Enumerating the Prime Alternating Links

Stuart Rankin and Ortho Flint[†]

Department of Mathematics, University of Western Ontario

srankin@uwo.ca, ortho@uwo.ca

Abstract

In [5], four knot operators were introduced and used to construct all prime alternating knots of a given crossing size. An efficient implementation of this construction was made possible by the notion of the master array of an alternating knot. The master array and an implementation of the construction appeared in [6]. The basic scheme (as described in [5]) is to apply two of the operators, D and $ROTS$, to the prime alternating knots of minimal crossing size $n - 1$, which results in a large set of prime alternating knots of minimal crossing size n, and then the remaining two operators, T and OTS, are applied to these n crossing knots to complete the production of the set of prime alternating knots of minimal crossing size n.

In this paper, we show how to obtain all prime alternating links of a given minimal crossing size. More precisely, we shall establish that given any two prime alternating links of minimal crossing size n, there is a finite sequence of T and OTS operations that transforms one of the links into the other. Consequently, one may select any prime alternating link of minimal crossing size n (which is then called the seed link), and repeatedly apply only the operators T and OTS to obtain all prime alternating links of minimal crossing size n from the chosen seed link. The process may be standardized by specifying the seed link to be (in the parlance of [5]) the unique link of n crossings with group number 1, the $(n, 2)$ torus link.

1. Introduction

In [5], four knot operators were introduced. Of the four, the two called D and $ROTS$ were simply specific instances of the general splicing operation (see Calvo [1] for an extensive discussion of the splicing operation). A form of D was also used by H. de Fraysseix and P. Ossona de Mendez (see [3]) in their work to characterize Gauss codes, and both T and OTS appeared in Conway's seminal paper [2]. These operators were used in [5] to present a method for the construction of all prime alternating knots of a given minimal crossing size. An efficient implementation of this method was presented in [6].

When the operators D and $ROTS$ are applied to the prime alternating knots of minimal crossing size $n - 1$, the result is a large set of prime alternating knots of minimal crossing size n (in the computational work we have done, about 98% of the total number of knots have been constructed by D and $ROTS$). If the remaining two operators, T and OTS, are then applied to these n crossing knots, the remaining prime alternating knots of minimal crossing size n are obtained.

In this paper, we extend this work to show how all prime alternating links of a given minimal crossing size may be obtained. More precisely, we shall establish that given any two prime alternating links of minimal crossing size n, there is a finite sequence of T and OTS operations that will transform one of the links into the other. Consequently, one may select any prime alternating link of minimal crossing size n (which is then called the seed link), and repeatedly apply the operators T and OTS to obtain all prime alternating links of minimal crossing size n from the chosen seed link. The process may be standardized by specifying the seed link to be the $(n, 2)$ torus link.

Definition 1. A *link* of n components is a smooth embedding of the disjoint union of n copies of S^1 into R^3. The image of each copy of S^1 is called a *component* of the link. A link L is said to be *split* if there exist disjoint open 3-balls U and V such that $L \subset U \cup V$, $L \cap U \neq \emptyset$ and $L \cap V \neq \emptyset$. A link diagram is a projection of the link into a plane such that the preimage of any point is of size at most two, and a point has preimage of size two only if the point is the image of a crossing, which also displays the over/under behaviour of both strands of each crossing. A crossing is said to be a *link crossing* if the two strands of the crossing belong to different components, otherwise it is said to be a *component crossing*.

We shall regard a link diagram as a 4-regular plane graph by considering each crossing as a vertex of the graph and the portion of the curve between two consecutive crossings as an edge between the two vertices. If a link is split, then there exists a diagram of the link which is not connected (as a graph). From now on, by link we shall mean non-split link.

By a link traversal, we mean the process of assigning an orientation to each component, then traversing each component in the direction of its orientation. A link is said to be *alternating* if there exists a diagram of the link such that as the link is traversed, the sequence of overpasses and underpasses alternates. Such a diagram is said to be an *alternating diagram* for the link. An alternating diagram of a link is said to be *reduced* if it is loop-free.

