On the fractal behavior of primes

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Abstract

Ken Ono et al [Ono] showed partition numbers exhibit fractal behavior which suggests that the primes upon which these numbers are based may also be found to exhibit fractal behavior. Here we confirm it is the case. Indeed we have experimental evidence confirming primes follow fractal patterns considering a recursive integer sequence encoding all possible integer factorisations. We then make conjectures having possible applications in additive number theory.

Introduction

Many sequences are related to primes and many of them involve the floor function which is naturally related to the divisors. A very simple example is $\sum_{k\geq 1}\left\lfloor\frac{n}{k}\right\rfloor$ which counts the number of divisors of all integers less or equal to n. Thus we have a formula for the number of divisors function:

$$\tau(n)=\sum_{k\geq 1}\left\lfloor\frac{n}{k}\right\rfloor-\left\lfloor\frac{n-1}{k}\right\rfloor$$

and then we have a caracterisation of prime numbers in term of the floor function since $\tau(n)=2\Leftrightarrow n$ is prime. However we have very few information about primes among other integers using this fact and the behaviour of the arithmetical function $\tau(n)$ is not regular. In an other hand it is well known that:

$$\sum\mu_k\left\lfloor\frac{n}{k}\right\rfloor=1\Rightarrow\mu_n=1-\sum_{k=1}^{n-1}\mu_k\left\lfloor\frac{n}{k}\right\rfloor$$

Hence we have a recurrence formula for the Moebius functions involving simply the floor function. This time we can say whether a given number has an odd or even number of prime factors or if n is squarefree. But again this gives few information about prime numbers and the Moebius function behaves quite erraticaly.

It appears there is a way to obtain less erratic sequences related to the factorisation of n using the floor function and in [Clo] we considered the function

$\theta(x) = (-1)^{\lfloor x \rfloor}$ and discovered it has many nice arithmetical properties. This is an example of function of good variation (FGV) (a tauberian concept introduced in [Clo2]). More precisely θ generates a sequence encoding informations about prime numbers and reveals a fractal structure. Since fractal is merging chao and order we agree G. Tenenbaum and M. Mendès-France [TM]. In section 1 we define this sequence $(a(n))_{n\geq 1}$ and provide graphs showing its fractal structure. In section 2 we give conjectural formulas relating the sequence to the factorisation of integers. In section 3 we state conjectures related to gaps in $(a(n))_{n\geq 1}$ with application to additive number theory. In section 4 we give another conjecture using properties of $(a(n))_{n\geq 1}$. Finally in section 5 we discuss the fractal aspects of the sequence. In the APPENDIX 3 we provide several examples of other functions θ yielding similar fractal aspects for sequences related to primes or to the factorisation of n.

1 The sequence a_n

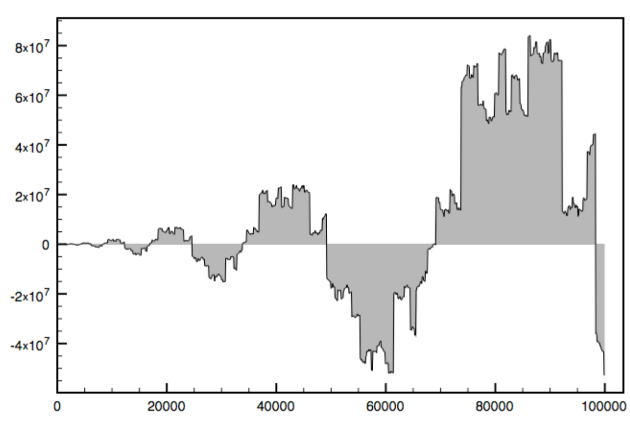
Taking $\theta(x) = (-1)^{\lfloor x \rfloor}$ we define the sequence $(a_n)_{n\geq 1}$ recursively as follows:

$$a_1 = 1 \text{ and } \sum_{k=1}^{n} a_k \theta(n/k) = 0 \text{ for } (n \geq 2)$$

This sequence $(a_n)_{n\geq 1}$ is unbounded and begins:

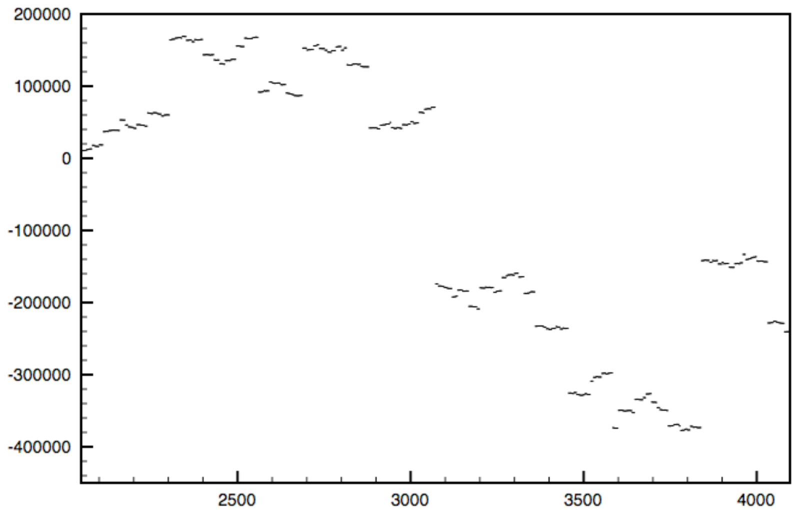
$1, 1, -2, 4, -2, -4, -2, 12, 2, -4, -2, -16, -2, -4, 6, 36, -2, 8, -2, -16, 6, -4, -2, -56, 2$

Although a_n has certainly some fractal structure, the direct plot of a_n is not very illuminating for the reader. A better way to see there is a fractal structure in the sequence consists in considering the summatory function $A(n)$. Here a big picture of $A(n)$.

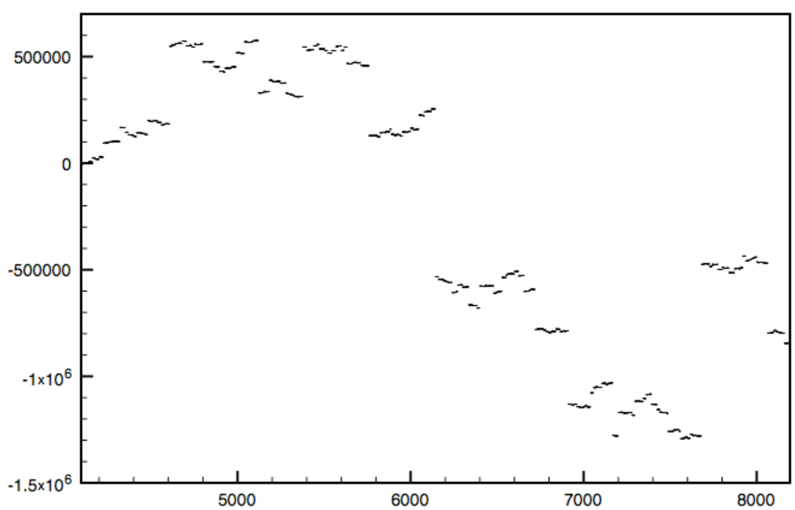


The following graphs are more striking (scatterplot).

