ma
the
ma
tisch

cen
trum

AFDELING ZUIVERE WISKUNDE   ZN 41/72   MARCH

A.E. BROUWER and A. VERBEEK

COUNTING FAMILIES OF MUTUALLY INTERSECTING SETS

amsterdam   **1972**

stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZN 41/72 MARCH

A.E. BROUWER and A. VERBEEK

COUNTING FAMILIES OF MUTUALLY INTERSECTING SETS

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat 49, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

Introduction

A family of sets is called linked if every two sets have a non-empty intersection. As application of a recent results of Kleitman estimating the number of antichains on an n-point-set, we derive asymptotic formula for the $^2\log$ of the following numbers:

the number $\lambda(n)$ of maximal linked families of subsets of $\{1,2,\ldots,n\}$

the number $\Lambda(n)$ of all linked families of subsets of $\{1,2,\ldots,n\}$

In his survey [1] p. 79, P. Erdös asked for an asymptotic formula for $\Lambda(n)$. Our concern came forth from an investigation [4], on maximal linked families of closed sets in topological spaces.

Notation

$S_n = \{1,2,\ldots,n\}$

$P_n = P(S_n)$ = powerset of S_n

$M \subset P_n$ is linked if $\forall S,S' \in M \quad S \cap S' \neq \emptyset$

$M \subset P_n$ is an antichain if $\forall S,S' \in M \quad S \not\subset S'$

an mls is a maximal linked (sub)system (of P_n)

$L_n = \{M \subset P_n \mid M \text{ is an mls}\}$

$A_n = \{M \subset P_n \mid M \text{ is a non-empty antichain}\}$

$I_n = \{M \subset P_n \mid \cap M \neq \emptyset\}$

$\lambda(n) = |L_n|$

$\Lambda(n) = |\{M \subset P_n \mid M \text{ is linked}\}|$

$\alpha(n) = |A_n|$

$i(n) = |I_n|$

For arbitrary $M \subset P_n$ we define

$$M_{MIN} = MIN(M) = \{S \in M \mid \forall T \subset S \quad T \in M \Rightarrow T = S\}$$

Finally for two function $f,g\colon N \to R$ we write

$$f \sim g$$

$$\text{iff} \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1 \, .$$

The first lemma is trivial.

Lemma 1

(a) A linked family $M \subset P_n$ is an mls iff M contains S_n and moreover (precisely) one set of each pair of complementary proper subsets of S_n.

(b) Each linked family is contained in (at least one) mls.

(c) Two mls's $M, M' \subset P_n$ are different iff $\exists S \in M \exists S' \in M'$ $S \cap S' = \emptyset$.

Lemma 2.

$$2^{\binom{n-1}{[n/2]-1}} \leq \lambda(n).$$

Proof.

We give slightly different proofs for even and for odd n.

Let $n = 2k$. Let $\{\{A_i, S_n \backslash A_i\} \mid 1 \leq i \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}\}$ be the family of all unordered pairs of complementary $k = n/2$-point-sets in S_n. If we choose one k-point-set from each pair then we obtain a linked system. Thus we obtain $2^{\binom{2k-1}{k-1}}$ different linked families, with the properties that for two such families, say A and A', $\exists A \in A\ \exists A' \in A'$ $A \cap A' = \emptyset$. By 1b + 1c it follows that there are at least $2^{\binom{2k-1}{k-1}}$ different mls's.

Let $n = 2k-1$. Consider the family $\{\{A_i, S_n \backslash A_i\} \mid 1 \leq i \leq \binom{2k-2}{k-2}\}$ of all pairs of complementary sets A_i, $S_n \backslash A_i$ satisfying $1 \in A_i$ and A_i has $k - 1$ points. The same reasoning as above leads to $2^{\binom{2k-2}{k-2}} \leq \lambda(n)$.

Lemma 3.

$$\lambda(n) \leq \alpha(n-1).$$

Proof.

Define $f: L_n \to A_{n-1}$ by

$$f(M) = \{S \mid S \in \mathrm{MIN}\{T \mid n \notin T \in M\}\}.$$

By 1a the family $M' = \{T \mid n \notin T \in M\}$ uniquely determines M (viz. $M = M' \cup \{S \mid n \in S \subset S_n \text{ and } S_n \setminus S \notin M'\}$), and hence also MIN M' uniquely determines M, as $M' = \{T \subset S_{n-1} \mid \exists\, S \in \mathrm{MIN}\, M' : S \subset T\}$. Finally MIN M' obviously is an antichain in P_{n-1}.