Definition 2. For any positive integer m, an *m-tangle* is a connected plane graph in which there exists exactly one face, called the *edge face* of the m-tangle, for which some number k of its boundary vertices have degree less than four, with the sum of the degrees of the boundary vertices equal to $4k - m$, while all other vertices have degree 4.

We shall consider each vertex v of degree less than four in an m-tangle to have $4 - \deg(v)$ arcs lying in the edge face and incident to v (which we then refer to as arcs incident to the tangle), such that except for the endpoint v, the arcs lie in the interior of the edge face, and no two incident arcs meet each other other than at a vertex in the boundary of the edge face in the case of two such curves incident to the same boundary vertex. With this convention, we have recovered the plane projection equivalent of the usual notion of a tangle.

Definition 3. Let G be a 4-regular plane graph. An induced subgraph T of G which is an m-tangle for some positive integer m is called an *m-tangle of G*.

A reduced diagram of an alternating link is a 4-regular plane graph, and our interest will be the m-tangles found in such a graph. For each vertex v of degree

less than four in a given m-tangle in a reduced diagram of an alternating link, we shall regard each arc that is incident to v in the diagram but is not an edge of the m-tangle as an initial segment of an edge curve incident to v.

We remark that since a tangle is a graph, it must contain at least one vertex. As well, the requirement of connectedness removes the unwanted situation of what were called pass-through arcs in [5].

With this terminology, an alternating link L is prime if and only if a reduced alternating diagram D of L has no 2-tangles (see Theorem 4.4 of [4]); equivalently, if and only if D is 3-edge-connected (that is, the removal of at most two edges does not disconnect the graph).

Lemma 1. If T is an m-tangle in a reduced alternating diagram of an alternating link, then m is even.

Proof. Suppose that T contains k crossings. Then the sum of the vertex degrees in T is $4k - m$, and by the handshake lemma, this sum is twice the number of edges of T. Thus m is even. □

Note that in a reduced alternating diagram of a prime alternating link, there are no 2-tangles, so $m = 4$ is the smallest positive integer for which there exists an m-tangle in such a diagram. We shall be primarily interested in 4-tangles and 6-tangles.

We are now in a position to introduce the general tangle turn operation, of which T and OTS are just particular instances.

Definition 4. Let D be a reduced alternating diagram of an alternating prime link, and let T be an m-tangle of L. Choose an edge incident to T and, starting with the selected edge, proceed in a clockwise direction around the tangle, labelling the incident edges as $1, 2, \ldots, m$ in order as they are encountered. Then for each edge incident to T, move further away from T on the edge, and if the edge was labelled i, place the label i' on the edge, and cut the edge between the labels i and i'. For each i from 1 to $m - 1$, attach the cut edge labelled i to the cut edge labelled $(i + 1)'$, and attach the edge labelled m to the cut edge labelled 1. Finally, perform the unknotting surgery on each crossing of T. The result is a diagram of a link that is said to have been obtained by *turning T*.

It is always the case that the result of turning an m-tangle in a alternating diagram of an alternating link is again an alternating diagram of some alternating link. This may be seen by considering any two consecutive edges incident to the tangle being turned. They must be edges in the boundary walk of a face of the plane graph that is the reduced alternating diagram of the alternating link in question. Start at one of the edges and follow the boundary walk into the tangle being turned until we reach the other incident edge, labelling the ends of each edge traversed with either u if the end of the edge is an underpass, or o if the end of the edge is an overpass. It should now be apparent that the two incident edges will have opposite labelling at the vertices of the tangle to which they are attached. The result of turning the tangle will then cause every crossing of the turned tangle to have the wrong over/under behaviour. The unknotting surgery remedies this situation.