Plot of $A(n)$ for $2048 \le n \le 4096$



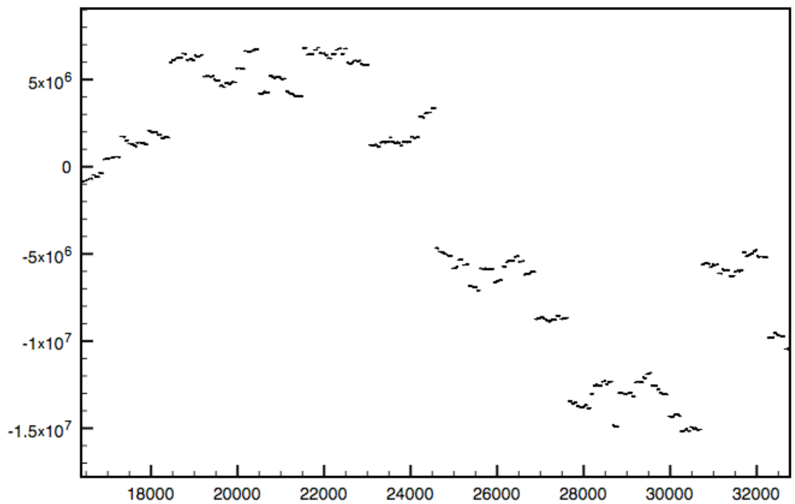
Plot of $A(n)$ for $4096 \le n \le 8192$



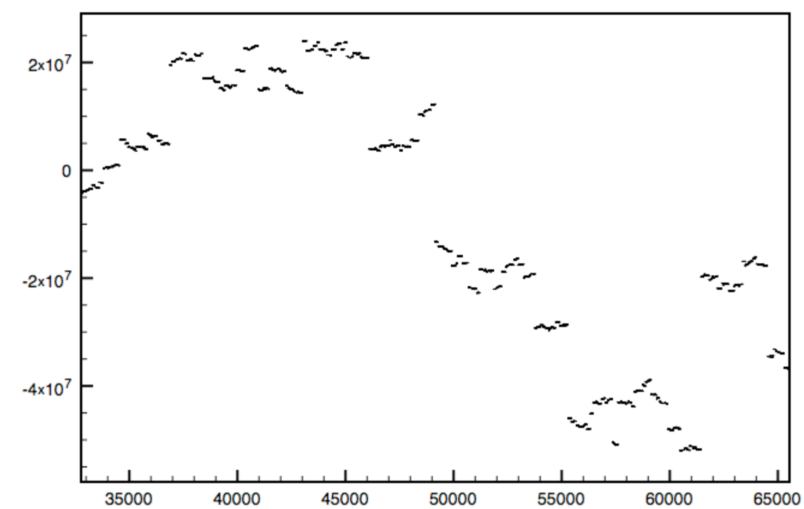
We can see there is a quasi-self-similarity[1] , not an exact one.

[1]"The fractal appears approximately (but not exactly) identical at different scales. Quasi-self-similar fractals contain small copies of the entire fractal in distorted and degenerate forms. Fractals defined by recurrence relations are usually quasi-self-similar. The Mandelbrot set is quasi-self-similar, as the satellites are approximations of the entire set, but not exact copies."

Plot of $A(n)$ for $2^{14} \le n \le 2^{15}$



Plot of $A(n)$ for $2^{15} \le n \le 2^{16}$

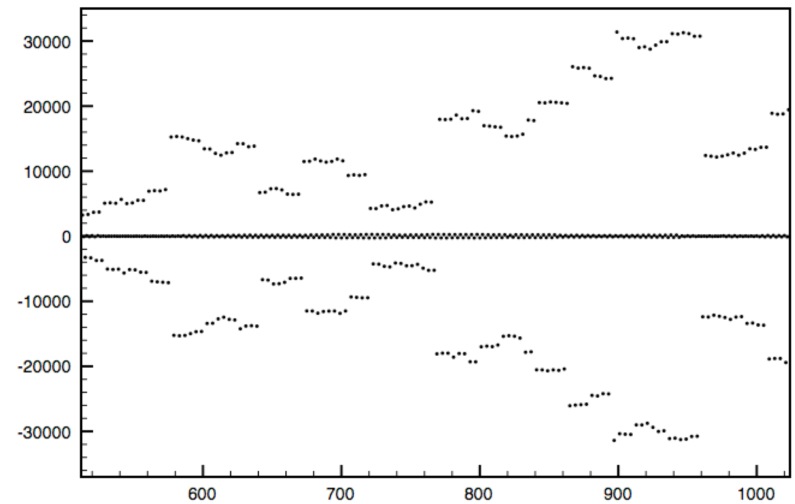


1.1 Another view

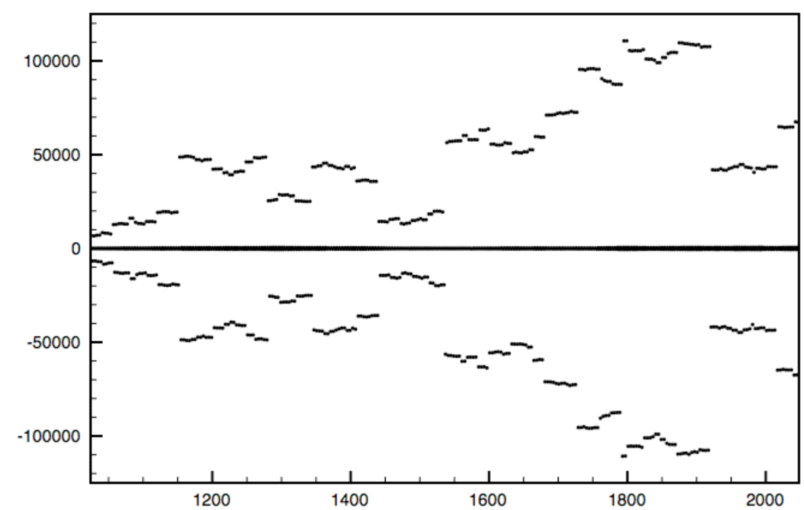
Forcing a symetrisation around the x axis we plot $\Re(S(n))$ where:

$$S(n+1) = iS(n) + a(n)$$

Plot of $\Re(S(n))$ **for** $512 \leq n \leq 1024$

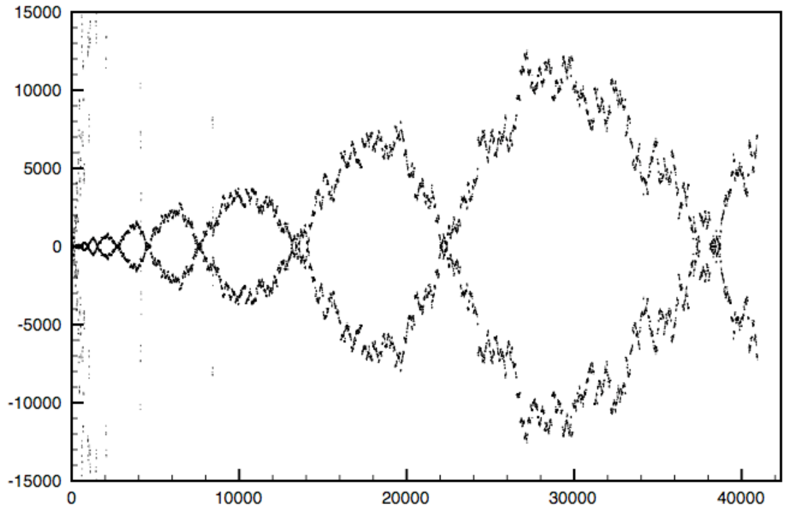


Plot of $\Re(S(n))$ **for** $1024 \leq n \leq 2048$

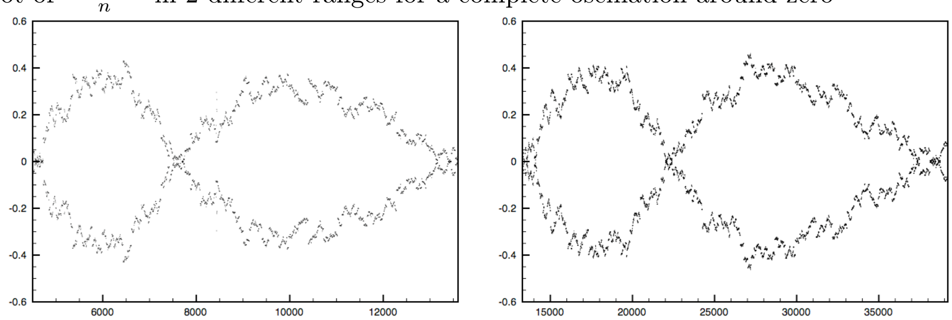


And it is interesting to make a zoom around the x axis where there is another pattern