Lemma 4. KLEITMAN [2]

$$^2\log \alpha(n) \sim \binom{n}{[n/2]}.$$

Lemma 5.

$$\binom{n-1}{[n/2]-1} \sim \binom{n-1}{[\frac{n-1}{2}]} \sim \frac{2^n}{\sqrt{2\pi n}} \sim \frac{2^n}{\sqrt{2\pi(n-1)}} \quad .$$

The last lemma is trivial. From 2, 3, 4 and 5 we immediately obtain our main result:

Theorem 6.

$$^2\log \lambda(n) \sim {}^2\log \alpha(n-1) \sim \frac{2^n}{\sqrt{2\pi n}} \ .$$

From this result it is easy to deduce an asymptotic formula for $^2\log \Lambda(n)$. First we observe that $\Lambda(n) \geq i(n)$. The wellknown expression for $i(n)$, see below, can be obtained by first counting for all $k \in S_n$ all families A with $\{k\} \subset \cap A$. Then for $k \neq k'$ the families with $\{k,k'\} \subset \cap A$ were counted twice, so their number should be subtracted and so on.

Lemma 7.

(a) $n.2^{2^{n-1}}(1-(n-1)/2.2^{2^{n-2}}) < \sum_{k=1}^{n} (-)^{k+1}\binom{n}{k}2^{2^{n-k}} = i(n) < \Lambda(n) < \lambda(n).2^{2^{n-1}}$.

(b) $(\lambda(n)/2^{\binom{n}{[n/2]}}).2^{2^{n-1}} < \Lambda(n) < \lambda(n).2^{2^{n-1}}$

Proof.
Let $M \subset P_n$ be an arbitrary mls. Then, by 1a, M has 2^{n-1} members, and, by Sperner's lemma [3], M_{MIN} , being an antichain, has at most $\binom{n}{[n/2]}$ members. Thus M contains $2^{2^{n-1}}$ linked subfamilies, which proves the right-hand inequality of (a) and (b). To prove the left-hand side of (b), we observe that M is the only mls containing M_{MIN} . This means that no linked system N satisfies $M_{MIN} \subset N \subset M$ and $M'_{MIN} \subset N \subset M'$ for different mls's M and M'. As there are at least $2^{n-1} - \binom{n}{[n/2]}$ many sets in $M \backslash M_{MIN}$, the left-hand inequality follows.

From 6 and 7a we see that

Theorem 7.

$$^2\log i(n) \sim {}^2\log \Lambda(n) \sim 2^{n-1} .$$

In the numerical results (see page 5) $\lambda(6)$ (and $\lambda(1) - \lambda(5)$) were computed by means of the bijection

$$\phi\colon L_n \to \{M \subset P_n \mid M \text{ is a linked antichain}\}$$

defined as follows. Let $A = \{S_i \mid 1 \leq i \leq 2^{n-1}\}$ be a selection of subsets of S_n of at most n/2 points such that A contains precisely one of each pair of complementary subsets of S_n. Then for $M \in L_n$:

$$\phi(M) = \mathrm{MIN}(M \cap A),$$

and

$$\phi^{-1}(N) = \{A \in P_n \mid \exists A' \in N \quad A' \subset A\} \cup$$
$$\cup \{A \in P_n \backslash A \mid \neg \exists A' \in N \quad A' \subset S_n \backslash A\}.$$

Moreover $\lambda(1) - \lambda(5)$ and $\alpha(1) - \alpha(4)$ were also computed by hand, and $\lambda(7)$, $\alpha(5)$ and $\alpha(6)$ have been obtained by means of a PDP-8 computer, but were evaluated only once.

Numerical results

n	$\lambda(n)$	$\alpha(n-1)$	$^2\log \lambda(n)$	$^2\log \alpha(n-1)$	$2^n/\sqrt{2\pi(n-1)}$	$2^n/\sqrt{2\pi n}$	$\binom{n-1}{[n/2]-1}$	$i(n)$	$\Lambda(n)$
0	1	-	0	-	-	∞	-	1	1
1	1	1	0	0	∞	.798	0	2	2
2	2	2	1	1	1.596	1.128	1	6	6
3	4	5	2	2.322	2.257	1.843	2	38	40
4	12	19	3.585	4.248	3.685	3.192	3	942	1.888
5	81	167	6.340	7.384	6.383	5.709	6	325.262	?
6	2.646	7.580	11.370	12.888	11.418	10.424	10	26.10^{9}	?
7	1.422.564	7.828.353	20.440	22.900	20.847	19.301	20	13.10^{19}	?
8	conjectured $7.10^{10}<\lambda(8)<4.10^{11}$	$<\alpha(7)<4.10^{12}$	?	?	38.602	36.102	35	27.10^{38}	?
9	$10^{21}<\lambda(9)<5.10^{21}$	$<\alpha(8)<10^{24}$	?	?	72.204	68.1	70	10.10^{77}	?

References

[1] P. ERDÖS Problems and results in combinatorial analysis.
Proc. of symp. in pure math. Amer. Math. Soc. 19 (1971), 77-90.

[2] D. KLEITMAN On Dedekind's problem: the number of monotone Boolean functions.
Proc. Amer. Math. Soc. 21 (1969), 677-682.

[3] A. SPERNER Ein Satz ueber Untermengen einer endlichen Menge.
Math. Z. 27 (1928), 544-548.

[4] A. VERBEEK Superextensions of topological spaces.
Mathematical Centre Tracts. Mathematisch Centrum, Amsterdam 1972 (in print).