While it is natural to enquire as to whether turning a tangle in a prime alternating link will always result in a prime link, it is not difficult to see that this is not

necessarily so. For example, in Figure 1, we turn a 6-tangle in a prime alternating knot and obtain an alternating knot which is the sum of two trefoils.
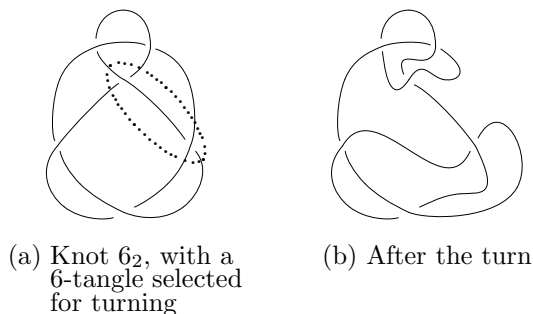

(a) Knot 6_2, with a 6-tangle selected for turning

(b) After the turn

Figure 1: Turning an m-tangle need not result in a prime link.

Proposition 1. Let D be a reduced alternating diagram of an alternating prime link L, let T be a 2-edge-connected m-tangle of D with $m \leq 6$, and let D' be the alternating diagram of an alternating link L' that is obtained by turning T. Then L' is a prime alternating link and D' is reduced.

Proof. Observe that T is a tangle in D', and that $D' - T = D - T$. Suppose that D' is not prime, and that T_1 is a 2-tangle in D'. By definition of tangle, there are vertices that don't belong to T, whence T_1 must meet both T and $D' - T = D - T$. Furthermore, since T is connected, there must be at least one edge of T joining a vertex of $T_1 \cap T$ to a vertex of $T - T_1$. Since any such edge is incident to T_1, there can be at most two such edges. If there were exactly one such edge, then that edge would be a cut-edge in T, which by hypothesis is not possible. Thus both edges incident to T_1 join vertices in $T_1 \cap T$ to vertices in $T - T_1$. Now $T_1 \cap (D' - T) = T_1 \cap (D - T)$ must have at least 4 incident edges since D is prime, and since no edge incident to $T_1 \cap (D' - T)$ can have an endpoint outside of T_1, there must be at least four edges in T_1 joining vertices in $T_1 \cap (D' - T)$ to vertices in $T_1 \cap T$. If all edges incident to T are found among these (that is to say, if all edges (four or six, as the case may be) incident to T actually join vertices in $T_1 \cap (D' - T)$ to vertices in $T_1 \cap T$), then $T - T_1$ is a 2-tangle contained in T, which is not possible since L is prime. Thus T must be a 6-tangle, and there must be two edges incident to T that join vertices in $T - T_1$ to vertices in $D' - (T \cup T_1) = D - (T \cup T_1)$. But then $D - (T \cup T_1)$ is a 2-tangle in D, which is not possible. Thus each case results in a contradiction, whence D' must be prime. $\square$

2. The T and OTS Operators

Both of these operators act on tangles; 4-tangles in the case of T, and 6-tangles for OTS. An application of either operator to an n-crossing reduced alternating diagram results in an n-crossing reduced alternating diagram. Furthermore, in every application of OTS and most applications of T, the number of link components in the resulting link will be unchanged from that of the original link. Certain applications of T will result in either an increase or a decrease of 1 in the number

of link components. As we shall see, this is enough to allow us to obtain all prime alternating links of minimal crossing size n.

