lambda_0, \lambda_1, \ldots, \lambda_{\lceil m/2 \rceil}$, such that*

$$\zeta_k(m) = \begin{cases} \lambda_0 + \sum_{i=1}^{\lceil m/2 \rceil} \lambda_i \zeta(2i), & \text{if } km \text{ is even;} \\ \lambda_0 + \sum_{i=1}^{\lceil m/2 \rceil} \lambda_i \zeta(2i-1), & \text{if } km \text{ is odd.} \end{cases}$$

1.5.2 Proof of Theorem 1.5.1

Let $F(x) = \binom{x+k-1}{k}^{-m}$. Then $\zeta_k(m) = \sum_{i=1}^{\infty} F(i)$. Since the polynomial $F(x)$ has k zeros $x = 0, 1, \ldots, k-1$,

$$F(x) = \frac{1}{\binom{x+k-1}{k}^m} = \sum_{j=0}^{k-1} \sum_{i=1}^{m} \frac{a_{i,j}}{(x+j)^i} \tag{1.14}$$

$$= \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j}}{(x+j)^i} + \frac{a_{i,k-1-j}}{(x+k-1-j)^i} \right)$$

for some rational numbers $a_{i,j}$ $(1 \le i \le m, 0 \le j \le k-1)$.

By Lemma 1.2.4, $1/F(x)$ is $(k-1)$-reflective if km is even, and $1/F(x)$ is anti-$(k-1)$-reflective, if km is odd. In other words, $F(x) = eF(-x-k+1)$ for almost all x (except zeros of denominator), where $e = \pm 1$. Thus, by (1.14) and based on the reflectivity property mentioned above,

$$\sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j}}{(x+j)^i} + \frac{a_{i,k-1-j}}{(x+k-1-j)^i} \right)$$

$$= e \sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j}}{(-x-k+1+j)^i} + \frac{a_{i,k-1-j}}{(-x-j)^i} \right)$$

or

$$\sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j} - e(-1)^i a_{i,k-1-j}}{(x+j)^i} + \frac{a_{i,k-1-j} - e(-1)^i a_{i,j}}{(x+k-1-j)^i} \right) = 0.$$

Therefore,

$$a_{i,j} - e(-1)^i a_{i,k-1-j} = 0$$

for all i, j $(1 \le i \le m, 0 \le j \le k-1)$. Hence, equation (1.14) can be rewritten as

$$F(x) = \frac{1}{\binom{x+k-1}{k}^m} = \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j}}{(x+j)^i} + e(-1)^i \frac{a_{i,j}}{(x+k-1-j)^i} \right).$$

Finally,

$$
\begin{aligned}
\zeta_k(m) &= \sum_{x=1}^{\infty} F(x) \\
&= \sum_{x=1}^{\infty} \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=1}^{m} \left(\frac{a_{i,j}}{(x+j)^i} + e(-1)^i \frac{a_{i,j}}{(x+k-1-j)^i} \right) \\
&= \frac{1}{2} \sum_{j=0}^{k-1} \sum_{i=1}^{m} a_{i,j} \left(\zeta(i) - \sum_{l=1}^{j} \frac{1}{l^i} + e(-1)^i \zeta(i) - e(-1)^i \sum_{l=1}^{k-1-j} \frac{1}{l^i} \right) \\
&= c_0 + \sum_{i=1}^{m} c_i (\zeta(i) + e(-1)^i \zeta(i)),
\end{aligned}
$$

for some rational constants c_i $(0 \leq i \leq m)$. Note that value $(\zeta(i) + e(-1)^i\zeta(i))$ vanishes if $e = 1$ and i is odd, or if $e = -1$ and i is even. $\square$

Presentations of infinite series as a linear combination of odd (even) values of zeta functions play an important role in studying the irrationality problems of zeta functions. See, for example, the proof of Rivoal [43] that the sequence $\zeta(3), \zeta(5), \ldots$ contains infinitely many irrational values, or result of Zudilin [52] that at least one of four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

1.5.3 Sums of reciprocals of powers of triangular numbers

Recall that

$$\zeta_2(m) = \sum_{x=1}^{\infty} \frac{2^m}{(x(x+1))^m}.$$

In this subsection we give exact presentation of $\zeta_2(m)$ as a combination of binomial coefficients and Bernoulli numbers. Namely, we prove the following result.

Theorem 1.5.2.

$$
\begin{aligned}
\sum_{x=1}^{\infty} \frac{1}{(x(x+1))^m} = (-1)^{m-1} \binom{2m-1}{m} \\
+ (-1)^m \sum_{i=1}^{\lfloor m/2 \rfloor} \binom{2m-2i-1}{m-1} (-1)^{i+1} \frac{(2\pi)^{2i}}{(2i)!} B_{2i}.
\end{aligned}
$$

To prove this theorem we need the following

Lemma 1.5.3.

$$\frac{1}{\binom{x+1}{2}^m} = 2^m \sum_{i=0}^{m-1} \binom{m+i-1}{i} \left((-1)^i \frac{1}{x^{m-i}} + (-1)^m \frac{1}{(x+1)^{m-i}} \right).$$

Proof. We use induction on m. For $m = 1$ we have

$$\frac{1}{\binom{x+1}{2}} = 2\left(\frac{1}{x} - \frac{1}{x+1}\right).$$

Assume that our statement holds for m and let us prove it for $m+1$. In our proof we need the following formulas (j is a positive integer):

$$\begin{aligned}\frac{1}{x^j(x+1)} &= \frac{1}{x^j} - \frac{1}{x^{j-1}} + \cdots + (-1)^j \frac{1}{x+1} \text{ and}\\ \frac{1}{x(x+1)^j} &= \frac{1}{x} - \frac{1}{x+1} - \cdots - \frac{1}{(x+1)^j},\end{aligned}$$

which follow from:

$$\begin{aligned}1 - (-x)^j &= (1+x)(1 + (-x) + \cdots + (-x)^{j-1}) \text{ and}\\ 1 - (x+1)^j &= -x(1 + (x+1) + \cdots + (x+1)^{j-1}).\end{aligned}$$

We have

$$\begin{aligned}\frac{1}{(x(x+1))^{m+1}} &= \frac{1}{(x(x+1))^m}\left(\frac{1}{x} - \frac{1}{x+1}\right)\\ &= \sum_{i=0}^{m-1}\binom{m+i-1}{i}\left(\frac{(-1)^i}{x^{m-i}} + \frac{(-1)^m}{(x+1)^{m-i}}\right)\left(\frac{1}{x} - \frac{1}{x+1}\right)\\ &= \sum_{i=0}^{m}\binom{m+i}{i}\left(\frac{(-1)^i}{x^{m-i+1}} + \frac{(-1)^{m+1}}{(x+1)^{m-i+1}}\right)\end{aligned}$$

(since $\sum_{j=0}^{i}\binom{m+j-1}{j} = \binom{m+i}{i}$). □

Proof of Theorem 1.5.2. By Lemma 1.5.3,

$$\begin{aligned}\sum_{x=1}^{\infty}\frac{1}{(x(x+1))^m} &= \sum_{x=1}^{\infty}\sum_{i=0}^{m-1}\binom{m+i-1}{i}\left(\frac{(-1)^i}{x^{m-i}} + \frac{(-1)^m}{(x+1)^{m-i}}\right)\\ &= \sum_{i=0}^{m-1}\binom{m+i-1}{i}\left(\sum_{x=1}^{\infty}\frac{(-1)^i}{x^{m-i}} + \sum_{x=1}^{\infty}\frac{(-1)^m}{(x+1)^{m-i}}\right).\end{aligned}$$

Thus, the last expression can be written as

$$\sum_{i=0}^{m-1}\binom{m+i-1}{i}\big((-1)^i\zeta(m-i)+(-1)^m(\zeta(m-i)-1)\big)$$

$$=(-1)^{m-1}\sum_{i=0}^{m-1}\binom{m+i-1}{i}$$

$$+\sum_{i=0,(m-i)\text{ even}}^{m-1}\binom{m+i-1}{i}(-1)^i 2\zeta(m-i)$$

$$=(-1)^{m-1}\binom{2m-1}{m}+(-1)^m 2\sum_{i=1}^{\lfloor m/2\rfloor}\binom{2m-2i-1}{m-1}\zeta(2i)$$

$$=(-1)^{m-1}\binom{2m-1}{m}+(-1)^m\sum_{i=1}^{\lfloor m/2\rfloor}\binom{2m-2i-1}{m-1}(-1)^{i+1}\frac{(2\pi)^{2i}}{(2i)!}B_{2i}.$$

□

1.6 Faulhaber coefficients and the coefficients of the polynomial $f_{2,m}(x)$

1.6.1 Duality between A- and B-Faulhaber coefficients

Recall that any number in the form $\binom{n}{2}=\frac{n(n-1)}{2}$ is called triangular (A000217 in [45]). In this section we study analogs of Faulhaber coefficients for power sums of triangular numbers. To stress similarities between Faulhaber coefficients $A_i^{(m)}$ and $B_i^{(m)}$ (see definitions below), we use special notations for formulas like $(A1)$, $(B1)$, $(A2)$, $(B2)$, etc.

We define Faulhaber coefficients [23, 35] as numbers $A_i^{(m)}$, such that

$$\sum_{i=0}^{N} i^{2m-1}=\frac{1}{2m}\sum_{i=0}^{m-1}A_i^{(m)}(N(N+1))^{m-i}. \qquad (A1)$$

These coefficients can be extended for all real numbers x by $A_i^{(x)}$. In fact, $A_i^{(x)}$ is a polynomial in x. In Sloane's OEIS [45], $A_i^{(m)}$ is referred to the sequences A093556, A093557 and

$$\widetilde{A}_i^{(m)}=\frac{m+1-i}{(2m+1)(m+1)}A_i^{(m+1)} \qquad (1.15)$$

is referred to the sequences A093558, A093559.

Knuth [35] has established the following properties of these coefficients,

$$A_0^{(x)}=1;\quad \sum_{j=0}^{k}\binom{x-j}{2k+1-2j}A_j^{(x)}=0,\quad k>0. \qquad (A2)$$

$$A_k^{(x)} = x(x-1)\cdots(x-k+1)\times q_k(x), \tag{A3}$$

where $q_k(x)$ is

- a polynomial of degree k
- with leading coefficient equal to $(2-2^{2k})B_{2k}/(2k)!$;
- and $q_k(k+1)=0$ if $k>0$.

Below we show that all zeros of this polynomial are real and distinct.

A recurrence formula due to Jacobi yields:

$$(2x-2k)(2x-2k+1)A_k^{(x)} + (x-k+1)(x-k)A_{k-1}^{(x)} = 2x(2x-1)A_k^{(x-1)}. \tag{A4}$$

Gessel and Viennot [23] obtained the following explicit formula:

$$A_k^{(m)} = (-1)^{m-k}\sum_j \binom{2m}{m-k-j}\binom{m-k+j}{j}\frac{m-k-j}{m-k+j}B_{m+k+j}, \tag{A5}$$

for $0\le k<m$.

In terms of determinants, this formula can be written as

$$A_k^{(m)} = (-1)^{(m-k)}\frac{1}{(m-1)\cdots(m-k)}\det\left|\binom{m-k+i}{2i-2j+3}\right|_{i,j=1,\ldots,k}. \tag{A6}$$

The last determinant has a combinatorial interpretation.

Theorem 1.6.1 (Gessel, Viennot, Theorem 14, [23])**.** *The number of sequences of positive integer numbers $a_1,\ldots,a_{3k}$ satisfying inequalities $a_{3i-2}<a_{3i-1}<a_{3i}$, $a_{3i+1}\le a_{3i-1}$, $a_{3i+2}\le a_{3i}$ and $a_{3i}\le m-k+i$, for all i, is equal to*

$$\det\left|\binom{m-k+i}{2i-2j+3}\right|_{i,j=1,\ldots,k}.$$

According to Theorem 1.2.1, the polynomial $f_{2,m}(x)$ is odd. Therefore we can consider the coefficients $B_i^{(m)}$ defined by

$$f_{2,m}(N)2^m = \sum_{i=0}^{N-1}(i(i+1))^m = \sum_{i=0}^{m} B_i^{(m)}N^{2m-2i+1}. \tag{B1}$$

The following relations are $B_i^{(m)}$ analogs of relations $(A2)-(A6)$,

$$B_0^{(m)} = \frac{1}{2m+1};\quad \sum_{j=0}^{k}\binom{m+2k-j}{2k+1-2j}(-1)^j B_j^{(m+j)} = 0. \tag{B2}$$

$$B_i^{(m)} = \frac{m(m-1)\cdots(m-i)}{2m-2i+1} \times h_i(m), \tag{B3}$$

where $h_i(m)$ is

- a polynomial of degree i
- with leading coefficient $(-1)^{i-1}(2-2^{2i})B_{2i}/(2i)!$;
- and $h_i(-1) = 0$ if $i > 0$.

Proof of these properties will be presented in Theorem 1.6.6 below. In fact, Theorem 1.6.2 tells us that $h_i(m) = q_i(i-m)$.

A recurrence formula for $B_i^{(m)}$ is given by

$$2(m-i)(2m-2i+1)B_i^{(m)} = 2m(2m-1)B_i^{(m-1)} + m(m-1)B_{i-1}^{(m-2)}. \tag{B4}$$

Coefficients $B_i^{(m)}$ are closely related to $A_k^{(m)}$.