Zoom on $\Re(S(n))$ for $1 \leq n \leq 45000$



Plot of $\frac{\Re(S(n))}{n}$ in 2 different ranges for a complete oscillation around zero

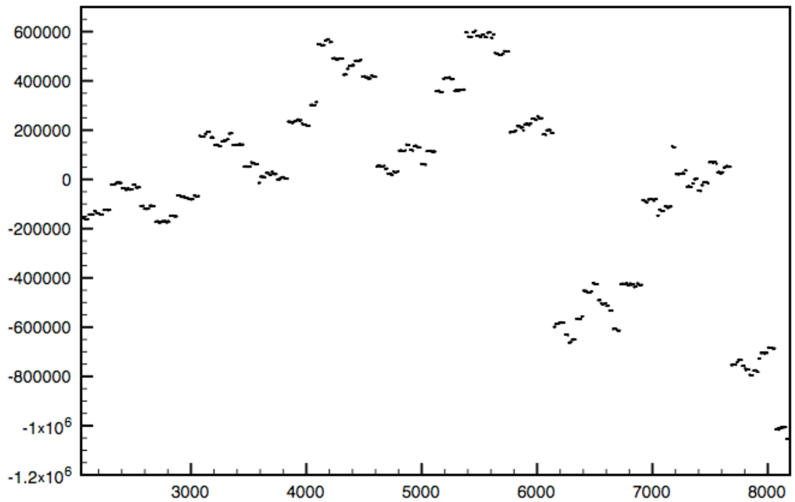


This is more chaotic than the global picture but this oscillation stays roughly self similar.

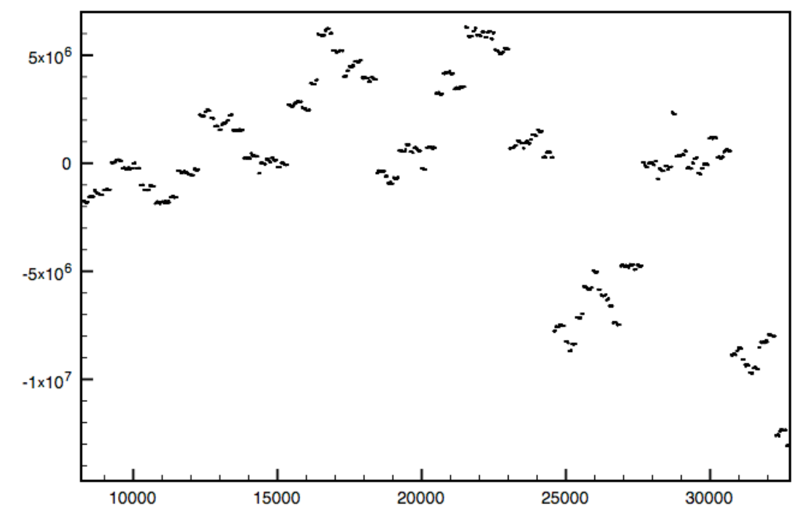
1.2 Combination with arithmetical functions

It is worth mentioning $a(n)\lambda_n$ (where $\lambda_n = (-1)^{\Omega(n)}$ is the Liouville function) appears to preserve a fractal structure. This time the scale factor is 4 not 2.

Plot of $\sum_{k=1}^{n} a(k)\lambda_k$ **for** $2048 \leq n \leq 8192$



Plot of $\sum_{k=1}^{n} a(k)\lambda_k$ **for** $8192 \leq n \leq 32768$



In the APPENDIX 4 we give several other examples. In general $\sum_{k=1}^{n} a(k)f(k)$ is fractal when f is an arithmetical function such as $f(k) = \varphi(k)$.

2 Analysis of the sequence

It appears the sequence $(a_n)_{n\geq 1}$ is clearly related to prime numbers since a quick check led us to suppose:

- $a_n = -4 \Leftrightarrow n = 2p$ where p is an odd prime
- $a_n = 2 \Leftrightarrow n = p^{2k}$ where p is an odd prime and $k \in \mathbb{N}^{\star}$.
- $a_n = -2 \Leftrightarrow n = p^{2k-1}$ where p is an odd prime and $k \in \mathbb{N}^{\star}$.

Hence we can say that a_n encapsulates informations about primes (specially $a_n = -4 \Leftrightarrow n = 2p$ where p is an odd prime) and much more information than the Moebius function. Thus the graph of $A(n)$ indicates that prime numbers follow fractal patterns among the set of integers. But much more seems true and the sequence yields all possible factorisations.

2.1 The sequence gives all factorisations

The above observations can be extended and if p, q, r, s any distinct odd primes we have also for instance:

- $a_n = -10 \Leftrightarrow n = p^2 q$.
- $a_n = -26 \Leftrightarrow n = pqr$.
- $a_n = 62 \Leftrightarrow n = p^2 qr$.
- $a_n = 150 \Leftrightarrow n = pqrs$.
- $a_n = -50 \Leftrightarrow n = p^3 q^2$.
- $a_n = -466 \Leftrightarrow n = p^2 qrs$.
- $a_n = -616 \Leftrightarrow n = 2^2 pqr$.
- $a_n = -35296 \Leftrightarrow n = 2^3 p^3 qr$.
- $a_n = -83312 \Leftrightarrow n = 2^4 p^3 q^2$.

In fact it appears the sequence $a(n)$ allows us to exhibit all integers n with a given factorisation (but not for prime powers p^n since we have in this case $a(p^n) = (-1)^n 2$). More precisely for any $r \geq 1$ and for any r odd distinct primes $(p_i)_{1\leq i\leq r}$ we claim that when $\alpha_0 \geq 2$ and $\alpha_1, \alpha_2, ... \geq 0$ we have 3 types of formulas for $a(n)$ involving only the exponents in the factorisation of n. Namely we must have from experiments something like:

$$a\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) = F_r(\alpha_1, \alpha_2, \ldots, \alpha_r)$$

$$a\left(2\prod_{i=1}^{r} p_i^{\alpha_i}\right) = G_r(\alpha_1, \alpha_2, \ldots, \alpha_r)$$

$$a\left(2^{\alpha_0}\prod_{i=1}^{r} p_i^{\alpha_i}\right) = h(\alpha_0)H_r(\alpha_1, \alpha_2, \ldots, \alpha_r)$$

Where F_r, G_r and H_r are distinct symmetric functions in r variables and h is a function in one variable (in fact it appears we can take $h(x) = 3^x$ as suggested in 2.2.2. below). Next we give an exact recurrence formula involving the divisors for integers n of form $2m-1$ or $2(2m-1)$. Then we compute special cases of F_r, G_r and H_r which are polynomials in a single variable.