2.1. *The OTS operator*

A full, proper, alternating 6-tangle which is a cycle graph on 3 vertices shall be called an OTS 6-tangle, and the OTS operator is simply the general tangle turn operation applied to an OTS 6-tangle. However, we prefer to visualize the OTS operation not as a tangle turn, but rather as the act of moving an arc joining two crossings of the OTS 6-tangle across the third crossing of the OTS 6-tangle, much like the Reidemeister III move. This point of view is illustrated in Figure 2. In (a), we have shown an OTS 6-tangle, and (b) and (c) illustrate how an OTS operation is performed on such a 6-tangle. The operator can be considered to consist of two stages: in the first stage, one of the three strands ad, be, or cf is chosen. The chosen strand is cut, moved to the other side of the crossing formed by the other two strands, and rejoined so as to preserve the over/under pattern on the strand that has been moved. This stage has been completed in Figure 2 (b), where we have illustrated the situation if strand cf was chosen. At this point, the crossing formed by the other two strands is now an over-pass when it should be an under-pass or vice-versa, so to complete the OTS operation, we must apply the unknotting surgery to this crossing (the crossing formed by strands d and e). The completed OTS operation is shown in Figure 2 (c).
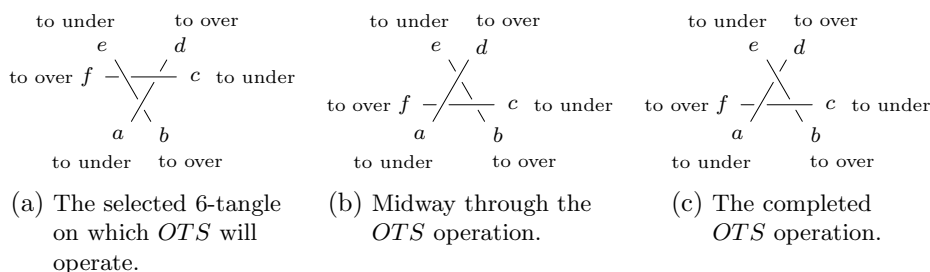

(a) The selected 6-tangle on which OTS will operate.

(b) Midway through the OTS operation.

(c) The completed OTS operation.

Figure 2: The OTS operator

The tangle turn interpretation of the OTS operation makes it clear that for a given OTS 6-tangle, the OTS operation yields the same diagram (not just flype equivalent) independently of which of the three strands is selected to move across the remaining crossing. A moment's reflection also reveals that OTS is self-inverse.

Since an OTS operation can be viewed as the cutting and rejoining of one strand, followed by the cutting and rejoining of a second strand, it follows that the end result is a diagram of a link with the same components as the original link, while the tangle turn interpretation establishes that the result of applying OTS to an alternating diagram is again an alternating diagram.

Finally, since an OTS 6-tangle has no cut-edges, it follows from Proposition 1 that the result of applying OTS to an n-crossing prime (hence reduced) alternating diagram is again a prime alternating diagram.

2.2. *The T operator*

The T operator works on tangles that we call (sub)groups of a link. The notion of group and subgroup was introduced in [5] for knots, but the same definition applies to links as well, although we do encounter a new scenario when we have a link of two or more components, which leads us to the notion of a link group.

Definition 5. A *group* in a link is a 4-tangle which is a maximal 2-braid in the link, and a *subgroup* of a group is simply a 2-braid contained in the group. At each end of a 2-braid, the two strands of the braid (both arcs incident to the tangle) are called *end arcs* of the subgroup. A group is said to be a *link group* if the two strands of the group belong to different link components, and a group that is not a link group is called a *component group*, or simply a group. A group is referred to as a *k-group* if it contains k crossings, and a group consisting of a single crossing is simply referred to as a *loner*. A component group of at least two crossings is said to be *positive* if during a traversal of the component, the two strands of the group are traversed in the same direction, otherwise it is said to be *negative*.

The T operator is simply the general tangle turn operation, but performed only on full, proper 4-tangles that are (sub)groups of the link. Since a (sub)group does not contain any cut-edges, it follows from Proposition 1 that the result of applying T to an n-crossing reduced alternating diagram again an n-crossing reduced alternating diagram. Furthermore, it is evident that T is its own inverse. We illustrate the T operator in Figure 3, where in Figure 3 (a), the torus knot of five crossings is shown, with a subgroup of size three singled out for turning. Note that the group of five crossings (and thus the subgroup of three crossings) is a positive group. Figure 3 (b), the turn has been initiated, but the unknotting has yet to be done. Finally, in Figure 3 (c), the completed turn is shown. The result is a prime alternating knot with a negative 2-group and the turned group, now a negative 3-group.
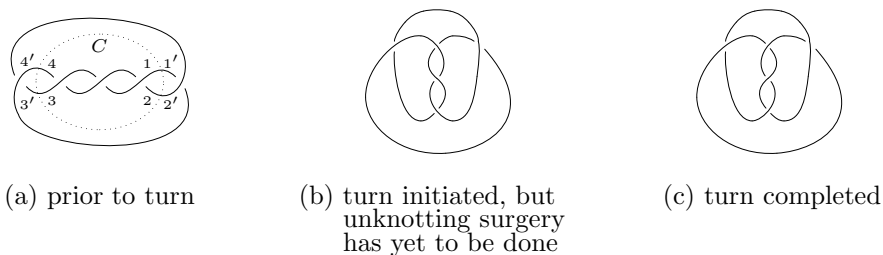