Theorem 1.6.2.

$$B_i^{(m+i)} = (-1)^{i-1}\frac{m+i}{m(2m+1)}A_i^{(-m)} \tag{1.16}$$

Another formulation of Theorem 1.6.2 is given by

$$B_i^{(m)} = (-1)^{i-1}\frac{m}{(m-i)(2m-2i+1)}A_i^{(i-m)}.$$

Proof. From relations $(A4)$ and $(B4)$, we can see that the sequences $B_i^{(m+i)}$ and $A_i^{(-m)}$ satisfy the same recurrence relations and they have equal initial values $A_0^{(x)} = B_0^{(x)} = 1$. □

Knuth [35] proved that

$$\sum_{i=0}^{N-1} i^{2m} = (N - \frac{1}{2}) \sum_{i=0}^{m} \widetilde{A}_i^{(m)} (N(N-1))^{m-i},$$

with relation (1.15) between the coefficients $\widetilde{A}_i^{(m)}$ and $A_i^{(m)}$.

Corollary 1.6.3.

$$B_{m+i-1,i} = (-1)^i \widetilde{A}_i^{(-m)}. \tag{1.17}$$

The following is a general formula for $B_i^{(m)}$

$$B_i^{(m)} = \frac{1}{(2m-2i+1)} \sum_{k=0}^{2i} \binom{m}{2i-k} \binom{2m-2i+k}{k} B_k. \qquad (B5)$$

Let us rewrite this formula in terms of determinants

$$B_k^{(m)} = \frac{(-1)^{(k-m+1)}(k-1)!}{(2m-2k+1)(m-1)!} \det \left| \binom{m+i-2j+2}{2i-2j+3} \right|_{i,j=1,\ldots,k}, \qquad (B6)$$

which follows from $(A6)$ and Theorem 1.6.2.

By Theorem 16 of [23], we obtain one more combinatorial interpretation for the last determinant.

Theorem 1.6.4. *The number of sequences of positive integers* $a_1, \ldots, a_{3k}$ *such that* $a_{3i-2} \geq a_{3i-1} \geq a_{3i}$, $a_{3i+1} \geq a_{3i-1}$, $a_{3i+2} \geq a_{3i}$ *and* $a_{3i-2} \leq m-i$, *for all* i, *is equal to*

$$\det \left| \binom{m+i-2j+2}{2i-2j+3} \right|_{i,j=1,\ldots,k}.$$

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be two non-increasing sequences of nonnegative integers. *Plane partition of shape* $\lambda - \mu$ is a filling of the corresponding diagram with integers which are weakly decreased in every row and column. Such a diagram can be obtained from Ferrer diagram for λ by removing the diagram for μ (for details, see [23]) .

By Theorem 16 of [23], the number of plane partitions of shape $\lambda - \mu$ whose parts in row i are at most A_i and at least B_i, where $A_i \geq A_{i+1} - 1$ and $B_i \geq B_{i+1} - 1$, is equal to

$$\det \left| \binom{A_j - B_i + \lambda_i - \mu_j}{\lambda_i - \mu_j + j - i} \right|_{i,j=1,\ldots,k}.$$

Taking $\lambda_i = k+3-i$, $\mu_j = k-j$, $A_j = m-j$ and $B_i = 1$, we obtain the result. □

1.6.2 The polynomial $f_{2,m}(x)$

Set $\lambda_{m,i} = B_i^{(m)}/2^m$.

Theorem 1.6.5. *The polynomial* $f_{2,m}(x) = \sum_{i=0}^{m} \lambda_{m,i} x^{2m-2i+1}$ *has the following properties.*

(I) *It satisfies the equation*

$$f''_{2,m}(x) = m(2m-1) f_{2,m-1}(x) + \frac{m(m-1)}{4} f_{2,m-2}(x), \qquad m \geq 2, \qquad (1.18)$$

$$f_{0,2}(x) = x, \qquad f_{2,1}(x) = \binom{x+1}{3}.$$

***(II)** The following recurrence relation holds for coefficients* $\lambda_{m,i}$,

$$\lambda_{m,i} = \frac{m(2m-1)}{2(m-i)(2m-2i+1)}\lambda_{m-1,i} + \frac{m(m-1)}{8(m-i)(2m-2i+1)}\lambda_{m-2,i-1}, \quad (1.19)$$

where $0 < i < m$.

***(III)** The following general formula holds*

$$\lambda_{m,i} = \frac{1}{2^m(2m-2i+1)} \sum_{k=0}^{2i} \binom{m}{2i-k}\binom{2m-2i+k}{k} B_k. \quad (1.20)$$

Proof. **(I)** Since $\Delta f_{2,m}(x) = \binom{x+1}{2}^m$, we have

$$\Delta f''_{2,m}(x) = m(2m-1)\binom{x+1}{2}^{m-1} + \frac{m(m-1)}{4}\binom{x+1}{2}^{m-2},$$

and

$$f''_{2,m}(0) = 0.$$

Therefore, (1.18) is true.

(II) (1.19) is a consequence of (1.18).

(III) We have

$$\begin{aligned}
2^m f_{2,m}(N) &= \sum_{i=0}^{N-1}(i(i+1))^m = \sum_{i=0}^{N-1}\sum_{j=0}^{m}\binom{m}{j} i^{m+j} = \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{N-1} i^{m+j} \\
&= \sum_{j=0}^{m}\frac{1}{m+j+1}\binom{m}{j}\sum_{k=0}^{m+j}\binom{m+j+1}{k} B_k N^{m+j+1-k} \\
&= \sum_{k=1}^{2m+1} N^k \left(\sum_{j=0}^{m}\frac{1}{m+j+1}\binom{m}{j}\binom{m+j+1}{k} B_{m+j+1-k}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_{m,i} &= \frac{1}{2^m}\sum_{j\geq m-2i}^{m}\frac{1}{m+j+1}\binom{m}{j}\binom{m+j+1}{2m-2i+1} B_{j+2i-m} \\
&= \frac{1}{2^m}\sum_{k=0}^{2i}\frac{1}{2m-2i+k+1}\binom{m}{m+k-2i}\binom{2m-2i+k+1}{k} B_k \\
&= \frac{1}{2^m(2m-2i+1)}\sum_{k=0}^{2i}\binom{m}{2i-k}\binom{2m-2i+k}{k} B_k.
\end{aligned}$$

□

1.6.3 The polynomial part of $B_i^{(m)}$

Let us consider the function $\widetilde{h}_i(m)$ defined as

$$\begin{aligned}\widetilde{h}_i(m) &= (-1)^i(i+1)(2m-2i+1)\frac{B_i^{(m)}}{\binom{m}{i}}\\ &= (-1)^i(i+1)\sum_{k=0}^{2i}\frac{\binom{m}{2i-k}\binom{2m-2i+k}{k}}{\binom{m}{i}}B_k.\end{aligned}$$

In fact, $\widetilde{h}_i(m) = h_i(m)/(i+1)!$, where $h_i(m)$ is defined in relation $(B3)$.

For example,

$$\begin{aligned}\widetilde{h}_0(m) &= 1,\\ \widetilde{h}_1(m) &= \frac{m+1}{3},\\ \widetilde{h}_2(m) &= \frac{1}{60}(7m-6)(m+1),\\ \widetilde{h}_3(m) &= \frac{(m+1)(31m^2-97m+60)}{630},\\ \widetilde{h}_4(m) &= \frac{(m+1)(127m^3-833m^2+1606m-840)}{5040}.\end{aligned}$$

Analogous to Knuth's relation $(A3)$, polynomial parts of $B_i^{(m)}$ are interesting by themselves. Below we study properties of $\widetilde{h}_i(x)$.

Theorem 1.6.6. *Let $i \geq 0$.*

(I) *The function $\widetilde{h}_i(x)$ is a polynomial of degree i.*

(II) *The following formula holds for $\widetilde{h}_i(m)$*

$$\begin{aligned}&\widetilde{h}_i(m) \qquad (1.21)\\ &= (-1)^i(i+1)\left(\sum_{k=0}^{i}\frac{\binom{m-i}{i-k}\binom{2m-2i+k}{k}}{\binom{2i-k}{i}}B_k + \sum_{k=i+1}^{2i}\frac{\binom{i}{k-i}\binom{2m-2i+k}{k}}{\binom{m-2i+k}{k-i}}B_k\right)\\ &= (-1)^i\frac{(i+1)!}{(2i)!}\left(\sum_{k=0}^{i}p_k(m)\binom{2i}{k}B_k + \sum_{k=i+1}^{2i}q_k(m)\binom{2i}{k}B_k\right).\end{aligned}$$

Here, $p_k(m), q_k(m)$ are polynomials given by

$$p_k(m) = (m-i)\cdots(m-2i+k+1)(2m-2i+k)\cdots(2m-2i+1)$$

and

$$\begin{aligned}q_k(m) &= \frac{(2m-2i+k)\cdots(2m-2i+1)}{(m-2i+k)\cdots(m-i+1)}\\ &= 2^{k-i}(2m-2i+1)(2m-2i+3)\cdots\\ &\quad\times(2m-4i+2k+1)\ldots(2m-2i+k).\end{aligned}$$

(If $k = 2i$, the last term $(2m - 2i + k)$ in the product of $q_k(m)$ is cancelled).

***(III)** The following recurrence relation holds for $\widetilde{h}_i(x)$*

$$(2i + 1 - 2x)\widetilde{h}_i(x) = \frac{1}{2}(i + 1)\widetilde{h}_{i-1}(x - 2) - (2x - 1)\widetilde{h}_i(x - 1). \tag{1.22}$$

***(IV)** The initial value for $\widetilde{h}_i(x)$ is given by:*

$$\widetilde{h}_i(0) = (-1)^i(i + 1)B_{2i}. \tag{1.23}$$

***(V)** If x approaches ∞, then*

$$\widetilde{h}_i(x) \sim (-1)^i \frac{(i + 1)!}{(2i)!}(2 - 2^{2i})B_{2i}x^i. \tag{1.24}$$

***(VI)** If $i > 0$, then all zeros of the polynomial $\widetilde{h}_i(x)$ are real. Moreover, it has one negative zero $x_0 = -1$ and if $i > 1$, the other $i - 1$ zeros $x_1, \ldots, x_{i-1}$ are positive, distinct and satisfy the following inequalities*

$$0 < x_1 < 1,\ \ 2 < x_2 < 3,\ \ 3 < x_3 < 4,\ \ \ldots,\ \ i - 1 < x_{i-1} < i.$$

Proof. **(I)** Note that the terms $\binom{m}{2i-k}$ and $\binom{2m-2i+k}{k}$ are polynomials in m of degree $2i - k$ and k respectively. Hence, $\binom{m}{2i-k}\binom{2m-2i+k}{k}$ is a polynomial of degree $2i$ and it vanishes at $m = 0, 1, \ldots, i - 1$ (if $i > 0$). This means that it is divisible by $m(m-1)\cdots(m-i+1)$ or $\binom{m}{i}$. So, $\frac{\binom{m}{2i-k}\binom{2m-2i+k}{k}}{\binom{m}{i}}$ is a polynomial in m of degree i. Therefore, $\widetilde{h}_i(m)$ is a polynomial in m of degree i.

(II) To prove (1.21), we have

$$\begin{aligned}
\widetilde{h}_i(m) &= (-1)^i(i + 1)\sum_{k=0}^{2i} \frac{\binom{m}{2i-k}\binom{2m-2i+k}{k}}{\binom{m}{i}} B_k \\
&= (-1)^i(i + 1)\sum_{k=0}^{2i} \frac{i!(m - i)!}{(2i - k)!(m - 2i + k)!}\binom{2m - 2i + k}{k} B_k \\
&= (-1)^i(i + 1)\left(\sum_{k=0}^{i} \frac{\binom{m-i}{i-k}\binom{2m-2i+k}{k}}{\binom{2i-k}{i}} B_k + \sum_{k=i+1}^{2i} \frac{\binom{i}{k-i}\binom{2m-2i+k}{k}}{\binom{m-2i+k}{k-i}} B_k\right) \\
&= (-1)^i(i + 1)\times \\
&\sum_{k=0}^{i} \frac{(m - i)\cdots(m - 2i + k + 1)(2m - 2i + k)\cdots(2m - 2i + 1)i!}{k!(2i - k)!} B_k \\
&+ (-1)^i(i + 1)\sum_{k=i+1}^{2i} \frac{i!(2m - 2i + k)\cdots(2m - 2i + 1)}{k!(2i - k)!(m - 2i + k)\cdots(m - i + 1)} B_k \\
&= (-1)^i \frac{(i + 1)!}{(2i)!}\left(\sum_{k=0}^{i} p_k(m)\binom{2i}{k} B_k + \sum_{k=i+1}^{2i} q_k(m)\binom{2i}{k} B_k\right).
\end{aligned}$$

(III) We use (1.19) for $x = m > i$, where m is a positive integer. We obtain (1.22) for positive integers $x = m > i$. Since $\widetilde{h}_i(x)$ is a polynomial, recurrence relation (1.22) must hold for all real numbers x.

(IV) A substitution $m = 0$ in (1.21) gives us

$$\widetilde{h}_i(0) = (-1)^i(i+1)\left(\sum_{k=0}^{i}\frac{\binom{-i}{i-k}\binom{-2i+k}{k}}{\binom{2i-k}{i}}B_k + \sum_{k=i+1}^{2i}\frac{\binom{i}{k-i}\binom{-2i+k}{k}}{\binom{-2i+k}{k-i}}B_k\right)$$
$$= (-1)^i(i+1)\sum_{k=0}^{2i}\frac{1}{2}\binom{2i}{k}B_k = (-1)^i(i+1)B_{2i}.$$

Thus, (1.23) is proved.