2.2 Conjecture : a_n in terms of the divisors of n

We found an exact conjectural recursive formula for 2 types of numbers which has been checked for $n \leq 300000$:

$$n = 2m - 1 \Rightarrow a(n) = -2 \sum_{d|n, d<n} a(d)$$

$$n = 2(2m - 1) \Rightarrow a(n) = -2 \sum_{d|n, 2d<n} a(d)$$

Despite these recursions are simple, we didn't find closed form formula for $a(2m-1)$ nor $a(2(2m-1))$ but it is easy to see by induction that F_r and G_r are indeed non trivial symmetric functions. Unfortunately we didn't catch similar simple recurrence formula for n of form $n = 2^k(2m-1)$ and $k \geq 2$ which seems to be a difficult case.

2.2.1 Two families of polynomials

The recursions above allows to give the following kind of specific formulas:

- $a(p_1^n p_2) = -2(-1)^n(2n+1)$
- $a(p_1^n p_2 p_3) = 2(-1)^n(4n^2+6n+3)$
- $a(p_1^n p_2^2) = 2(-1)^n(2n^2+2n+1)$
- $a(p_1^n p_2 p_3 p_4) = -2(-1)^n(8n^3+24n^2+30n+13)$
- $a(p_1^n p_2^2 p_3) = -2(-1)^n(4n^3+10n^2+12n+5)$

- $a(p_1^n p_2^3) = -2(-1)^n(\frac{4}{3}n^3 + 2n^2 + \frac{8}{3}n + 1)$

and

- $a(2p_1^n p_2) = -4(-1)^n(2n^2 + 2n + 1)$
- $a(2p_1^n p_2 p_3) = 4(-1)^n(4n^3 + 10n^2 + 12n + 5)$
- $a(2p_1^n p_2^2) = 4(-1)^n(2n^3 + 4n^2 + 5n + 2)$

In general there are some polynomials $P_{\alpha_1,\alpha_2,...,\alpha_r}$ of degree $\alpha_1 + ... + \alpha_r$ and $Q_{\alpha_1,\alpha_2,...,\alpha_r}$ of degree $\alpha_1 + ... + \alpha_r + 1$ such that for $k \geq 0$ we have:

$$a\left(p_0^k \prod_{i=1}^{r} p_i^{\alpha_i}\right) = 2(-1)^{n-r} P_{\alpha_1,\alpha_2,...,\alpha_r}(k)$$

$$a\left(2p_0^k \prod_{i=1}^{r} p_i^{\alpha_i}\right) = 4(-1)^{n-r} Q_{\alpha_1,\alpha_2,...,\alpha_r}(k)$$

In the APPENDIX 1 we provide an array for $P_{\alpha_1,\alpha_2,...,\alpha_r}(k)$ and some $(\alpha_1, \alpha_2, ...)$. For numbers of form $2^k(2m-1)$ with $k \geq 2$ and $m \geq 1$ we unearthed in the next subsection another family of polynomials suggesting there is also a formula like 2.2. involving the divisors of n.

2.2.2 A third family of polynomials

As said before the case $n = 2^k(2m-1)$ and $k \geq 2$ is not easy. However we succeded to find the following conjectured formula explaining somewhat the growing fractal picture of $A(n)$ since for any fixed n odd $a(2^k n)$ is growing like 3^k. Firstly we observe a simple formula for powers of 2:

$$k \geq 2 \Rightarrow a(2^k) = 4.3^{k-2}$$

Next for $k \geq 2$, $m \geq 2$ and $n = 2^k \prod_{i=1}^{r} p_i^{\alpha_i}$ we claim there is the following formula:

$$a\left(2^k \prod_{i=1}^{r} p_i^{\alpha_i}\right) = (-1)^r 3^{k-2} R_{\alpha_1,\alpha_2,...,\alpha_r}(k)$$

where $R_{\alpha_1,\alpha_2,...,\alpha_r}$ is a polynomial in k of degree $\alpha_1 + ... + \alpha_r$. For instance we have for $k \geq 2$ and p, q, r distinct odd primes:

- $a(2^k p) = -3^{k-3}(8k + 32)$
- $a(2^k pq) = 3^{k-4}(16k^2 + 184k + 360)$
- $a(2^k pqr) = -3^{k-5}(32k^3 + 720k^2 + 4072k + 5352)$
- $a(2^k p^2 q) = -3^{k-5}(16k^3 + 336k^2 + 1760k + 2136)$

In the APPENDIX 2 we list few more polynomials $R_{\alpha_1,\alpha_2,...,\alpha_r}$.

3 Gap conjectures

We guess that the sequence $a(n)$ has intrinsic properties due to its fractal behavior, i.e., not coming from number theory subtle results, allowing us to derive something in the realm of additive number theory. To do this we state 3 conjectures related to gaps in the sequence $a(n)$. We define firstly the following set. Suppose that $\lambda, \mu \geq 1$ are fixed integers then we define

$$E_{\lambda,\mu} := \{n \in \mathbb{N} \mid a(\lambda) = a(n) = a(n+\mu)\}$$

In other words, due to the arithmetical properties of the sequence given in section 2, the set $E_{\lambda,\mu}$ contains all integers n such that n and $n+\mu$ have a same given factorisation. In general we suspect that all possible gaps are reached infinetely many time. This is more precisely described thereafter.

3.1 The gap conjecture for odd n

Suppose $x, y \geq 1$ are fixed integers and suppose $2x-1$ is not a square then we have

$$|E_{2x-1,2y}| = +\infty$$

If $2x-1$ is a square we have $|E_{2x-1,2y}| = 0$.

Examples

$E_{2\times 3-1,2\times 1} = \{3, 5, 11, 17, 27, 29, 41, 59, 71, 101, 107, 125, 137, 149, 179, 191, ...\}$

$E_{2\times 3-1,2\times 2} = \{3, 7, 13, 19, 23, 27, 37, 43, 67, 79, 97, 103, 109, 127, 163, 193, 223, ...\}$

$E_{2\times 3-1,2\times 3} = \{5, 7, 11, 13, 17, 23, 31, 37, 41, 47, 53, 61, 67, 73, 83, 97, 101, ...\}$

$E_{2\times 38-1,2\times 1} = \{423, 475, 603, 637, 845, 925, 1773, 2007, 2523, 2525, ...\}$

$E_{2\times 38-1,2\times 2} = \{171, 275, 927, 1175, 1179, 2057, 2299, 2421, 2523, 2525, ...\}$

$E_{2\times 38-1,2\times 3} = \{147, 363, 867, 925, 1519, 2523, 3751, 4107, 5547, 5819, ...\}$

3.2 The gap conjecture for n of form $2(2m-1)$

Suppose $x, y \geq 1$ are fixed integers and suppose $2x-1$ is not a square then we have

$$\left|E_{2(2x-1),4y}\right| = +\infty$$

If $2x-1$ is a square we have $\left|E_{2(2x-1),4y}\right| = 0$.

Examples

$E_{2(2\times 35-1),4\times 1} = \{66, 110, 170, 182, 186, 282, 286, 318, 366, 370, 402, 406, ...\}$

$E_{2(2\times 35-1),4\times 2} = \{70, 102, 130, 174, 182, 222, 230, 238, 258, 282, 310, 366, ...\}$

$E_{2(2\times 35-1),4\times 3} = \{30, 66, 102, 170, 174, 246, 310, 354, 406, 418, 426, 430, ...\}$

$E_{2(2\times 116-1),4\times 1} = \{1326, 2618, 3090, 3770, 4026, 4070, 4130, 4182, 4466, ...\}$

$E_{2(2\times 116-1),4\times 2} = \{2002, 2470, 2982, 3094, 3190, 3534, 4270, 4522, 4810, ...\}$

$E_{2(2\times 116-1),4\times 3} = \{858, 1110, 1218, 1290, 1794, 2478, 3090, 3198, 3306, ...\}$

3.3 The gap conjecture for n of form $2^k(2m-1)$ with $k \geq 2$

Suppose $x, y \geq 1$ are fixed integers and suppose $2x-1$ is not a square and $k \geq 2$ then we have

$$\left|E_{2^k(2x-1),2^{k+1}y}\right| = +\infty$$

If $2x-1$ is a square we have $\left|E_{2^k(2x-1),2^{k+1}y}\right| = 0$.