(V) Let us calculate the leading coefficient A of $\widetilde{h}_i(m)$. By (1.21), we have

$$A = (-1)^i\frac{(i+1)!}{(2i)!}\sum_{k=0}^{2i}2^k\binom{2i}{k}B_k.$$

Let $B_n(x)$ be the Bernoulli polynomial,

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}.$$

It is well known that $B_n(1/2) = (2^{1-n}-1)B_n$. Therefore,

$$A = (-1)^i\frac{(i+1)!}{(2i)!}2^{2i}\sum_{k=0}^{2i}(1/2)^{2i-k}\binom{2i}{k}B_k$$
$$= (-1)^i\frac{(i+1)!}{(2i)!}2^{2i}B_{2i}(1/2)$$
$$= (-1)^i\frac{(i+1)!}{(2i)!}2^{2i}(2^{1-2i}-1)B_{2i} = (-1)^i\frac{(i+1)!}{(2i)!}(2-2^{2i})B_{2i},$$

which yields (1.24).

(VI) We have

$$\widetilde{h}_i(-1) = (-1)^i\frac{(i+1)}{\binom{-1}{i}}\sum_{k=0}^{2i}\binom{-1}{2i-k}\binom{-2-2i+k}{k}B_k$$
$$= (i+1)\sum_{k=0}^{2i}\binom{2i+1}{k}B_k = 0.$$

The part of the statement **(VI)** that relates to positive roots is evident for $i = 2$. Suppose that $i \geq 3$.

Let $\operatorname{sgn}(x) = -1$, if $x < 0$; $\operatorname{sgn}(x) = 0$, if $x = 0$; and $\operatorname{sgn}(x) = 1$, if $x > 0$.

Putting $m = 1$ in (1.22), we have

$$(2i-1)\widetilde{h}_i(1) = -\widetilde{h}_i(0).$$

Equation (1.23) gives us

$$\widetilde{h}_i(0) = (-1)^i (i+1) B_{2i}, \tag{1.25}$$

$$\widetilde{h}_i(1) = (-1)^{i+1} \frac{i+1}{2i-1} B_{2i}. \tag{1.26}$$

Therefore,

$$\operatorname{sgn}(\widetilde{h}(0)) = -\operatorname{sgn}(\widetilde{h}(1)).$$

Hence, there exists a real zero $x_1 \in (0,1)$ of $\widetilde{h}_i(x)$.

Putting $m = 2$ in (1.22), by (1.25) and (1.26), we have

$$\widetilde{h}_i(2) = \frac{1}{2i-3}\left(\frac{1}{2}(-1)^{i-1} i B_{2i-2} + 3(-1)^i \frac{i+1}{2i-1} B_{2i}\right).$$

By (1.13), for any positive integer i,

$$\operatorname{sgn}(B_{2i}) = (-1)^{i+1}.$$

So,

$$\operatorname{sgn}(\widetilde{h}_i(2)) = \operatorname{sgn}(\widetilde{h}_i(1)).$$

Therefore, relation (1.22) for $m = 3$ gives

$$\widetilde{h}_i(3) = \frac{1}{2i-5}\left((-1)^i \frac{11}{2}\frac{i}{2i-3} B_{2i-2} + 3(-1)^{i+1} \frac{i+1}{2i-1} B_{2i}\right).$$

Hence,

$$\operatorname{sgn}(\widetilde{h}(3)) = -\operatorname{sgn}(\widetilde{h}(2)).$$

Induction on i and on k shows that for any $k \in [2, i]$

$$\operatorname{sgn}(\widetilde{h}_i(k)) = (-1)^{i+k} \operatorname{sgn}(B_{2i}).$$

As we discussed above, this relation is also true for $k = 2, 3$.

By inductive hypothesis,

$$\operatorname{sgn}(\widetilde{h}_i(k-1)) = -\operatorname{sgn}(\widetilde{h}_{i-1}(k-2)).$$

Recurrence relation (1.22) yields

$$\begin{aligned}
\operatorname{sgn}(\widetilde{h}_i(k)) &= \operatorname{sgn}\left(\frac{1}{2i-5}\left(\frac{1}{2}\widetilde{h}_{i-1}(k-2) - (2k-1)\widetilde{h}_i(k-1)\right)\right) \\
&= \operatorname{sgn}(\widetilde{h}_{i-1}(k-2) - \widetilde{h}_i(k-1)) \\
&= \operatorname{sgn}((-1)^{i+k-3} B_{2i-2} - (-1)^{i+k-1} B_{2i}) \\
&= (-1)^{i+k} \operatorname{sgn}(B_{2i}).
\end{aligned}$$

It remains to note that

$$\operatorname{sgn}(\widetilde{h}(2)) = -\operatorname{sgn}(\widetilde{h}(3)) = \cdots = (-1)^{i-1}\operatorname{sgn}(\widetilde{h}(i-1)) = (-1)^{i}\operatorname{sgn}(\widetilde{h}(i)).$$

□

2 STIRLING NUMBERS ON MULTISETS

Stirling numbers are well-known and have many remarkable properties. By combinatorial definition:

Stirling numbers of the first kind $c(n,m)$ count the number of permutations of $(1,\ldots,n)$ having m cycles. They can be computed, e.g., by the following recurrence relation

$$c(n,m) = (n-1)c(n-1,m) + c(n-1,m-1)$$

or by a polynomial identity

$$x(x+1)\cdots(x+n-1) = \sum_{i=0}^{n} c(n,i)x^i.$$

Stirling numbers of the second kind $S(n,m)$ count the number of partitions of $\{1,\ldots,n\}$ into m blocks. They can also be computed, e.g., by the recurrence relation

$$S(n,m) = mS(n-1,m) + S(n-1,m-1)$$

or by a polynomial identity

$$x^n = \sum_{i=0}^{n} S(n,i)x(x-1)\cdots(x-i+1).$$

Moreover, these numbers are *dual*, they connected by orthogonal identity

$$\sum_{k\geq 0}(-1)^{n-k}c(n,k)S(k,m) = \delta_{n,m},$$

where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$, otherwise.

Many other properties can be found in [11, 25].

In this chapter we introduce Stirling numbers defined for multisets. As an analog of partitions we consider the *compositions* (ordered partitions, covers) of multisets by the usual sets or cycles and introduce the corresponding generalizations of Stirling numbers. In the further part we show that the similar argument can be applied for restricted partitions and permutations of sets. For instance, we change the repeated elements i^k of multiset by distinguished elements $i_1,\ldots,i_k$ and make the restriction that $i_1,\ldots,i_k$ cannot be in one set in a partition; or in one cycle in a permutation.

The compositions (ordered partitions) of multisets into multiset blocks were first studied by MacMahon [37] and after in the works of Riordan [42], Simion [44]. Broder [6] introduced generalization of Stirling numbers with special restrictions of elements in a set, called the r-Stirling numbers. Our case generalizes these numbers in the notion of restricted partitions [51] or permutations on sets. This type of

Stirling numbers of the second kind we can also meet (with another combinatorial interpretation) in the problem of boson normal ordering [38], which has the origin related to the composition of differential operators (or annihilation and creation in bosons terminology).

2.1 Definition

Let $\mathbf{m} = 1^{k_1} \cdots m^{k_m}$ be a multiset, where any element i repeats k_i times, for any $i = 1, \ldots, m$.

We define the union (merge) of two multisets by the rule:

$$1^{k_1} \cdots m^{k_m} \uplus 1^{l_1} \cdots m^{l_m} = 1^{k_1+l_1} \cdots m^{k_m+l_m}$$

and difference by:

$$1^{k_1} \cdots m^{k_m} \setminus 1^{l_1} \cdots m^{l_m} = 1^{\max(k_1-l_1,0)} \cdots m^{\max(k_m-l_m,0)}.$$

Definition 2.1.1 (Striling number of the second kind)**.** For a multiset $\mathbf{n}$, let $S(\mathbf{n}, k)$ be the number of ordered k-tuples $(S_1, \ldots, S_k)$ of nonempty sets (not necessarily disjoint) having the property that

$$S_1 \uplus \cdots \uplus S_k = \mathbf{n}.$$

So, $S(\mathbf{n}, k)$ expresses the number of ways to *cover* multiset $\mathbf{n}$ by an ordered k-tuple of sets.

Note that if $\mathbf{n} = \{1, \ldots, n\}$, then we obtain the usual ordered Stirling numbers of the second, because $S_1 \uplus \cdots \uplus S_k = \{1, \ldots, n\}$ implies that sets $S_1, \ldots, S_k$ form the ordered partition of $\{1, \ldots, n\}$.

Example 2.1.2. Suppose that $\mathbf{n} = 1^3 2^2$.

Then all possible 3-tuples of sets which cover $\mathbf{n}$ are:

$$(\{1\}, \{1,2\}, \{1,2\});$$
$$(\{1,2\}, \{1\}, \{1,2\});$$
$$(\{1,2\}, \{1,2\}, \{1\});$$

which gives $S(1^3 2^2, 3) = 3$.

All possible 4-tuples of sets which cover $\mathbf{n}$ are:

$$(\{1,2\}, \{1\}, \{1\}, \{2\}); (\{1\}, \{1,2\}, \{1\}, \{2\}); (\{1\}, \{1\}, \{1,2\}, \{2\});$$
$$(\{1,2\}, \{1\}, \{2\}, \{1\}); (\{1\}, \{1,2\}, \{2\}, \{1\}); (\{1\}, \{1\}, \{2\}, \{1,2\});$$
$$(\{1,2\}, \{2\}, \{1\}, \{1\}); (\{1\}, \{2\}, \{1,2\}, \{1\}); (\{1\}, \{2\}, \{1\}, \{1,2\});$$
$$(\{2\}, \{1,2\}, \{1\}, \{1\}); (\{2\}, \{1\}, \{1,2\}, \{1\}); (\{2\}, \{1\}, \{1\}, \{1,2\});$$

which gives $S(1^3 2^2, 4) = 12$.

2.2 Properties

2.2.1 Recurrence relations

Theorem 2.2.1.

$$S(\mathbf{n},k)=\sum_{S\subseteq\{1,\dots,n\}} S(\mathbf{n}\setminus S,k-1).$$

Proof. We may form the last set S_k in $(S_1,\dots,S_k)$ by any subset S of $\{1,\dots,n\}$. The number of corresponding $(k-1)$-tuples $(S_1,\dots,S_{k-1})$ is equal to $S(\mathbf{n}\setminus S,k-1)$, which gives the needed sum. □

Corollary 2.2.2.

$$S(n,k)=\frac{1}{k}\sum_{i=1}^{n}\binom{n}{i}S(n-i,k-1).$$

Theorem 2.2.3. *For* $\mathbf{n}=1^{l_1}\cdots n^{l_n}$ *the following recurrence relation holds*

$$S(\mathbf{n}\uplus(n+1)^{l_{n+1}},k)=\sum_{j=0}^{l_{n+1}}\binom{k}{j}\binom{k-j}{l_{n+1}-j}S(\mathbf{n},k-j), \tag{2.1}$$

where $S(1^k,k)=1$ *and* $S(1^k,i)=0$ *if* $i\neq k$.

Proof. If exactly j $(0\le j\le l_{p+1})$ sets of $(S_1,\dots,S_k)$ are equal to $\{n+1\}$, then it provides $\binom{k}{j}$ ways to choose these j sets, $\binom{k-j}{l_{n+1}-j}$ ways to put remained $(l_{n+1}-j)$ elements $n+1$ into the other $(k-j)$ sets, and the number of such $(k-j)$-tuples is equal to $S(\mathbf{n},k-j)$. Thus, for any j $(0\le j\le l_{n+1})$ we have the number of $\binom{k}{j}\binom{k-j}{l_{n+1}-j}S(\mathbf{n},k-j)$ ways to form the k-tuple which yields the needed recurrence. □

2.2.2 General formula for $S(\mathbf{n},k)$

Theorem 2.2.4. *For* $\mathbf{n}=1^{l_1}\cdots n^{l_n}$ *the following general formula holds*

$$S(\mathbf{n},k)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\binom{i}{l_1}\cdots\binom{i}{l_n}. \tag{2.2}$$

Proof. Let A_t be the set of k-tuples of sets $(S_1,\dots,S_k)$ such that S_t is empty (other sets can be empty or not); and all S_i $(1\le i\le k)$ are subsets of $\{1,\dots,n\}$. Note that

$$|A_{t_1}\cap\cdots\cap A_{t_i}|=\binom{k-i}{l_1}\cdots\binom{k-i}{l_n},$$

because for remained $k-i$ sets we can arbitrarily choose l_1 sets for 1^{l_1}, l_2 sets for 2^{l_2}, etc. Then, according to the inclusion-exclusion principle we get

$$\begin{aligned}|A_1 \cup \cdots \cup A_k| &= \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| \\ &+ \sum_{i<j<l} |A_i \cap A_j \cap A_l| - \cdots - (-1)^k |A_1 \cap \cdots \cap A_k| \\ &= \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} \binom{k-i}{l_1} \cdots \binom{k-i}{l_n}\end{aligned}$$

Note that

$$S(\mathbf{n}, k) = \binom{k}{l_1} \cdots \binom{k}{l_n} - |A_1 \cup \cdots \cup A_k|,$$

which gives the formula. □

Corollary 2.2.5. *For $l_1 = \cdots = l_n = 1$ we obtain the known formula for ordered Stirling numbers of the second kind:*

$$k! S(n, k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

2.2.3 Polynomial identity

Theorem 2.2.6. *For $\mathbf{n} = 1^{l_1} \cdots n^{l_n}$ the following polynomial identity holds*

$$\prod_{i=1}^{n} \binom{x}{l_i} = \sum_{i=0}^{l_1+\cdots+l_n} S(\mathbf{n}, i) \binom{x}{i}. \tag{2.3}$$

Proof. From the formula (2.2), the sequences $a_i = \binom{i}{l_1} \cdots \binom{i}{l_p}$ and $b_k = S(\mathbf{n}, k)$ are related by the binomial transform [36], i.e.,

$$b_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} a_i.$$

Thus, the inverse transform yields

$$a_k = \sum_{i=0}^{k} \binom{k}{i} b_i,$$

or

$$\binom{k}{l_1} \cdots \binom{k}{l_n} = \sum_{i=0}^{k} \binom{k}{i} S(\mathbf{n}, i).$$

The last identity can be considered as polynomial identity and positive integer k can be changed to any x. □

Combinatorial proof. If we prove that polynomial identity (2.3) is true for any positive integer $x = m$, then as a result it holds for any (complex) x.