Examples

$E_{2^3(2\times 35-1),2^4\times 1} = \{264, 440, 680, 728, 744, 1128, 1144, 1272, 1464, 1480, ...\}$

$E_{2^3(2\times 35-1),2^4\times 2} = \{280, 408, 520, 696, 728, 888, 920, 952, 1032, 1128, 1240, ...\}$

$E_{2^3(2\times 35-1),2^4\times 3} = \{120, 264, 408, 680, 696, 984, 1240, 1416, 1624, 1672, 1704, ...\}$

$E_{2^3(2\times 116-1),2^4\times 1} = \{5304, 10472, 12360, 15080, 16104, 16280, 16520, 16728, ...\}$

$E_{2^3(2\times 116-1),2^4\times 2} = \{8008, 9880, 11928, 12376, 12760, 14136, 17080, 18088, ...\}$

$E_{2^3(2\times 116-1),2^4\times 3} = \{3432, 4440, 4872, 5160, 7176, 9912, 12360, 12792, 13224, ...\}$

3.4 Corollary: there are infinitely many twin primes

Using the conjecture 4.2 we have $|E_{6,4}| = +\infty$ which means there are infinitely many odd n such that $a(2n) = a(2n+4) = -4$. Since we made the claim

$$a_n = -4 \Leftrightarrow n = 2p$$

where p is an odd prime, there are infinitely many twin primes.

4 Conjecture : additional arithmetical properties of $a(n)$

Of course our gap conjectures allow us to derive much more results in additive number theory and here we give another conjecture. From the conjectured properties of the sequence there are two strictly increasing sequences $(b_1(n))_{n\geq1}$ and $(b_2(n))_{n\geq1}$ of odd integers such that for any fixed $k \geq 0$ we have

$$a\,(2n) = a\left(n + 2^k\right) \Leftrightarrow n \in \left\{2^k b_1(i)\right\}_{i\geq1}$$

$$a\,(2n) = a\left(n - 2^k\right) \Leftrightarrow n \in \left\{2^k b_2(i)\right\}_{i\geq1}$$

Proof $2n$ and $n \pm 2^k$ must have the same factorisation thus n must be of the form $n = 2^k(2m-1)$.

Conjecture related to the sequences b_1 and b_2

Similarly as was suggested in section 3 we guess the fractal properties of the sequence a allow us to claim that for any fixed $k \geq 0$ and any integer value x there are infintely many values of n such that we have

$$a\,(2n) = a\left(n + 2^k\right) = a(2^k(2x-1))$$

and also there are infintely many values of n such that we have

$$a\,(2n) = a\left(n - 2^k\right) = a(2^k(2x-1))$$

Consequently taking $x = 2$ there are infinitely many values of n such that:

- $b_1(n)$ and $\frac{b_1(n)+1}{2^k}$ are primes.

Also there are infinitely many values of n such that:

- $b_2(n)$ and $\frac{b_2(n)-1}{2^k}$ are primes.

In other words there are infinitely many primes p such that $2^k p - 1$ is prime and there are infinitely many primes q such that $2^k q + 1$ is prime. As a corollary there are infinitely many Sophie Germain primes.

Examples If $k = 1$ the sequence b_1 begins:

$1, 5, 13, 37, 49, 61, 65, 69, 73, 77, 129, 157, 185, 193, 221, 237, 265, 277, 309, ...$

And primes in the sequence b_1 $(5, 13, 37, 61, 73, ...)$ are primes p such that $\frac{p+1}{2}$ is also prime ($A005383$ in [Slo]). The sequence b_2 begins:

$7, 11, 23, 47, 59, 83, 107, 111, 115, 155, 167, 179, 183, 187, 227, 247, 259, 263, 267, 287, ...$

And primes in the sequence b_2 $(7, 11, 23, 47, 59, 83, ...)$ are primes p such that $\frac{p-1}{2}$ is also prime ($A005385$ in [Slo]).

5 The fractal property of a_n

The previous formulas relating the sequence to the factorisation of integers is helpless to understand why there is a fractal structure due to the mysterious nature of prime numbers. The reccurence in 2.2. shows the sequence $a(n)$ is somewhat a self referential sequence but this reccurence alone can't be used to say wether there is a fractal structure. Hence we claim the fractal structure of $A(n)$ comes simply from the discontinuity of the function $\theta(x) = (-1)^{\lfloor x \rfloor}$ at 2. Loosely speaking there is a way to see this. Let us consider a smooth example of function that is:

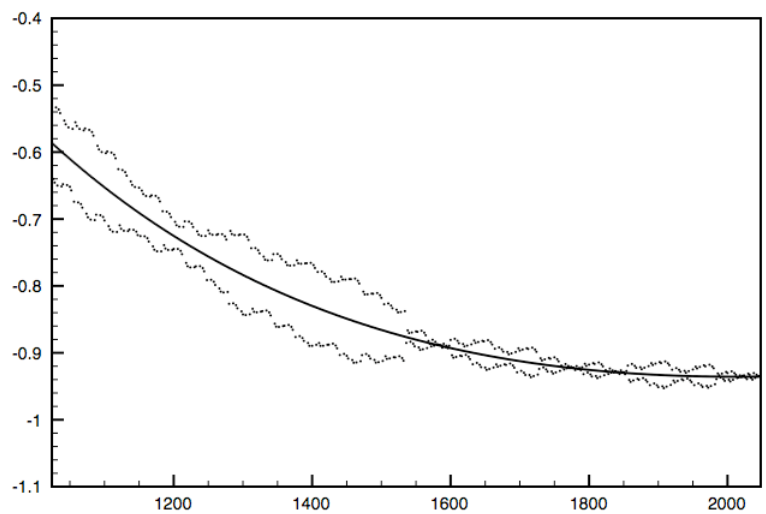
- $\theta(1/x) = 1 - 2x(1-x)$

We still define $a_1 = 1$ and $\sum_{k=1}^{n} a_k \theta(n/k) = 0$ for $(n \geq 2)$. Then this is an proved example of FGV of index 1/2 and $A(n)n^{1/2}$ is bounded and behaves very smoothly (see [Clo2]). Now consider the modified function having a discontinuity at 2:

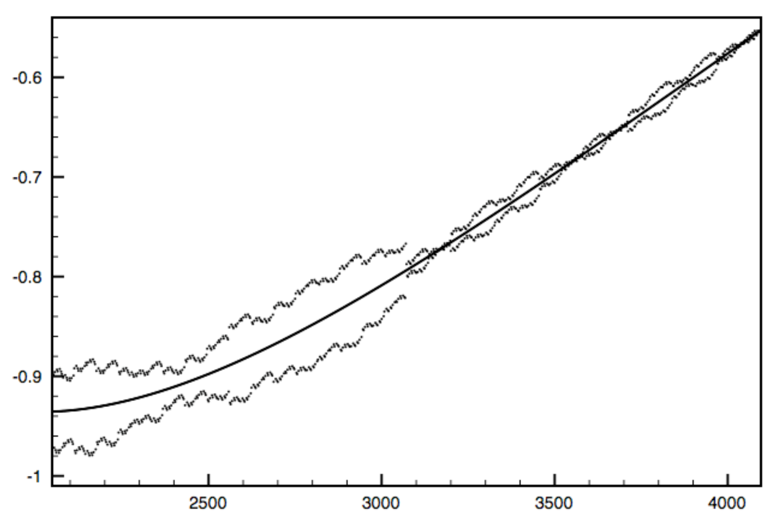
- $\theta^{\star}(1/x) = 1 - 2x(1-x)$ if $x \neq 1/2$ and $\theta^{\star}(2) = 1$.

and again let $a_1 = 1$ and $\sum_{k=1}^{n} a_k \theta^{\star}(n/k) = 0$ for $(n \geq 2)$. We then observe this perturbation generates clear fractal pattern in the vincinity of a smooth curve as shown thereafter.