Let us consider the set

$$\overline{\mathbf{S}} = \{(S_1, \dots, S_m) | S_i \subseteq \{1, \dots, n\}, 1 \le i \le m, S_1 \uplus \dots \uplus S_m = \mathbf{n}\}.$$

Then, firstly,

$$|\overline{\mathbf{S}}| = \binom{m}{l_1} \cdots \binom{m}{l_n},$$

because for any $1 \le i \le n$ we can arbitrarily put l_i repetitions of element i in any i of m sets $S_1, \dots, S_m$ in $\binom{m}{l_i}$ ways.

On the other hand,

$$|\overline{\mathbf{S}}| = \sum_{i=0}^{l_1+\dots+l_n} S(\mathbf{n}, i)\binom{m}{i},$$

because for any $0 \le i \le l_1 + \dots + l_n$ if $m - i$ sets of $S_1, \dots, S_m$ are empty, then the number of ways to form (cover) $\mathbf{n}$ by other i sets is equal (by definition) to $S(\mathbf{n}, i)$; and the number of ways to choose $m - i$ empty sets is equal to $\binom{m}{m-i} = \binom{m}{i}$. □

2.2.4 Composition of differential operators

For positive integer k let us consider the differential operator

$$D_k = \frac{x^k}{k!} \cdot \frac{d^k}{dx^k}.$$

If $k = 1$, then $D = D_1 = x\frac{d}{dx}$ has good properties. For instance, it is well known (see, e.g., [11]) that

$$D^n = \sum_{k=1}^{n} S(n, k)x^k \frac{d^k}{dx^k} = \sum_{k=1}^{n} S(n, k)k!D_k. \tag{2.4}$$

Note that operator D_k is commutative and associative:

$$D_a D_b = D_b D_a,$$

$$D_a(D_b D_c) = (D_a D_b)D_c.$$

It implies from the following

Lemma 2.2.7.

$$D_a D_b = \sum_{i=0}^{\min(a,b)} \binom{a+b-i}{a-i, b-i} D_{a+b-i},$$

where $\binom{a+b-i}{a-i,b-i} = \frac{(a+b-i)!}{(a-i)!(b-i)!i!}$ *is a multinomial coefficient.*

Proof. Using the Leibniz formula for the n-th derivative

$$d^n(uv) = \sum_{i=0}^{n} \binom{n}{i} d^i(u) d^{n-i}(v),$$

where $d(u) = \frac{du}{dx}$, we have

$$\begin{aligned} D_a D_b &= \frac{x^a}{a!} d^a \left(\frac{x^b}{b!} d^b \right) = \frac{x^a}{a!} \sum_{i=0}^{a} \binom{a}{i} d^i(x^b) d^{a-i}(d^b) \\ &= \frac{1}{a!b!} \sum_{i=0}^{\min(a,b)} \binom{a}{i} b \cdots (b-i+1) x^{a+b-1} d^{a+b-1} \\ &= \sum_{i=0}^{\min(a,b)} \binom{a+b-i}{a-i, b-i} D_{a+b-i}. \end{aligned}$$

□

In the next theorem we show that property similar to (2.4) holds for Stirling numbers on multisets.

Theorem 2.2.8. *For* $\mathbf{n} = 1^{l_1} \cdots n^{l_n}$ *the following identity holds.*

$$D_{l_1} D_{l_2} \cdots D_{l_n} = \sum_{i=0}^{l_1+\cdots+l_n} S(\mathbf{n}, i) \frac{x^i}{i!} \frac{d^i}{dx^i} = \sum_{i=0}^{l_1+\cdots+l_n} S(\mathbf{n}, i) D_i. \tag{2.5}$$

Proof. By induction on n. For $n = 1$ we have $S(1^{l_1}, i) = 1$ if $i = l_1$ and $S(1^{l_1}, i) = 0$, otherwise. Then

$$D_{l_1} = \sum_{i=0}^{l_1} S(1^{l_1}, i) D_i = S(1^{l_1}, l_1) D_{l_1}.$$

If (2.5) is true for n, then for $n+1$ using Lemma 2.2.7 and recurrence relation

(2.1) we have

$$\begin{aligned}
D_{l_1}D_{l_2}\cdots D_{l_{n+1}} = D_{l_{n+1}}D_{l_1}\cdots D_{l_n} &= D_{l_{n+1}}\sum_{i=0}^{l_1+\cdots+l_n} S(\mathbf{n},i)D_i \\
&= \sum_{i=0}^{l_1+\cdots+l_n} S(\mathbf{n},i)D_{l_{n+1}}D_i \\
&= \sum_{i=0}^{l_1+\cdots+l_n} S(\mathbf{n},i)\sum_{j\geq 0}\binom{l_{n+1}+i-j}{l_{n+1}-j,i-j}D_{l_{n+1}+i-j} \\
&= \sum_{m=0}^{l_1+\cdots+l_{n+1}} D_m\sum_{j=0}^{l_{n+1}}\binom{m}{j,m-l_{n+1}}S(\mathbf{n},m-j) \\
&= \sum_{m=0}^{l_1+\cdots+l_{n+1}} D_m S(\mathbf{n},m).
\end{aligned}$$

□

2.3 Generalization of Stirling numbers of the first kind

Definition 2.3.1 (Stirling numbers of the first kind)**.** For a multiset $\mathbf{n}$, let $c(\mathbf{n},k)$ be the number of ordered k-tuples $(C_1,\ldots,C_k)$ of nonempty cycles of distinct elements having the property that

$$C_1 \uplus \cdots \uplus C_k = \mathbf{n}.$$

Example 2.3.2. Suppose that $\mathbf{n} = 1^2 2^2 3^1$.

Then possible 2-tuples of cycles which cover $\mathbf{n}$ are:

$$((123),(12));$$

$$((12),(123));$$

$$((132),(12));$$

$$((12),(122));$$

which gives $c(1^2 2^2 3^1, 2) = 4$.

Theorem 2.3.3.

$$c(\mathbf{n},k) = \sum_{S\subseteq\{1,\ldots,n\}} (|S|-1)!c(\mathbf{n}\setminus S, k-1).$$

Proof. We may form the last cycle C_k in $(C_1,\ldots,C_k)$ by any subset S of $\{1,\ldots,n\}$ in $(|S|-1)!$ ways. The number of corresponding $(k-1)$-tuples $(C_1,\ldots,C_{k-1})$ is equal to $c(\mathbf{n}\setminus S,k-1)$, which gives the needed sum. □

2.4 Restricted partitions of sets

Notice that covers of multisets can be viewed as partitions of sets with some restrictions. In this section we introduce the connections and definitions for objects of that kind.

Suppose that $k_1, \dots, k_n$ be positive integers and $K = k_1 + \dots + k_n$.

Let $\left\{ {(k_1,\dots,k_n) \atop m} \right\}$ be the number of partitions of the set $\{1, 2, \dots, K\}$ into m blocks such that

- first k_1 elements ($1, \dots, k_1$) are in distinct blocks;
- next k_2 elements ($k_1 + 1, \dots, k_2$) are in distinct blocks
- and so on;
- last k_n elements ($k_1 + \dots + k_{n-1} + 1, \dots, k_1 + \dots + k_n$) are in distinct blocks.

In fact, it is easy to state that

$$\left\{ {(k_1, \dots, k_n) \atop m} \right\} = S(1^{k_1} \cdots n^{k_n}, m) \frac{k_1! \cdots k_n!}{m!}$$

Example 2.4.1. $\left\{ {(2,2,1) \atop 2} \right\} = 4$; the pairs of lements $1, 2$ and $3, 4$ cannot be in one block and corresponding partitions are:

$$\{1, 3\}, \{2, 4, 5\};$$

$$\{1, 3, 5\}, \{2, 4\};$$

$$\{1, 4\}, \{2, 3, 5\};$$

$$\{1, 4, 5\}, \{2, 3\}.$$

The number $\left\{ {(k_1,\dots,k_n) \atop m} \right\}$ can be considered as a generalization of Stirling number of the second kind because $\left\{ {(1,\dots,1) \atop m} \right\} = S(n, m)$, where $S(n, m)$ is Stirling number of the second kind. The case $\left\{ {(r,1,\dots,1) \atop m} \right\}$ is a generalization of r-Stirling number of the second kind introduced by Broder [6].

Note that the value $\left\{ {(k_1,\dots,k_n) \atop m} \right\}$ will remain the same if we arbitrarily permute the numbers $(k_1, \dots, k_n)$.

Let $a_{(j)} = a(a-1)\dots(a-j+1)$ be a falling factorial.

The recurrence relations are shown in

Theorem 2.4.2. *The following properties are true:*

- *if* $n = 1$ *then* $\left\{ {(k) \atop k} \right\} = 1$ *(and* $\left\{ {(k) \atop i} \right\} = 0$ *if* $i \neq k$*);*

- *and the recurrence relations*

$$\left\{ \begin{matrix} (k_1, \dots, k_n) \\ m \end{matrix} \right\} = \left\{ \begin{matrix} (k_1, \dots, k_n - 1) \\ m-1 \end{matrix} \right\} + (m - k_n + 1) \left\{ \begin{matrix} (k_1, \dots, k_n - 1) \\ m \end{matrix} \right\}; \tag{2.6}$$

or

$$\left\{ \begin{matrix} (k_1, \dots, k_n, k_{n+1}) \\ m \end{matrix} \right\} = \sum_{j=0}^{k_{n+1}} \binom{k_{n+1}}{j} (m + j - k_{n+1})_{(j)} \left\{ \begin{matrix} (k_1, \dots, k_n) \\ m + j - k_{n+1} \end{matrix} \right\}. \tag{2.7}$$

Proof. Equation (2.6). Consider the last element K. It can form one separate block and this provides $\left\{ \begin{smallmatrix} (k_1, \dots, k_n - 1) \\ m-1 \end{smallmatrix} \right\}$ ways; or the last element is contained in some block with other elements and this gives $(m - k_n + 1) \left\{ \begin{smallmatrix} (k_1, \dots, k_n - 1) \\ m \end{smallmatrix} \right\}$ because there are only $(m - k_n + 1)$ proper blocks which do not contain elements $\{k_1 + \dots + k_{n-1} + 1, \dots, k_1 + \dots + k_n - 1\}$.

Equation (2.7). Consider the group of last k_{n+1} elements. Note that for each $j = 0, \dots, k_{n+1}$ exactly j elements can share common blocks with other elements and thus, other $(k_{n+1} - j)$ elements form $(k_{n+1} - j)$ separate blocks. Therefore, it clearly gives that

$$\left\{ \begin{matrix} (k_1, \dots, k_n, k_{n+1}) \\ m \end{matrix} \right\} = \sum_{j=0}^{k_{n+1}} \binom{k_{n+1}}{j} (m + j - k_{n+1})_{(j)} \left\{ \begin{matrix} (k_1, \dots, k_n) \\ m + j - k_{n+1} \end{matrix} \right\}.$$

□

The general formula is present in the next

Theorem 2.4.3.

$$\left\{ \begin{matrix} (k_1, \dots, k_n) \\ m \end{matrix} \right\} = \frac{1}{m!} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} i_{(k_1)} \dots i_{(k_n)}.$$

Proof. Similar to Theorem 2.2.4. □

For establishing the properties with composition of differential operators let us redefine now the operator D_k as

$$D_k = x^k D^k = x^k \frac{d^k}{dx^k}.$$

Theorem 2.4.4.

$$D_{k_1} D_{k_2} \dots D_{k_n} = \sum_{i=0}^{K} \left\{ \begin{matrix} (k_1, \dots, k_n) \\ i \end{matrix} \right\} x^i D^i = \sum_{i=0}^{K} \left\{ \begin{matrix} (k_1, \dots, k_n) \\ i \end{matrix} \right\} D_i, \tag{2.8}$$

Proof. By induction on n and using the recurrence relation (2.7), similar to the proof of Theorem 2.2.8. □

Note that numbers $\left\{ {(k_1,\ldots,k_n) \atop i} \right\}$ were established with another combinatorial interpretation in the problem of boson normal ordering [38].

2.4.1 The case for Stirling numbers of the first kind

Similarly, let $\left[{(k_1,\ldots,k_n) \atop m} \right]$ be the number of permutations of $(1, 2, \ldots, K)$ with m cycles such that

- first k_1 elements ($1, \ldots, k_1$) are in distinct cycles;
- next k_2 elements ($k_1 + 1, \ldots, k_2$) are in distinct cycles
- and so on;
- last k_n elements ($k_1 + \cdots + k_{n-1} + 1, \ldots, k_1 + \cdots + k_n$) are in distinct cycles.

Then we pose the following problem concerning generalized Stirling numbers of the first kind.

Problem. *The following polynomial identity holds*

$$(x + K - 1)_{(K)} = \sum_{i=0}^{K} \left[{(k_1, \ldots, k_n) \atop i} \right] (x + k_1 - 1)_{(k_1)} \cdots (x + k_j - 1)_{(k_j)} \quad (2.9)$$
$$\times (x + (i - k_1 - \cdots - k_j) - 1)_{((i-k_1-\cdots-k_j))},$$

where j *is an index for which* $k_1 + \cdots + k_{j+1} \geq i > k_1 + \cdots + k_j$.

Then the following orthogonality relation is a consequence of (2.9)

$$\sum_{i \geq 0} (-1)^{K-i} \left[{(k_1, \ldots, k_n) \atop i} \right] \left\{ {(k_1, \ldots, k_j, i - k_1 - \cdots - k_j)) \atop m} \right\} = \delta_{n,m},$$

where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0,$ otherwise.

3 STIRLING PERMUTATIONS

Well-known identity

$$\sum_{m=0}^{\infty} m^n x^m = \frac{\sum_{i=1}^{n} A_{n,i} x^i}{(1-x)^{n+1}}$$

relates the function m^n with Eulerian numbers $A_{n,i}$ (e.g., [25]). Combinatorial meaning of $A_{n,i}$ known as the number of permutations $(a_1, \dots, a_n)$ of $(1, \dots, n)$ having exactly $i-1$ descents, i.e., indices j $(1 \le j \le n-1)$ for which $a_j > a_{j+1}$.

In a similar way, Stirling permutations were introduced by Gessel and Stanley [22], concerning multipermutations of $(1,1,2,2,\dots,n,n)$, which have the property that only greater elements can be placed between any two equal elements in a permutation. Their result shows an interesting connection between Eulerian numbers defined on these permutations and Stirling numbers:

If now $A_{n,i}$ denotes the number of Stirling permutations $(a_1, \dots, a_{2n})$ of $(1,1,2,2,\dots,n,n)$ that have exactly $i-1$ descents, then

$$\sum_{k=0}^{\infty} S(n+k,k) x^k = \frac{\sum_{i=1}^{n} A_{n,i} x^i}{(1-x)^{2n+1}},$$

where $S(n+k,k)$ is a Stirling number of the second, which counts the number of partitions of $\{1, \dots, n+k\}$ into k nonempty blocks.

In the present chapter we explore combinatorial properties in a general case, by considering Stirling permutations of the multiset $\{1^{k_1}, \dots, n^{k_n}\}$.

Let us fix the numbers $k_1, \dots, k_n$, set $K = k_1 + \dots + k_n$ and similarly define

(i) Eulerian numbers $A_{n,i}$ as the number of Stirling permutations $(a_1, \dots, a_K)$ of $\{1^{k_1}, \dots, n^{k_n}\}$ that have exactly $i-1$ descents;

(ii) and numbers $B_{K,m}$ obtained from the generating function:

$$\sum_{m=0}^{\infty} B_{K,m} x^m = \frac{\sum_{i=1}^{n} A_{n,i} x^i}{(1-x)^{K+1}}.$$

The main object of our study are the numbers $B_{K,m}$.

We obtain (Theorem 3.4.4) that $B_{K,m}$ have a compact recurrence relation:

$$B_{K,m} = \begin{cases} B_{K,m-1} + B_{K-1,m}, & \text{if } k_n > 1; \\ mB_{K-1,m}, & \text{if } k_n = 1. \end{cases}$$

We establish several combinatorial properties of $B_{K,m}$ with relations to the barred permutations (as they were originally considered by Gessel and Stanley [22]), weighted lattice paths, bipartite multigraphs, P-partitions.

Next, we show that generating function $\sum_m B_{K,m} x^m$ corresponds to the triangular generating function $\sum_m S_{(k_1,\dots,k_n)}(n+m,m)x^m$ or that $B_{K,m} = S_{(k_1,\dots,k_n)}(n+m,m)$. In a general case, the class $S_{(k_1,\dots,k_{p-q})}(p,q)$ depends on a sequence $(k_1,\dots,k_{p-q})$ and enumerates some $(l+1)$-tuple $(\pi_0,\dots,\pi_l)$, where $\pi_1,\dots,\pi_l$ are partitions of set $\{1,\dots,p\}$ into q blocks, π_0 is a special family of multisets under some additional constraints and properties given for $\pi_0,\dots,\pi_l$ (Definition 3.5.10).

As we derive it further, in particular cases for $k_1,\dots,k_n$, the general construction of $S_{(k_1,\dots,k_{p-q})}(p,q)$ can be refined with nice combinatorial interpretations. The numbers $B_{K,m}$ can be specified, e.g., as the function m^K, the sum of powers of consecutive integers $\sum_{i=1}^m i^n$, the binomial coefficient $\binom{K+m-1}{K}$, the Stirling number $S(n+m,m)$, the central factorial number $T(2n+2m,2m)$ (defined by Riordan [42]). We also introduce Stirling numbers of odd type (for $(k_1,\dots,k_n) = (1,2,\dots,2)$) and a natural generalization of the central factorial numbers (for $(k_1,\dots,k_n) = (1,\dots,1,2,\dots,1,\dots,1,2)$).

Background

The case $k_1 = \dots = k_n = k$ was studied by Klingsberg and Schmalzried [33], Park [39, 40] has explored and denoted such permutations of $\{1^r,\dots,n^r\}$ by r-multipermutations. Particularly, they obtained properties and combinatorial interpretation for $B_{kn,m}$ in the theory of P-partitions of a poset. In this chapter we generalize that construction for any multiset.

Bóna [2] has considered the Eulerian polynomial $\sum_i A_{n,i}x^i$ over Stirling permutations of $\{1^2,\dots,n^2\}$ and proved that it has only real zeros. Brenti [5] proved this fact in a general case $\{1^{k_1},\dots,n^{k_n}\}$. Haglund and Visontai [28] have considered Eulerian multivariate polynomial and proved its stability property, which implies the result on real zeros.

Janson (et al.) [31, 32] have shown the connection of generalized Stirling permutations with the plane recursive trees and an urn model.

Egge [19] has presented a similar theory of the Legendre–Stirling permutations. Gessel et al. [24] have applied the theory of P-partitions for the Jacobi-Stirling numbers.

3.1 Definitions, notation

For fixed positive integers $k_1,\dots,k_n,$ (throughout the chapter we suppose that these numbers are fixed) consider a multiset $\{1^{k_1},2^{k_2},\dots,n^{k_n}\}$, where every element i $(1 \le i \le n)$ has k_i repetitions and $K = k_1 + \dots + k_n$.

Definition 3.1.1. A permutation $\sigma = \sigma(1)\dots\sigma(K)$ of $\{1^{k_1},2^{k_2},\dots,n^{k_n}\}$ is called *Stirling permutation* [1] if for all indices i,t,j $(1 \le i < t < j \le K)$ with $\sigma(i) = \sigma(j),$ we have $\sigma(t) \ge \sigma(i)$.

[1] Or, a generalized Stirling permutation

Example 3.1.2. 1122332 is a Stirling permutation and 1212233 is not.

Example 3.1.3. All Stirling permutations of $\{1^1, 2^2, 3^3\}$ are

$$122333, 123332, 133322, 333122,$$

$$221333, 223331, 233321, 333221.$$

Proposition 3.1.4. *Let $\mathcal{GS}_n$ be the set of Stirling permutations of $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$. Then*

$$|\mathcal{GS}_n| = (k_{n-1} + \ldots + k_1 + 1)(k_{n-2} + \ldots + k_1 + 1) \cdots (k_1 + 1). \tag{3.1}$$

Proof. By induction on n. There is only one Stirling permutation of 1^k. For the induction step notice that by definition, the elements n^{k_n} cannot be separated. Thus, we can put the entire block n^{k_n} into any $(k_{n-1} + \cdots + k_1 + 1)$ spaces of a Stirling permutation of $\mathcal{GS}_{n-1}$. This gives that

$$\begin{aligned} |\mathcal{GS}_n| &= (k_{n-1} + \cdots + k_1 + 1)|\mathcal{GS}_{n-1}| \\ &= (k_{n-1} + \ldots + k_1 + 1)(k_{n-2} + \ldots + k_1 + 1) \cdots (k_1 + 1). \end{aligned}$$

□

Definition 3.1.5 (Eulerian numbers)**.** For a permutation $\sigma = \sigma(1) \cdots \sigma(K) \in \mathcal{GS}_n$, i is a *descent* index if $i = K$ or $\sigma(i) > \sigma(i+1)$ for $i < K$ and a descent number $\text{des}(\sigma)$ is a number of descent indices of σ. Eulerian number $A_{n,i}$ expresses the number of Stirling permutations having exactly i descents, i.e.,

$$A_{n,i} = |\{\sigma \in \mathcal{GS}_n | \text{des}(\sigma) = i\}|.$$

Definition 3.1.6 (The B numbers)**.** The generating function $G_n(x)$ and the numbers $B_{K,m}$ are defined by

$$G_n(x) = \frac{\sum_{i=1}^{n} A_{n,i} x^i}{(1-x)^{K+1}} = \sum_{m=0}^{\infty} B_{K,m} x^m. \tag{3.2}$$

3.2 Inversion codes for Stirling Permutations

Definition 3.2.1. For a Stirling permutation $\sigma \in \mathcal{GS}_n$, let

$$\text{cinv}(\sigma) = (\text{cinv}_1, \dots, \text{cinv}_n)$$

be the *inversion code*, such that for all i $(1 \le i \le n)$, cinv_i is equal to the number of elements of σ which are less than i, and placed to the right of i in a permutation.

Example 3.2.2.

$$\mathrm{cinv}(233211) = (0, 2, 3),$$

$$\mathrm{cinv}(1123444321) = (0, 2, 2, 3).$$

Proposition 3.2.3. *There is a bijection between the set of all sequences* $(\mathrm{cinv}_1, \ldots, \mathrm{cinv}_n)$ *which satisfy the properties:*

$$\mathrm{cinv}_1 = 0, 0 \le \mathrm{cinv}_2 \le k_1, \ldots, 0 \le \mathrm{cinv}_n \le k_1 + \cdots + k_{n-1},$$

and the set $\mathcal{GS}_n$ *of* $(k_1+1)\cdots(k_1+\cdots+k_{n-1}+1)$ *Stirling permutations.*

Proof. The bijection is very natural. First, it is easy to see that for any $\sigma \in \mathcal{GS}_n$, $\mathrm{cinv}(\sigma)$ forms the sequence satisfying all the above inequalities.

For the inverse, we may construct a permutation with the given sequence $(\mathrm{cinv}_1, \ldots, \mathrm{cinv}_n)$:

At each step i $(1 \le i \le n)$ put the block i^{k_i} before exactly cinv_i elements of a current permutation. It is always possible, because at the beginning of any step i, the current permutation has exactly $k_1 + \cdots + k_{i-1}$ elements (and 0 elements at the first step). □

Corollary 3.2.4.

$$\sum_{\sigma \in \mathcal{GS}_n} q_0^{\mathrm{cinv}_1} \cdots q_{n-1}^{\mathrm{cinv}_n} = (1 + q_1 + \cdots + q_1^{k_1}) \cdots (1 + q_{n-1} + \cdots + q_{n-1}^{k_1+\cdots+k_{n-1}}).$$

3.3 Some properties of Eulerian numbers $A_{n,i}$

Proposition 3.3.1. *The numbers* $A_{n,i}$ *satisfy the following recurrence relation*

$$A_{n,i} = i \cdot A_{n-1,i} + (k_1 + \ldots + k_{n-1} + 1 - (i-1)) \cdot A_{n-1,i-1} \tag{3.3}$$

with initial value $A_{1,1} = 1$.

Proof. The block n^{k_n} can be inserted:

(i) in any of i descents of $A_{n-1,i}$ permutations without producing a new descent, or

(ii) create a new descent at any of $(k_1 + \ldots + k_{n-1} + 1 - (i-1))$ positions (with no descent) of $A_{n-1,i-1}$ permutations. □

Theorem 3.3.2 ([5, 28])**.** *Eulerian polynomial* $\sum_{i=1}^{n} A_{n,i} x^i$ *over Stirling permutations has only real zeros.*

This theorem can also be proved using the same argument as in [2] for case $k_1 = \cdots = k_n = 2$ and using the recurrence (3.3).

3.4 Properties of the B numbers

Proposition 3.4.1. *The numbers $B_{K,m}$ and $A_{n,i}$ are related by:*

$$B_{K,m} = \sum_{i\geq 1} \binom{m-i+K}{K} A_{n,i}.$$

Proof. We have

$$\begin{aligned} G_n(x) &= \sum_{m=0}^{\infty} B_{K,m}x^m = \frac{\sum_{i=1}^{n} A_{n,i}x^i}{(1-x)^{K+1}} = \left(\sum_{i=1}^{n} A_{n,i}x^i\right) \sum_{j=0}^{\infty} \binom{K+j}{K} x^j \\ &= \sum_{m=0}^{\infty} x^m \sum_{i\geq 1} A_{n,i} \binom{K+m-i}{K}. \end{aligned}$$

□

Proposition 3.4.2. *The function $G_n(x)$ satisfies the following differential equation:*

$$G_n(x) = \frac{x}{(1-x)^{k_n-1}} \frac{d(G_{n-1}(x))}{dx}. \tag{3.4}$$

Proof. By recurrence relation (3.3), we have

$$\begin{aligned} G_n(x) &= \frac{\sum_{i=1}^{n} A_{n,i}x^i}{(1-x)^{K+1}} \\ &= \frac{x}{(1-x)^{k_n-1}} \frac{\sum_{i=1}^{n}(iA_{n-1,i}x^{i-1} + (k_1+\cdots+k_{n-1}+2-i)A_{n-1,i-1}x^{i-1})}{(1-x)^{k_1+\cdots+k_{n-1}+2}} \\ &= \frac{x}{(1-x)^{k_n-1}} d\left(\frac{\sum_{i=1}^{n-1} A_{n-1,i}x^i}{(1-x)^{k_1+\cdots+k_{n-1}+1}}\right)/dx \\ &= \frac{x}{(1-x)^{k_n-1}} \frac{d(G_{n-1}(x))}{dx}. \end{aligned}$$