Plot of $A(n)n^{1/2}$ for $1024 \leq n \leq 2048$



Plot of $A(n)n^{1/2}$ for $2048 \leq n \leq 4096$



So we see that a single discontinuity at 2 yields a fractal behaviour (around a smooth curve behaving like the graph generated by the function without discontinuity at 2 $\theta(1/x) = 1 - 2x(1-x)$). This phenomenom is discussed elsewhere [Clo].

Concluding remarks

The sequence a_n has many connections with combinatorics since we have for instance from our conjectured formula in 2.2. $a\left(p_1 p_2 ... p_n\right) = 2(-1)^n A000670(n)$, $a\left(2p_1 p_2 ... p_n\right) = 4(-1)^n A069321(n)$ or $a\left(p_0^2 p_1 p_2 ... p_n\right) = 2(-1)^n A069321(n+1)$ where $A000670$ is the number of preferential arrangements of n labeled elements and $A069321$ is the number of compatible bipartitions of a set of cardinality n for which at least one subset is not underlined [Slo]. Thus it could be closed form formulas or nice generating functions for the functions F, G and H described above.

Perhaps advances regarding our FGV concept [Clo2] could give rise to many new results in analytic number theory using suitable θ functions (see the APPENDIX 3 for examples of other functions).

To us this fractality of numbers is satisfying from a philosophical view point but not really from a mathematical view point if we restrein our goal to asymptotic considerations, i.e., the study of the behaviour of $A(n)$. Indeed the fractality is not important regarding the asymptotic behaviour and is forced by the recurrence formula once the function θ has a discontinuity at 2 as said in section 3 and we elaborate upon this in [Clo]. Hence the mystery of numbers is not explained by any kind of fractal theory and our belief is that the concept of good variation [Clo, Clo2] makes fractality natural when we consider specific FGV. Arithmetic makes sometime an appearance and our approach could have important application such as mentioned in section 3, but it looks like a nice accident and a deeper and general mathematical theory is working.

References

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[Clo] Benoit Cloitre, *Good variation*, preprint, 2011

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APPENDIX 1

Here $p_0, p_1, \ldots, p_r$ are distinct odd primes.

r	$(\alpha_i)_{1\leq i\leq r}$	$\frac{(-1)^{n-r}}{2} a\left(p_0^n \prod_{i=1}^r p_i^{\alpha_i}\right)$
1	(1)	$2n+1$
1	(2)	$2n^2+2n+1$
1	(3)	$(4/3)n^3+2n^2+(8/3)n+1$
1	(4)	$(2/3)n^4+(4/3)n^3+(10/3)n^2+(8/3)n+1$
1	(5)	$(4/15)n^5+(2/3)n^4+(8/3)n^3+(10/3)n^2+(46/15)n+1$
2	(1,1)	$4n^2+6n+3$
2	(1,2)	$4n^3+10n^2+12n+5$
2	(1,3)	$(8/3)n^4+(28/3)n^3+(58/3)n^2+(56/3)n+7$
2	(1,4)	$(4/3)n^5+6n^4+(56/3)n^3+30n^2+26n+9$
2	(2,3)	$(8/3)n^5+(44/3)n^4+(140/3)n^3+(238/3)n^2+(212/3)n+25$
3	(1,1,1)	$8n^3+24n^2+30n+1$
3	(1,1,2)	$8n^4+36n^3+78n^2+80n+31$
3	(1,1,3)	$(16/3)n^5+32n^4+(308/3)n^3+178n^2+160n+57$
3	(1,2,2)	$8n^5+52n^4+172n^3+306n^2+280n+101$
3	(2,2,2)	$8n^6+72n^5+336n^4+904n^3+1422n^2+1198n+409$
4	(1,1,1,1)	$16n^4+80n^3+180n^2+190n+75$
4	(1,1,1,2)	$16n^5+112n^4+380n^3+690n^2+640n+233$
4	(1,1,1,3)	$(32/3)n^6+96n^5+(1336/3)n^4+(3580/3)n^3+1870n^2+(4712/3)n+535$
5	(1,1,1,1,1)	$32n^5+240n^4+840n^3+1560n^2+1470n+541$

APPENDIX 2

r	$(\alpha_i)_{1\leq i\leq r}$	$a\left(2^n\prod_{i=1}^r p_i^{\alpha_i}\right)\times(-1)^r\times 3^{(2-n-\alpha_1-\alpha_2-\ldots-\alpha_r)}$
1	(1)	$8n+32$
1	(2)	$8n^2+80n+132$
1	(3)	$\frac{16}{3}n^3+96n^2+\frac{1280}{3}n+448$
1	(4)	$\frac{8}{3}n^4+\frac{224}{3}n^3+\frac{1864}{3}n^2+\frac{5344}{3}n+1412$
1	(5)	$\frac{16}{15}n^5+\frac{128}{3}n^4+\frac{1696}{3}n^3+\frac{9184}{3}n^2+\frac{32968}{5}n+4320$
2	$(1,1)$	$16n^2+184n+360$
2	$(2,1)$	$16n^3+336n^2+1760n+2136$
2	(3,1)	$\frac{32}{3}n^4+\frac{1040}{3}n^3+\frac{10048}{3}n^2+\frac{32896}{3}n+9680$
2	(4,1)	$\frac{16}{3}n^5+\frac{736}{3}n^4+\frac{11168}{3}n^3+\frac{68288}{3}n^2+54344n+38544$
2	(2,2)	$16n^4+544n^3+5528n^2+19088n+17724$
2	(3,2)	$\frac{32}{3}n^5+\frac{1568}{3}n^4+\frac{25456}{3}n^3+\frac{166720}{3}n^2+141584n+106128$
3	$(1,1,1)$	$32n^3+720n^2+4072n+5352$
3	(2,1,1)	$32n^4+1136n^3+12064n^2+43456n+41856$
3	(3,1,1)	$\frac{64}{3}n^5+\frac{3232}{3}n^4+\frac{54032}{3}n^3+\frac{363584}{3}n^2+316064n+241296$
3	(2,2,1)	$32n^5+1664n^4+28720n^3+199888n^2+539280n+424728$
4	(1,1,1,1)	$64n^4+2368n^3+26288n^2+99128n+99768$

APPENDIX 3
More fractal properties of prime numbers

For various suitable functions θ we still define the recursion

$$a_1 = 1 \text{ and } \sum_{k=1}^{n} a_k \theta(n/k) = 0 \text{ for } (n \geq 2)$$

For each function we give a property showing the sequence a_n is closely related to primes or to the factorisation of n.

The function $\theta(x) = \lfloor x \rfloor - 2 \left\lfloor \frac{x}{2} \right\rfloor$

This is $\lfloor x \rfloor$ modulo 2. Among properties of a_n there is this simple one showing the sequence is connected to the factorisation of n. We have:

- $a_n = -2 \Leftrightarrow \frac{n}{8}$ is an odd squarefree number for which the number of prime divisors is odd.