□

Theorem 3.4.3. *The following recurrence relations hold for numbers $B_{K,m}$ if $k_n > 1$:*

$$B_{K,m} = \sum_{i=0}^{m} iB_{K-k_n,i} \binom{k_n+m-i-2}{m-i}, \tag{3.5}$$

$$B_{K-k_n,m} = \frac{1}{m} \sum_{j=0}^{k_n-1} B_{K,j}(-1)^{m-j} \binom{k_n-1}{m-j}. \tag{3.6}$$

Proof. From equation (3.4), we have

$$\sum_{m=0}^{\infty} B_{K,m}x^m = \frac{1}{(1-x)^{k_n-1}} \sum_{j=0}^{\infty} jB_{k-k_n,j}x^j.$$

So, equation (3.5) implies on expansion $\frac{1}{(1-x)^{k_n-1}} = \sum_{i=0}^{\infty} \binom{k_n+i-2}{i} x^i$ and equation (3.6) from $(1-x)^{k_n-1} = \sum_{i=0}^{k_n-1} (-1)^i \binom{k_n-1}{i} x^i$. ☐

Theorem 3.4.4. *For any positive integer* m*, we have*
$B_{0,m} = 1, B_{i,0} = 0$*; and*

$$B_{K,m} = \begin{cases} B_{K,m-1} + B_{K-1,m}, & \text{if } k_n > 1; \\ mB_{K-1,m}, & \text{if } k_n = 1. \end{cases} \tag{3.7}$$

(Here we suppose that if $k_n > 1$*, then* $B_{K-1,m} = B_{k_1+\ldots+(k_n-1),m}$ *is applied for the sequence* $(k_1, \ldots, k_{n-1}, k_n - 1)$ *and if* $k_n = 1$*, then* $B_{K-1,m} = B_{k_1+\ldots+k_{n-1},m}$ *for the sequence* $(k_1, \ldots, k_{n-1})$*)*

Proof. In relation (3.5) setting $m \to m+1$ gives the following:

$$B_{K,m+1} = \sum_i iB_{K-k_n,i} \binom{k_n + m - i - 1}{m + 1 - i}.$$

We have

$$\begin{aligned} B_{K,m+1} - B_{K,m} &= \sum_i iB_{K-k_n,i} \left(\binom{k_n + m - i - 1}{m + 1 - i} - \binom{k_n + m - i - 2}{m - i} \right) \\ &= \sum_i iB_{K-k_n,i} \binom{(k_n - 1) + (m+1) - i - 2}{(m+1) - i} = B_{K-1,m+1} \end{aligned}$$

and thus the following recurrence relation:

$$B_{K,m+1} = B_{K,m} + B_{K-1,m+1}.$$

The last formula works for $k_n > 1$. If $k_n = 1$ then we get the second case of formula (3.7). ☐

3.4.1 Barred permutations

Generating function (3.2) provides a direct combinatorial meaning for $B_{K,j}$. It is known as the number of Stirling permutations with j bars, so that any descent position of a permutation receives at least one bar.

So, we consider Stirling permutations where some positions contain a special sign $'/'$, called *bar*. We specialize only on permutations whose descent indices must contain a bar.

Example 3.4.5. $1//22/1344/33/$ has 5 bars and $/2/33//2//15/1//$ has 9 bars.

Proposition 3.4.6. *$B_{K,m}$ is equal to the number of barred permutations $\sigma \in \mathcal{GS}_n$ having m bars so that any descent index of σ receives at least one bar.*

Proof. Let $P_{K,m}$ be the number of barred permutations having the described property. Every such permutation can be obtained by putting one bar in each descent position and any number of bars in any of $K+1$ positions between elements in a permutation.

Therefore,

$$\sum_{m\geq 0} P_{K,m}x^m = \left(\sum_{\sigma\in\mathcal{GS}_n} x^{\operatorname{des}(\sigma)}\right)(1+x+x^2+\cdots)^{K+1}$$
$$= \frac{\sum_{i=1}^{n} A_i x^i}{(1-x)^{K+1}} = \sum_{m=0}^{\infty} B_{K,m}x^m.$$

So, $P_{K,m} = B_{K,m}$. □

3.4.2 Weighted lattice paths

Consider paths in a lattice grid $\mathcal{L}_{K,m}$ from $(0,0)$ to (K,m), such that
(i) moves that allowed are
up: $(x,y)\to(x,y+1)$ or *right*: $(x,y)\to(x+1,y)$
(ii) and *weight* w of any such move is equal to 1, except the values of type:

$$w((k_1+\ldots+k_{i-1},j)\to(k_1+\ldots+k_{i-1}+1,j)) = j,$$

$$w((k_1+\ldots+k_{i-1}+1,j)\to(k_1+\ldots+k_{i-1}+1,j+1)) = 0,$$

for all $1\leq i\leq n$, $0\leq j\leq m$ (if $i=1$, then $w((0,j)\to(1,j)=j)$).

Example 3.4.7. A grid and a path:

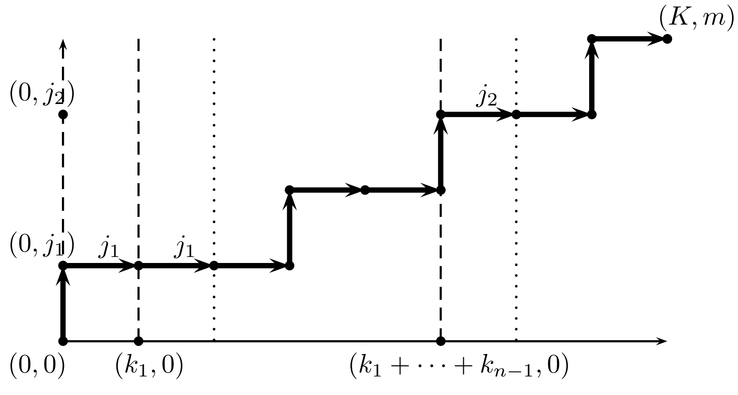


Figure 1 – A path from $(0,0)$ to (K,m). Labeled moves have weights j_1, j_2 with respect to their y-coordinate. Dotted vertical lines (if x dashed, then $(x+1)$ dotted) have weight 0. Other weights are equal to 1

Definition 3.4.8. For any path π in $\mathcal{L}_{K,m}$ from $(0,0)$ to (K,m), which consists of steps $\pi_1, \dots, \pi_{K+m}$, the *weight* of that path $w(\pi)$ is defined by

$$w(\pi) = w(\pi_1) \cdots w(\pi_{K+m}).$$

Example 3.4.9. Suppose that $k_1 = 1, k_2 = 4, k_3 = 3$. The weight of path given below is equal to $2 \cdot 2 \cdot 3 = 12$.

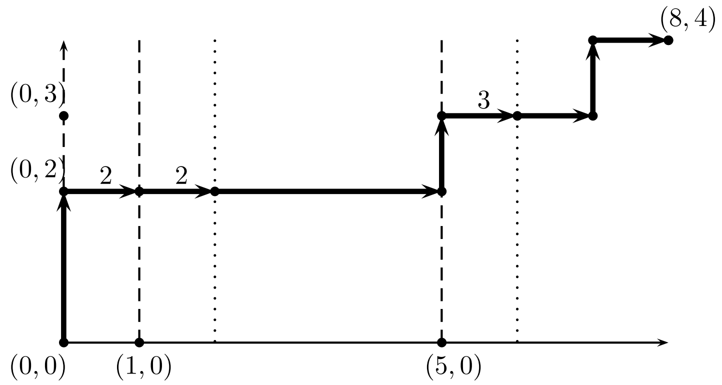


Figure 2 – A grid $\mathcal{L}_{8,4}$ with $k_1 = 1, k_2 = 4, k_3 = 3$

Theorem 3.4.10. *The following formula holds*

$$B_{K,m} = \sum_{\pi((0,0)\to(K,m))\in\mathcal{L}_{K,m}} w(\pi).$$

Proof. By induction on $K+m$. We can reach point (K,m) in two ways: from $(K-1,m)$ or $(K,m-1)$. Thus, if $k_n \neq 1$, then by inductive hypothesis we have $B_{K-1,m} + B_{K,m-1}$ ways; and if $k_n = 1$, then we can reach (K,m) only from $(K-1,m)$ with weight m. □

3.4.3 Bipartite multigraphs

Definition 3.4.11. Consider the set $\mathcal{G}_{n,m}$ of bipartite multigraphs (whose edges may duplicate) which satisfy the following properties:

1. The first part contains n vertices labeled as $1, \dots, n$
2. The second part contains m vertices labeled as $1, \dots, m$
3. The degree of i-th vertex $(1 \le i \le n)$ of the first part is equal to k_i

Definition 3.4.12. We say that edge (l,x) of the multigraph $G_{n,m} \in \mathcal{G}_{n,m}$ is *bad* if there exists an edge (k,y) of the same multigraph such that

$$k < n \text{ and } x < y.$$

(And (k,y) is not necessarily bad)

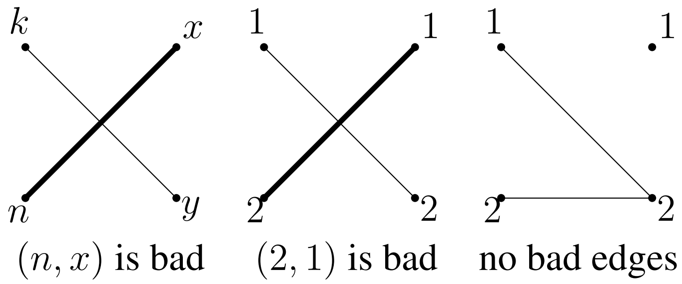


Figure 3 – Examples of bad edges

Definition 3.4.13. We say that the graph $G_{n,m} \in \mathcal{G}_{n,m}$ is *almost-ordered* if each vertex of the first part has at most one bad edge.

Theorem 3.4.14. *$B_{K,m}$ if equal to the number of almost-ordered graphs of $\mathcal{G}_{n,m}$.*

Proof. Let $\Omega_{K,m}$ be the number of almost-ordered elements of $\mathcal{G}_{n,m}$. We will prove that $\Omega_{K,m}$ satisfies the same recurrence (3.7) and initial values.

Let us consider a certain element $G_{n,m} \in \mathcal{G}_{n,m}$.

If $k_n = 1$ then it is obvious that $\Omega_{K,m} = m\Omega_{K-1,m}$.

Now, assume that $k_n > 1$. Then two cases are possible.

Case 1. There is an edge (n, m). Then the number of such graphs is to be equal $\Omega_{K-1,m}$, because this edge is not bad.

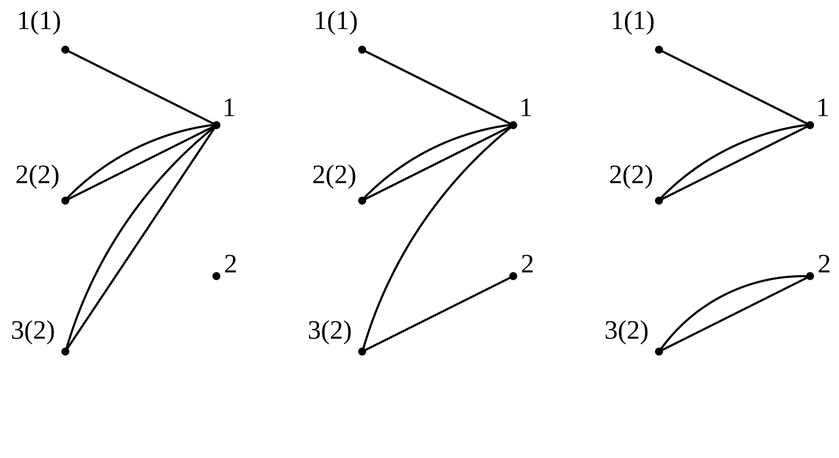
Case 2. There is no any edge (n, m). Now, we should understand that it means that there is no any other edge of the type (i, m) $1 \leq i \leq n$. It is true, because otherwise at least two edges of the vertex n ($k_n > 1$) are bad (will "intersect" by the edges (i, m)). And the needed number is equal to $\Omega_{K,m-1}$.

Therefore, $\Omega_{0,m} = 1$; and

$$\Omega_{K,m} = \begin{cases} \Omega_{K,m-1} + \Omega_{K-1,m}, & \text{if } k_n > 1 \\ m\Omega_{K-1,m}, & \text{if } k_n = 1. \end{cases}$$

So, $\Omega_{K,m} \equiv B_{K,m}$. □

Example 3.4.15. Consider $\mathcal{G}_{\{1,2^2,3^2\},2}$. We construct 11 almost-ordered graphs.



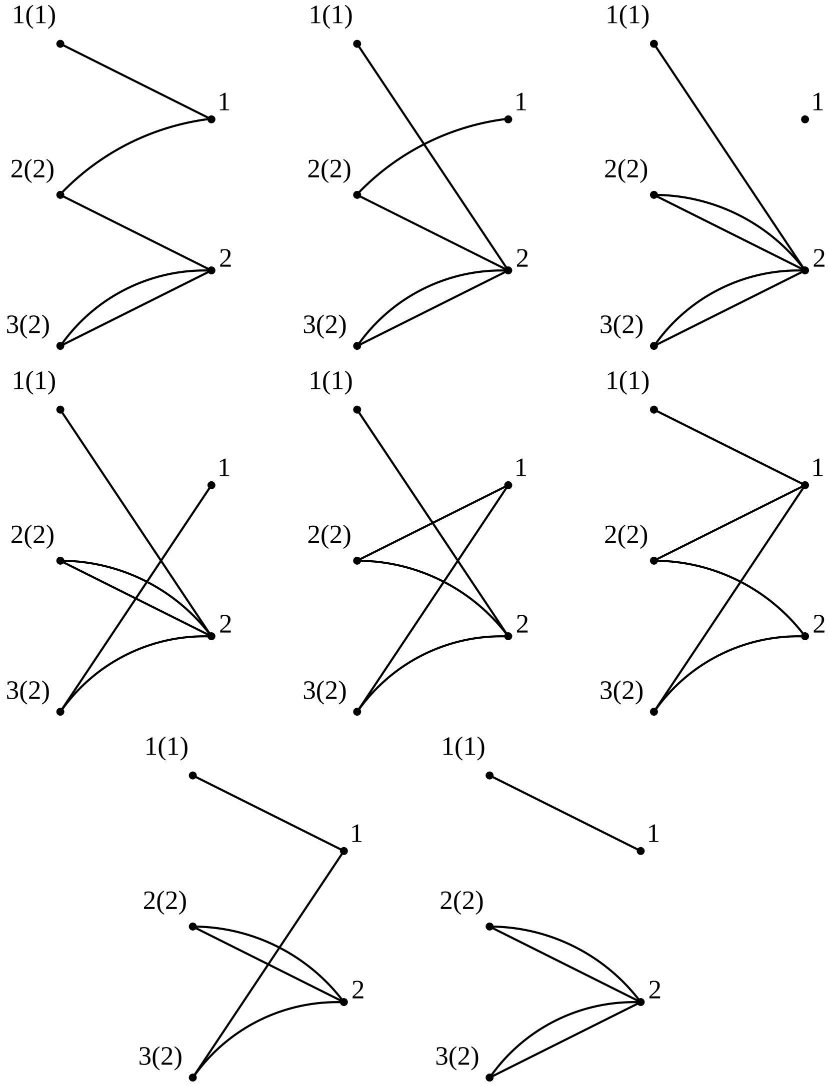


Figure 4 – Examples of almost-ordered graphs

3.4.4 The order polynomial

We show that combinatorial meaning of B numbers can be expressed using the theory of P-partitions of Stanley [47].

Definition 3.4.16. Consider a finite (labeled) partially ordered set $\mathbf{P}$ with partial order $<_p$. We define the *order polynomial* $\Omega(\mathbf{P}, m)$ [47] as the number of order-preserving maps

$$\sigma : \mathbf{P} \to \{1, \ldots, m\},$$

i.e., if $x <_p y$ then $\sigma(x) \leq \sigma(y)$.

The goal of this section is to construct some poset whose order polynomial values coincide with B numbers.

For instance, the poset which induces Stirling numbers $B_{2n,m} = S(n+m, m)$ (of the multiset $\{1^2, \ldots, n^2\}$), .i.e., $\Omega(\mathbf{P}, m) = B_{2n,m}$, is the following [39, 33]:

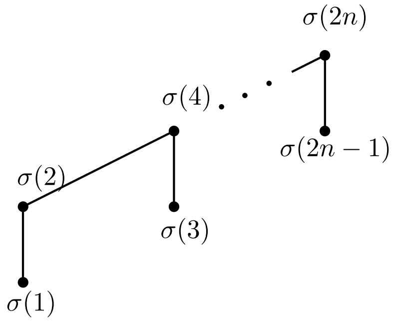


Figure 5 – a poset with $\Omega(\mathbf{P}, m) = S(n+m, m)$

It gives

$$\Omega(\mathbf{P}, m) = \sum_{1 \le \sigma(2) \le \cdots \le \sigma(2n) \le m} \sigma(2) \cdots \sigma(2n).$$

In case $\{1, 2^2, \ldots, n^2\}$, the poset for which $\Omega(\mathbf{P}, m) = B_{2n-1,m}$ slightly differs

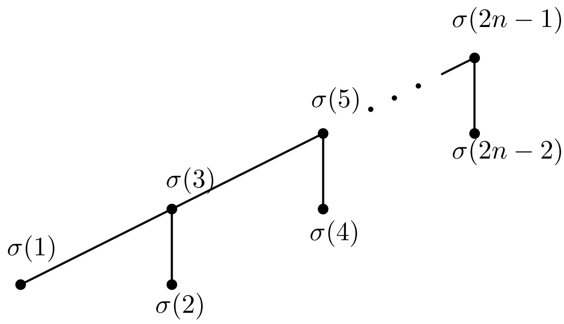


Figure 6 – a poset for the case $\{1, 2^2, \ldots, n^2\}$

which gives

$$\Omega(\mathbf{P}, m) = \sum_{1 \le \sigma(3) \le \sigma(5) \le \cdots \le \sigma(2n-1) \le m} \sigma(3)^2 \sigma(5) \cdots \sigma(2n-1).$$

The case $\{1^k, \ldots, n^k\}$ was introduced in [33] with the poset below, so that $\Omega(\mathbf{P}, m) = B_{nk,m}$:

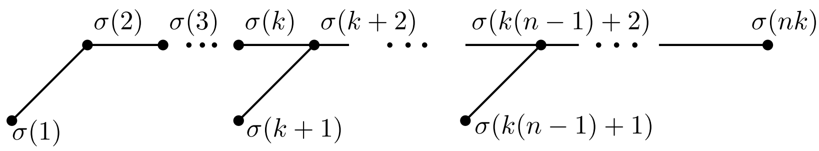


Figure 7 – a poset for the case $\{1^k, \ldots, n^k\}$

Next we describe the structure of poset which induces the numbers $B_{K,m}$ for any given $k_1, \ldots, k_n$.

Particularly, if all $k_i > 1$ $(1 \le i \le n)$, then the poset $\mathbf{P}_K$ looks as:

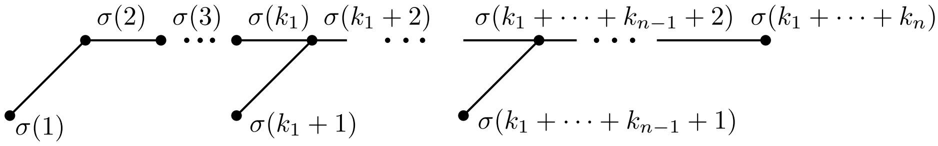


Figure 8 – poset in the general case

If for some i, $k_i = 1$, then diagram slightly differs. Suppose that

$$(k_1, \dots, k_n) = (\underbrace{1, \dots, 1}_{a_1-1 \text{ ones}}, t_1, \dots, \underbrace{1, \dots, 1}_{a_r-1 \text{ ones}}, t_r),$$

where $t_1 > 1, \dots, t_r > 1$. Then we present the following poset $\mathbf{P}_K$:

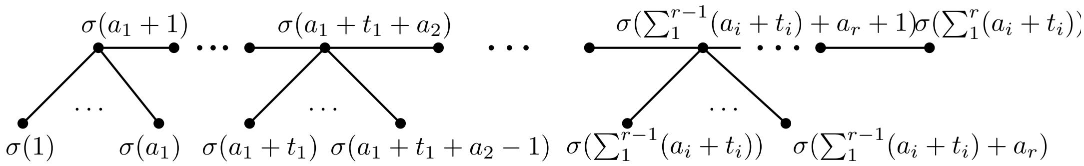


Figure 9 – a diagram of the poset $\mathbf{P}_k$

In the general case we construct the poset $\mathbf{P}_K$ (according to the numbers $k_1, \dots, k_n$) by the following algorithm:

1. Initially add to $\mathbf{P}$ one temporary node with label 0.
2. For all $1 \le i \le n$ do
   - Let i_m be the maximal value of label in the current poset
   - Join i_m with 0
   - If $k_i = 1$ then add new label $i_m + 1$
   - Else if $k_i > 1$ then
     - replace the label 0 to $i_m + 2$.
     - add new label $i_m + 1$ and join with $i_m + 2$.
     - if $(k_i > 2)$ then create new nodes labeled by $i_m + 3, \dots, i_m + k_i$
     - add the chain $i_m \to i_m + 2 \to i_m + 3 \to \dots \to i_m + k_i$
   - If $i < n$ then add one temporary node with label 0 (if there is no such element)

Example 3.4.17. Consider $\{1^2, 2^3, 3^1, 4^2\}$. Then $\mathbf{P}_K$ is constructed as follows:

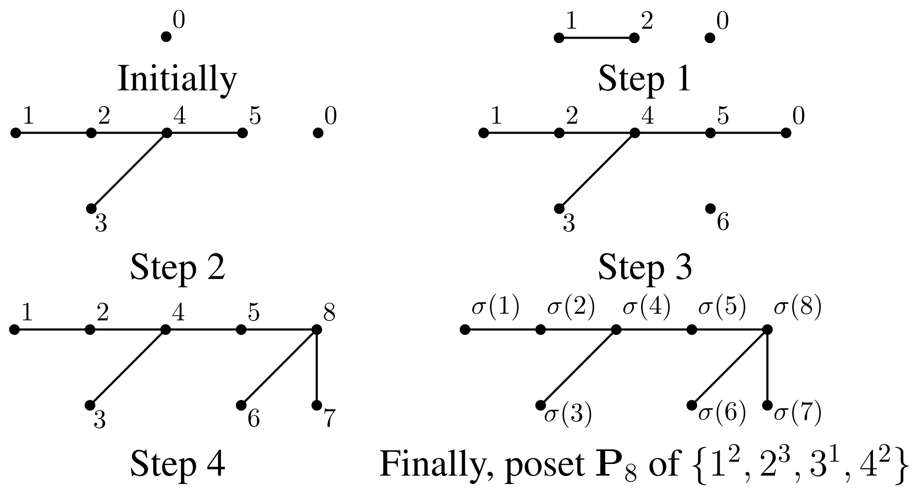


Figure 10 – steps of obtaining the poset $\mathbf{P}_K$ in case $\{1^2, 2^3, 3^1, 4^2\}$

Theorem 3.4.18. $B_{K,m} = \Omega(\mathbf{P}_K, m)$.

Proof. Let l be the last label added to the poset $\mathbf{P}_K$ in described algorithm.

Case 1. If $k_n = 1$, then l has no relation to other elements and $\sigma(l)$ can therefore take any of m values. Thus, $\Omega(\mathbf{P}_K, m) = m\Omega(\mathbf{P}_{K-1}, m)$ in this case.

Case 2. If $k_n > 1$, then $\sigma(l)$ is the maximal value in the map and two cases are possible:

Case 2a. if $\sigma(l) \leq m-1$, then the number of maps is equal to $\Omega(\mathbf{P}_K, m-1)$;

Case 2b. if $\sigma(l) = m$, then if we remove l it gives us $\Omega(\mathbf{P}_{K-1}, m)$ ways to map remained elements.

Hence, in case 2 we have

$$\Omega(\mathbf{P}_K, m) = \Omega(\mathbf{P}_{K-1}, m) + \Omega(\mathbf{P}_K, m-1).$$

So, $\Omega(\mathbf{P}_K, m)$ satisfies the same recurrence relation as $B_{K,m}$ (3.7) and it easy to check that initial values are also equal. □

3.5 $B_{K,m}$ as the number of set partitions

Denote $[n] = \{1, \ldots, n\}$.

Definition 3.5.1. For a family $\mathcal{F} = \{B_1, \ldots, B_m\}$ of m sets (or multisets) let

- $\min(B_i)$ be the minimal element of B_i $(1 \leq i \leq m)$;
- $\min(\mathcal{F}) = \{\min(B_1), \ldots, \min(B_m)\}$.

Example 3.5.2. For a parition $\pi = \{1,4\}\{2,3,7\}\{5,6\}$ we have $\min(\pi) = \{1,2,5\}$.

Definition 3.5.3. We say that a family $\mathcal{F} = \{B_1, \ldots, B_m\}$ of m sets (or multisets) is *non-intersecting* if any two of its representatives B_i, B_j (where $1 \leq i \neq j \leq m$) are

(i) *non-crossing*, i.e., if $a < b$ are in B_i and $c < d$ are in B_j, then the inequality $a < c < b < d$ does not hold;

(ii) *non-nested*, i.e., if $a < b$ are in B_i and $c < d$ are in B_j, then the inequality $a < c < d < b$ does not hold.

Example 3.5.4. Partition $\{1,2\}\{3,4,6,7\}\{5\}\{8,9\}$ is non-intersecting.

Definition 3.5.5. Let $S_3(n,m)$ be the number of ordered pairs (π_1, π_2) which satisfy the following properties:

(i) π_1 is a partition of $[n]$ into m blocks and π_2 is a non-intersecting partition of $[n+1]$ into m blocks;

(ii) $\min(\pi_1) = \min(\pi_2)$;

(iii) if x is the least element of $[n] \setminus \min(\pi_1)$, then x and 1 are in the same block of π_2.

Example 3.5.6. If $n = 4, m = 2$, then there are $S_3(4,2) = 10$ possible ordered pairs (π_1, π_2) which satisfy the properties above:

$$(\{1\},\{2,3,4\}),(\{1,3,4,5\},\{2\});$$

$$(\{1,3\},\{2,4\}),(\{1,3,4,5\},\{2\});$$
$$(\{1,4\},\{2,3\}),(\{1,3,4,5\},\{2\});$$
$$(\{1,3,4\},\{2\}),(\{1,3,4,5\},\{2\});$$
$$(\{1,2\},\{3,4\}),(\{1,2,4,5\},\{3\});$$
$$(\{1,2\},\{3,4\}),(\{1,2\},\{3,4,5\});$$
$$(\{1,2,4\},\{3\}),(\{1,2,4,5\},\{3\});$$
$$(\{1,2,4\},\{3\}),(\{1,2\},\{3,4,5\});$$
$$(\{1,2,3\},\{4\}),(\{1,2,3,5\},\{4\});$$
$$(\{1,2,3\},\{4\}),(\{1,2,3\},\{4,5\});$$

Theorem 3.5.7. *For* $k_1 = \cdots = k_n = 3$ *we have* $B_{3n,m} = S_3(n+m, m)$.