The function $\theta(x) = \lfloor x \rfloor - 3 \left\lfloor \frac{x}{3} \right\rfloor$

This is $\lfloor x \rfloor$ modulo 3. Among properties of a_n there is this one showing the sequence is connected to semi-primes. Namely we have for $n > 6$:

- $a_n = -4 \Leftrightarrow n = 6pq$ where p and q are 2 distinct odd prime numbers greater than or equal to 5.

This time this is a fractal with scale factor 3 not 2.

The function $\theta(x) = (-1)^{\sum_{k\geq 1} \tau(k) \left\lfloor \frac{x}{k} \right\rfloor}$

Here τ is the Ramanujan tau function (A000594 in [Slo]). Although the arithmetical properties of the tau function are not easy to unearth we provide here a very simple connection with prime numbers. We claim that:

- $|a_n| = 2 \Longleftrightarrow n$ is an odd prime number.

Moreover

$a_n = -2$ for those n:

$3, 5, 7, 19, 23, 29, 31, 47, 53, 67, 71, 79, 83, 89, 101, 103, 107, 137, 139, 149, 157, 163, 167, ...$

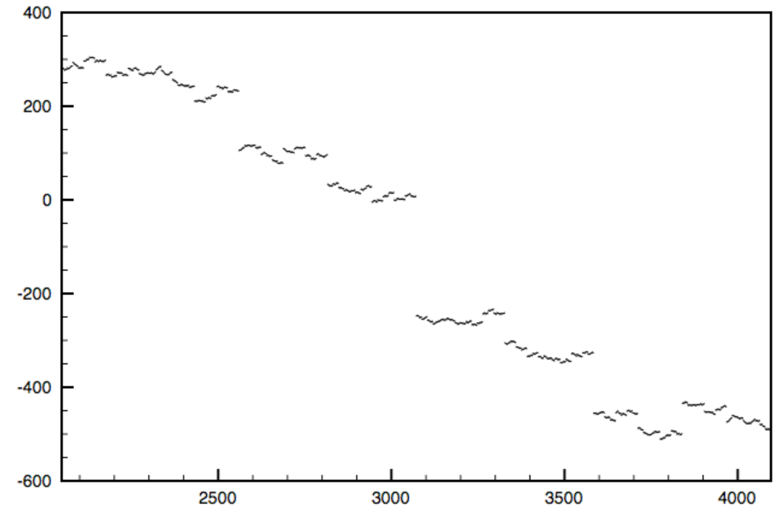
$a_n = +2$ for those n:

$11, 13, 17, 37, 41, 43, 59, 61, 73, 97, 109, 113, 127, 131, 151, 179, 191, 193, 199, 211, ...$

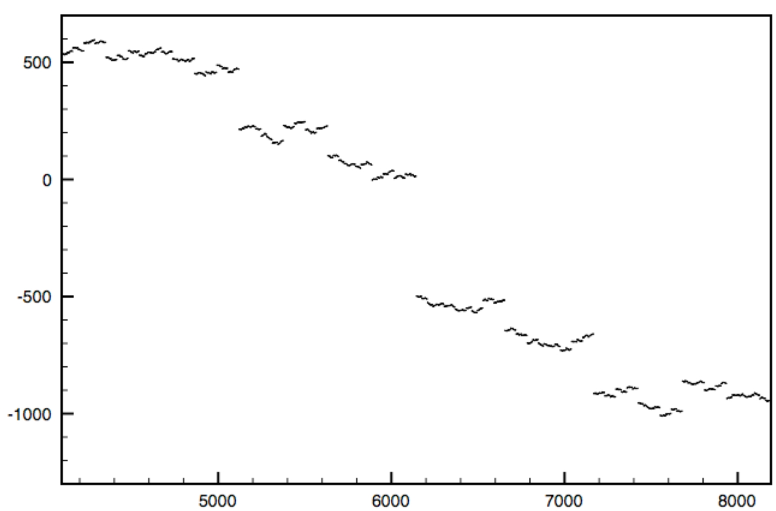
Thereafter we provide graphics for $A(n)$ and these 3 functions.