Proof. It is proved further in the general case. □

Definition 3.5.8. For a positive integer $k > 2$, let $S_k(n,m)$ be the number of ordered pairs (π_1, π_2) which satisfy the following properties:

(i) π_1 is a partition of $[n]$ into m blocks;

(ii) π_2 is a non-intersecting partition of a multiset $\{1^{k-2}, \ldots, (n+1)^{k-2}\}$ into m (multiset) blocks so that any block contains all $(k-2)$ copies of its minimal element;

(iii) $\min(\pi_1) = \min(\pi_2)$;

(iv) if x is the least element of $[n] \setminus \min(\pi_1)$, then $x^{(k-2)}$ belongs to the first block of π_2.

Theorem 3.5.9. *For* $k_1 = \cdots = k_n = k > 2,$ *we have* $B_{kn,m} = S_k(n+m, m)$.

Proof. It is proved further in the general case. □

For given positive integers n, m (with $n \geq m$) let us consider the sequence $\mathbf{t}$ of the form

$$\mathbf{t} = (\underbrace{1,\ldots,1}_{a_1-1 \text{ ones}}, t_1, \ldots, \underbrace{1,\ldots,1}_{a_{n-m}-1 \text{ ones}}, t_{n-m}),$$

where $a_1, \ldots, a_{n-m}, t_1, \ldots, t_{n-m}$ are positive integers and $t_1, \ldots, t_{n-m} > 1$. Set $M = \max(a_1, \ldots, a_{n-m})$.

Definition 3.5.10. Let $S_{\mathbf{t}}(n,m)$ be the number of ordered $(M+1)$-tuples $(\pi_0,\pi_1,\dots,\pi_M)$ which satisfy the following properties:

(i) $\pi_1,\dots,\pi_M$ are partitions of $[n]$ into m blocks;

(ii) π_0 is a non-intersecting family of m multisets whose elements are from $[n+1]$ and minimal elements in each multiset have multiplicity 1;

(iii) $\min(\pi_0)=\min(\pi_1)=\cdots=\min(\pi_M)$;

(iv) if $\{x_1,\dots,x_{n-m}\}=[n]\setminus\min(\pi_1)$, so that $x_1<\cdots<x_{n-m}$, then for any i $(1\le i\le n-m)$, x_i and 1 are in the same block in all π_j for which $j>a_i$;

(v) $x_1, 1$ are in the same multiset in π_0 and in a family π_0 the elements $(x_2,\dots,x_{n-m},n{+}1)$ have multiplicities equal to $(t_1-2,\dots,t_{n-m}-2)$, respectively.

Theorem 3.5.11. *Suppose that sequence* $(k_1,\dots,k_n)$ *has the form*

$$\mathbf{t}=(k_1,\dots,k_n)=(\underbrace{1,\dots,1}_{a_1-1\ ones},t_1,\dots,\underbrace{1,\dots,1}_{a_r-1\ ones},t_r),$$

where $t_1>1,\dots,t_r>1$. *Then* $S_{\mathbf{t}}(r+m,m)=B_{K,m}$.

Proof. Set $M=\max(a_1,\dots,a_r)$. Let us fix m minimal elements from $[r+m]$, which common for $(\pi_0,\dots,\pi_M)$ and consider the elements $\{x_1,\dots,x_r\}=[r+m]\setminus\min(\pi_1)$, so that $x_1<\ldots<x_r$. For any j $(1\le j\le r)$ denote

$$i_j=|\{x<x_j\mid x\in\min(\pi_1)\}|.$$

Then, according to the property (iv) from the definition above, the element x_j can be placed in i_j blocks in any of partitions $(\pi_1,\dots,\pi_{a_j})$, for any $1\le j\le m$. This provides $i_j^{a_j}$ ways to place x_j and totally $i_1^{a_1}\dots i_r^{a_r}$ ways to place all the elements $x_1,\dots,x_r$ if minimal elements are fixed.

Now consider the number of ways to form π_0. The element x_1 is already placed with minimal element 1. Let $x_{r+1}=r+m+1$ and $p_1=1$, then for any j $(2\le j\le r+1)$, according to the non-crossing and non-nested property, the element x_j can be placed only to multisets whose minimal elements p_j are greater than x_{j-1} or to the same multiset as x_{j-1}. Because inequalities of types $p_{j-1}<p_j<x_{j-1}<x_j$ and $p_j<p_{j-1}<x_{j-1}<x_j$ cannot hold. So, for any j $(2\le j\le r+1)$ there are $i_j-i_{j-1}+1$ vacant positions to put $t_{j-1}-2$ copies of element x_j, which gives $\binom{t_{j-1}-2+i_j-i_{j-1}}{i_j-i_{j-1}}$ ways.

Thus, for any fixed arrangement of minimal elements, we have

$$i_1^{a_1}\dots i_r^{a_r}\binom{t_1+i_2-i_1-2}{i_2-i_1}\cdots\binom{t_r+m-i_r-2}{m-i_r}.$$

ways to form $(M+1)$-tuple $(\pi_0,\dots,\pi_M)$. Note that it holds for the arbitrary sequence satisfying $1\le i_1\le\cdots\le i_r\le m$. Therefore,

$$S_{\mathbf{t}}(r+m,m)=\sum_{1\le i_1\le\cdots\le i_r\le m} i_1^{a_1}\dots i_r^{a_r}\binom{t_1+i_2-i_1-2}{i_2-i_1}\cdots\binom{t_r+m-i_r-2}{m-i_r}.$$

Iterative use of equation (3.5) gives the same formula for $B_{K,m}$:

$$B_{K,m} = \sum_{1 \le i_1 \le \cdots \le i_r \le m} i_1^{a_1} \dots i_r^{a_r} \binom{t_1 + i_2 - i_1 - 2}{i_2 - i_1} \dots \binom{t_r + m - i_r - 2}{m - i_r}.$$

□

3.6 Special cases if $k_i = 1, 2$

3.6.1 The power function m^n

If $k_1 = \cdots = k_n = 1$, then $B_{n,m} = m^n$. This refers to the classical result about the usual Eulerian numbers:

$$\sum_{m=0}^{\infty} m^n x^m = \frac{\sum_{i=1}^{n} A_{n,i} x^i}{(1-x)^{n+1}}.$$

3.6.2 Sums of powers of consecutive integers

If $k_1 = \cdots = k_{n-1} = 1$ and $k_n = 2$, then $B_{n+1,m} = \sum_{i=1}^{m} i^n$.

3.6.3 The binomial coefficient $\binom{K+m-1}{K}$

If $n = 1$ and $k_1 = K$, then $B_{K,m} = \binom{K+m-1}{K}$. $G_n(x)$ becomes

$$\sum_{m=0}^{\infty} \binom{K+m-1}{K} x^m = \frac{x}{(1-x)^{K+1}}.$$

3.6.4 The usual Stirling numbers

If $k_1 = \cdots = k_n = 2$ then $B_{2n,m} = S(n+m, m)$.

3.6.5 Stirling numbers of odd type

If $k_1 = 1$ and $k_2 = \cdots = k_n = 2$, then $B_{2n-1,m} = S_{\text{odd}}(n+m, m)$.

The recurrence relation which derived from the Theorem 3.7 is given by

$$S_{\text{odd}}(n, k) = S_{\text{odd}}(n-1, k-1) + k S_{\text{odd}}(n-1, k),$$

if $n > k$ and $S_{\text{odd}}(n, n) = n$ (which differs it from the usual Stirling number).

Interpretation with P-partitions provides the formula

$$S_{\text{odd}}(n+m, m) = \sum_{1 \le i_2 \le \cdots \le i_n \le m} i_2^2 i_3 \cdots i_n.$$

In fact, one can prove that $S_{\text{odd}}(n, k)$ are related with the r-Stirling numbers introduced by Broder [6]. Namely,

$$S_{\text{odd}}(n, k) = \left\{ {n \atop k} \right\}_1 + \left\{ {n \atop k} \right\}_2 + \cdots + \left\{ {n \atop k} \right\}_k,$$

where $\left\{ {n \atop k} \right\}_r$ is r-Stirling number of the second kind, which counts the number of partitions of $\{1, \ldots, n\}$ into k blocks such that the numbers $1, \ldots, r$ are in distinct blocks.

The following theorem can be obtained from the Definition 3.5.10.

Theorem 3.6.1. *$S_{odd}(n, k)$ is equal to the number of partitions of $[n]$ into k blocks such that the least element x, which is not a block minimum has two distinguished copies x_1, x_2.*

Note that these numbers $S_{\text{odd}}(n, k)$ were discussed by Knuth [34] as half-integer Stirling numbers.

3.6.6 Generalization of central factorial numbers

If we consider the multiset $\{1^{k_1}, \ldots, (nt)^{k_{nt}}\}$ with the sequence $(k_1, \ldots, k_{nt})$ of special type

$$(\underbrace{\overbrace{1, 1, \ldots, 1, 2}^{t \text{ numbers}}, \overbrace{1, 1, \ldots, 1, 2}^{t \text{ numbers}}, \ldots, \overbrace{1, 1, \ldots, 1, 2}^{t \text{ numbers}}}_{n \text{ blocks}}),$$

then the numbers $S_t(n + m, m) = B_{(t+1)n,m}$ have good properties.

The recurrence relation becomes

$$S_t(n, m) = S_t(n - 1, m - 1) + m^t S_t(n - 1, m),$$

which is a generalization of the central factorial numbers defined by Riordan [42] as $T(2n, 2m)$ in case if $t = 2$.

Now we introduce the generalizations of central factorial numbers of the first and second kinds $\{c_t(n, k)\}, \{S_t(n, k)\}$, which can be defined by the recurrence relations:

$$\begin{aligned} c_t(n, k) &= c_t(n - 1, k - 1) + (n - 1)^t c_t(n - 1, k), \\ S_t(n, k) &= S_t(n - 1, k - 1) + k^t S_t(n - 1, k). \end{aligned}$$

These numbers have natural combinatorial interpretations. For $t = 2$ it reduces to the interpretation of Dumont [13] of the central factorial numbers.

Theorem 3.6.2. *$c_t(n, k)$ is equal to the number of ordered t-tuples $(\sigma_1, \ldots, \sigma_t)$ of permutations of $(1, \ldots, n)$ such that all permutations σ_i $(1 \le i \le t)$ have exactly k cycles and the same set of cycle minima.*

Proof. Consider element n. If it forms a separate cycle (n), then as n occur as a minimal element, this cycle should appear in every permutation $\sigma_1, \ldots, \sigma_t$, so the number of ways $c_t(n - 1, k - 1)$. Otherwise, if n is present in cycles with other elements, then for any permutation σ_i $(1 \le i \le t)$, there are $n - 1$ ways to put n in k cycles of σ_i, and $(n - 1)^t c_t(n - 1, k)$ corresponding ways. $\square$

Theorem 3.6.3. *$S_t(n,k)$ is equal to the number of ordered t-tuples $(\pi_1,\ldots,\pi_t)$ of partitions of $\{1,\ldots,n\}$ such that all partitions π_i $(1 \le i \le t)$ have exactly k blocks and the same set of block minima.*

Proof. Consider element n. If it forms a separate block $\{n\}$, then as n occur as a minimal element, this block should appear in every partition $\pi_1,\ldots,\pi_t$, so the number of ways $S_t(n-1,k-1)$. Otherwise, if n is present in blocks with other elements, then for any partition π_i $(1 \le i \le t)$, there are k ways to put n in k blocks of π_i, and $k^t S_t(n-1,k)$ corresponding ways. □

CONCLUSION

Throughout the work we have presented several generalizations and extensions of known results.

For instance, the detailed investigation of properties for the sums of powers of binomial coefficients $\sum_i \binom{i}{k}^m$, led us to the analog of Faulhaber's theorem. Separately, we have focused on the triangular case $k = 2$ of the Faulhaber coefficients and introduced the B coefficients, which in fact inverse to the A existing coefficients studied by Knuth [35].

For the problem of partitions of multisets by the usual sets we have obtained a natural generalization of the Stirling number of the second kind. We also show that this approach is very close to the restricted partitions or permutation of sets, where some given blocks of elements cannot be in one subset or cycle. In that direction definition of the Stirling numbers of the first kind provides and interesting problem about orthogonality, polynomial identity.

We have studied the problem of obtaining the combinatorial interpretations for the B numbers that based on Stirling permutations. For Stirling permutations of any multiset we have presented the general construction $S_{\mathbf{k}}(p, q)$ which enumerates the tuples of partitions of sets under some properties. For the special cases of multisets this definition induce, e.g., the generalization of the central factorial numbers.

In the same time, this work covers just some partial variety of problems in combinatorics of multisets. When we change the basis from n^m to the product of binomial coefficients, many interesting problems can be posed for future research. For example, there is a connection of Bernoulli and Stirling numbers by the formula

$$B_n = \sum_{i=0}^{n} (-1)^i \frac{i! S(n, i)}{i + 1}.$$

Then our approach tends to consider the following generalization of Bernoulli numbers for multiset $\mathbf{n}$:

$$B_{\mathbf{n}} = \sum_{i=0}^{n} (-1)^i \frac{S(\mathbf{n}, i)}{i + 1},$$

with consequent questions about properties of these numbers, such as connections to sums of powers, generating functions, etc.

We can also meet Stirling numbers on multisets in the probabilistic analysis towards possible application in data structures (such as bloom filters). For example, the following problem:

There is an array B of m bits (all initially set to 0). For any i $(1 \le i \le n)$ at step i we choose k_i distinct bits of B and set them to 1. What is the probability of event that finally all bits of B are equal to 1? The answer is

$$\frac{S(\{1^{k_1} \cdots n^{k_n}\}, m)}{\binom{m}{k_1} \cdots \binom{m}{k_n}}.$$

Other directions of research may deal with relations between generalized Stirling numbers presented in chapter 2 and B numbers for Stirling permutations.

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