$$\theta(x) = \lfloor x \rfloor - 2\left\lfloor \frac{x}{2} \right\rfloor$$

Plot of $A(n)$ **for** $2048 \leq n \leq 4096$

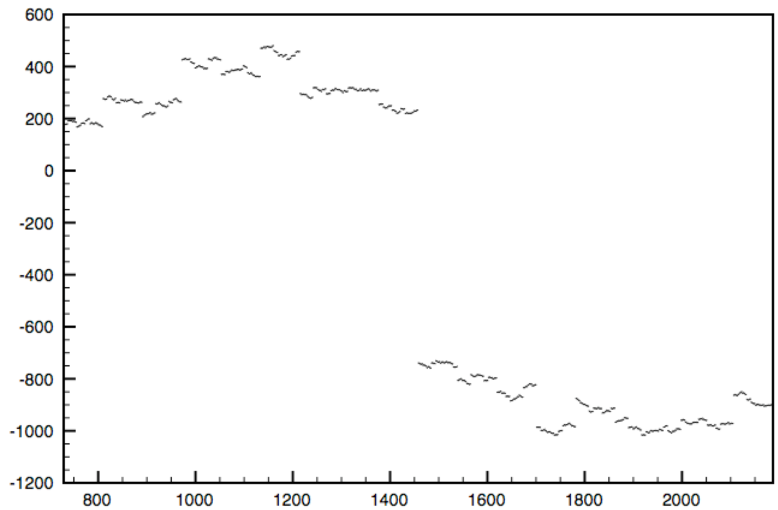


Plot of $A(n)$ **for** $4096 \leq n \leq 8192$

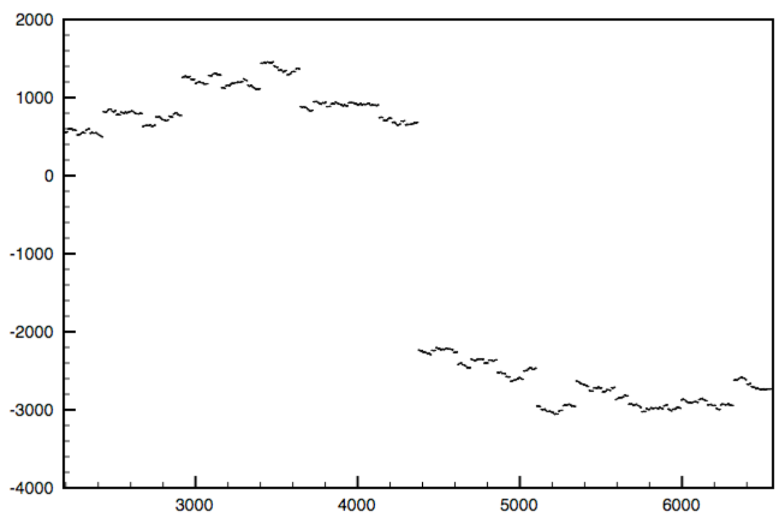


$$\theta(x) = \lfloor x \rfloor - 3\left\lfloor \frac{x}{3} \right\rfloor$$

Plot of $A(n)$ for $3^6 \le n \le 3^7$

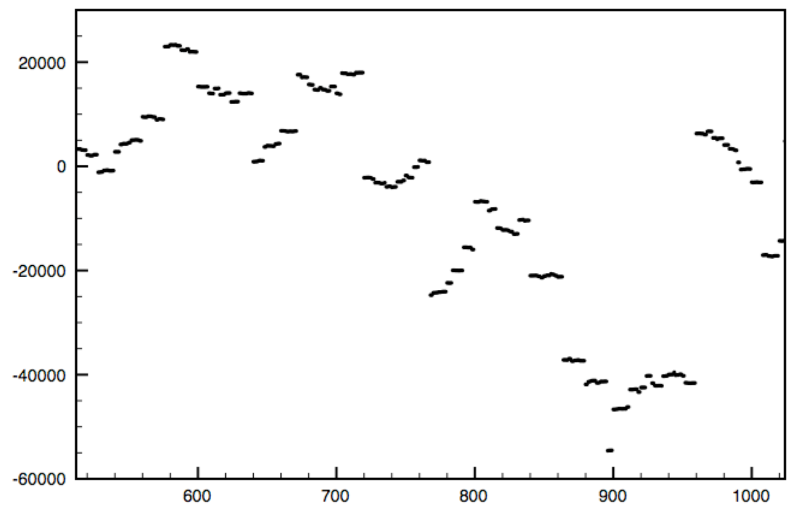


Plot of $A(n)$ for $3^7 \le n \le 3^8$

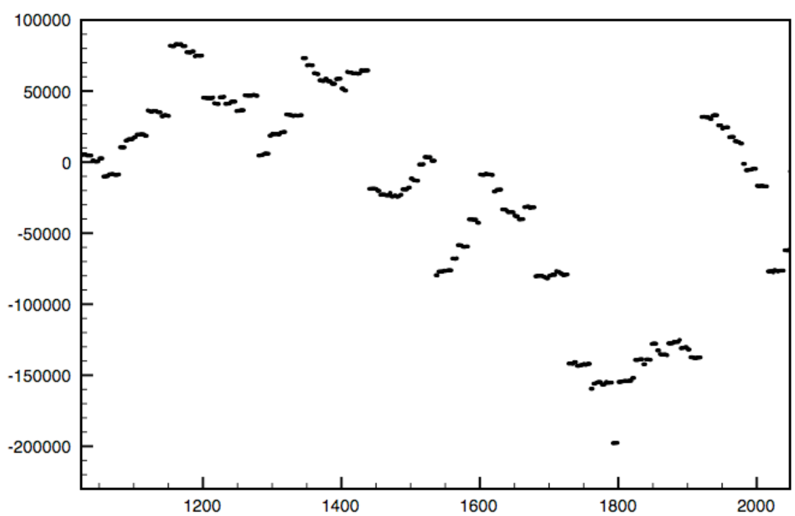


$$\theta(x) = (-1)^{\sum_{k\geq 1} \tau(k) \left\lfloor \frac{x}{k} \right\rfloor}$$

Plot of $A(n)$ for $512 \leq n \leq 1024$



Plot of $A(n)$ for $1024 \leq n \leq 2048$

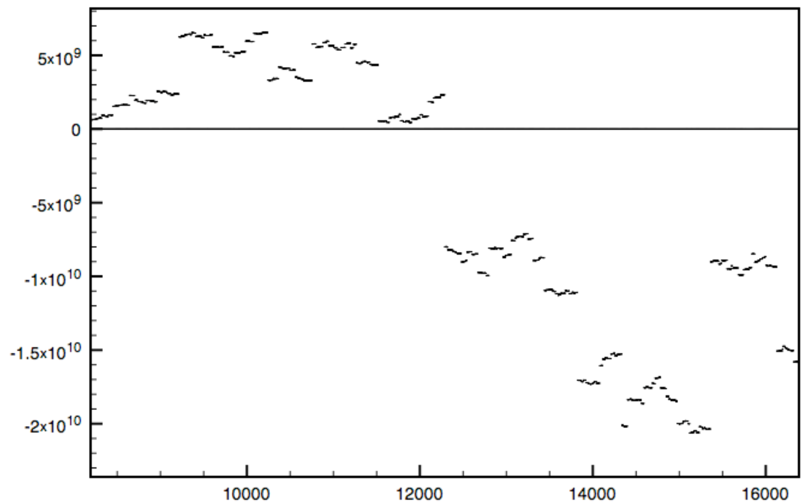


We have an asymptotic fractal picture. Small detail differ but we suspect that the plot of $A(n)$ becomes very similar for $2^k \leq n \leq 2^{k+1}$ and $k \to \infty$.

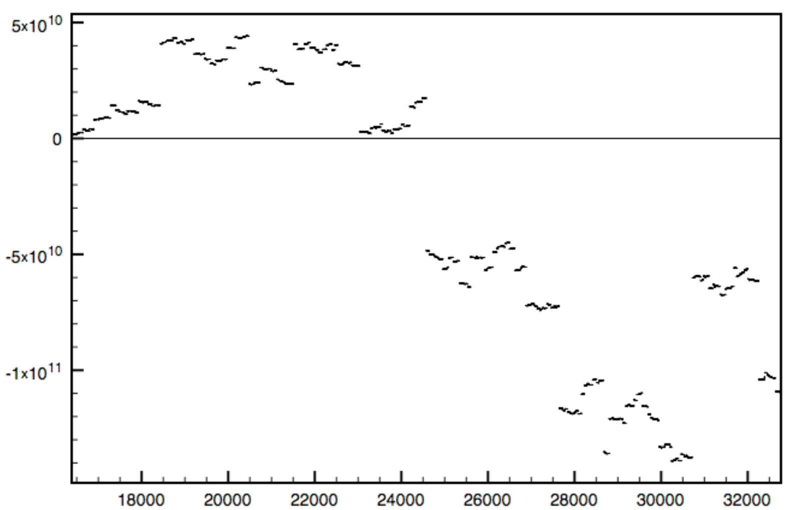
APPENDIX 4

The patterns are very similar to $A(n)$ (see section 1 to compare).

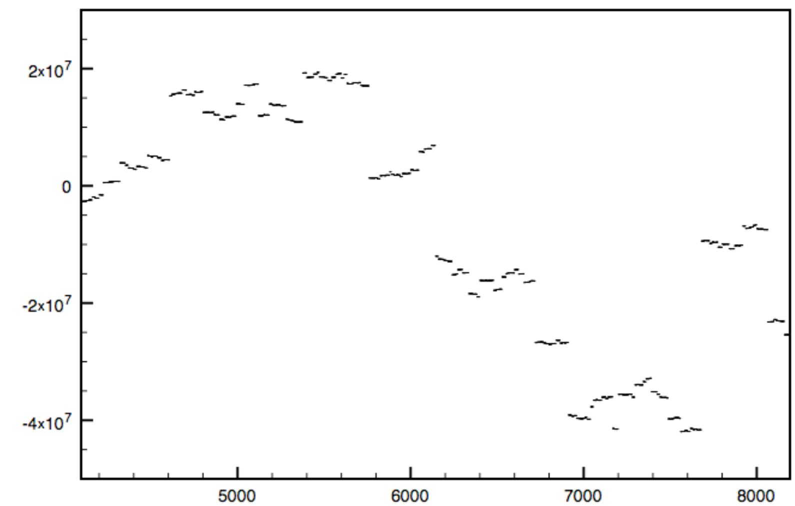
Plot of $\sum_{k=1}^{n} a(k)\varphi(k)$ **for** $8192 \leq n \leq 16384$



Plot of $\sum_{k=1}^{n} a(k)\varphi(k)$ **for** $16384 \leq n \leq 32768$



Plot of $\sum_{k=1}^{n} a(k)\tau(k)$ **for** $4096 \leq n \leq 8192$



Plot of $\sum_{k=1}^{n} a(k)\tau(k)$ **for** $8192 \leq n \leq 16384$