(a) prior to turn   (b) turn initiated, but   (c) turn completed
                             unknotting surgery
                             has yet to be done

Figure 3: The T operator: turning a subgroup of size 3.

The T operator can change the component count for a link. For example, when we apply T to the subgroup consisting of the top two crossings of the negative 3-group in the knot shown in Figure 3 (c), the result is the 2-component prime link shown in Figure 4 (a).

In Figure 4 (b), a 2-component prime alternating link is shown, with a link 2-group selected for turning. The effect of performing T on this link 2-group is seen in Figure 4 (c). Note that in this example, the turned 2-group has become a subgroup of a negative 3-group of the 1-component link. Note also that the link
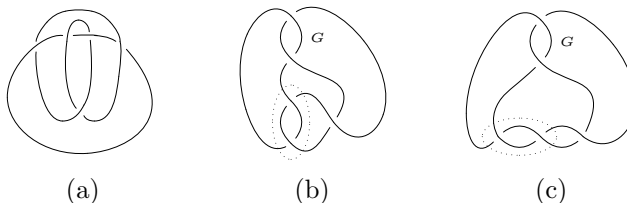
Figure 4: T can change the link component count.

2-group labelled by G in (b) has necessarily become a component 2-group in (c), positive as it turns out. It is a simple matter to modify the example in (b) by adding a crossing so that the negative group that results when the link 2-group is turned is a 4-group, but now G is a negative group.

In the preceding examples, we have seen several of the situations that may result when a group is turned. The next proposition, the proof of which is straightforward and is omitted, describes all of the situations that may occur when a group is turned.

Proposition 2. Let D be a reduced alternating diagram of an alternating link L, and let G be a (sub)group of D. Let D' denote the reduced alternating diagram that results when T is applied to G, and let L' denote the prime alternating link that is represented by D'. Then

(a) if G is an even link (sub)group, then D' is a prime alternating link with one more component than L, and G is now a negative group in D' (possibly a subgroup of a larger negative group);

(b) if G is an odd link (sub)group, then L' is a prime alternating link with the same number of components as L, and G is a link group in D' (possibly a subgroup of a larger link group);

(c) if G is an even component (sub)group, then L' is a prime alternating link with one fewer components than L, and G is now a link group in D' (possibly now a subgroup of a larger link group);

(d) if G is an odd component (sub)group, then L' is a prime alternating link with the same number of components as L, and G is still a component (sub)group, though now with the opposite sign to that which it had in D (and again, G might now be a subgroup of a larger group in D'.

We also remark that if the (sub)group that is to have T applied to it is a loner, then the result is the identical diagram. Thus in practice, we shall never apply T to a loner. Furthermore, if an $(n-1)$-subgroup of the n-group of the $(n,2)$ torus link is turned, the result is the mirror image of the $(n,2)$ torus link.

3. OTS and T make all prime alternating links

We are now in a position to establish our main objective in this paper; namely, to establish that given any two prime alternating links of the same minimal crossing size, there is a finite sequence of T and OTS operations that will transform one into the other. Since T and OTS are each self-inverse, it will suffice to prove that for any prime alternating link L of a given minimal crossing size n, there is a finite

sequence of T and OTS operations that will tranform the $(n,2)$ torus link into L. This work relies on Theorem 5 of [5], a graph-theoretic result which is applicable to reduced alternating diagrams of prime alternating links as well as to those of prime alternating knots, as was the case in [5].

For the reader's convenience, we present below the main graph-theoretic notions and results from [5] that will be required for the subsequent work.

Definition 6. A plane graph G whose edges are piecewise smooth curves is called a 2-region, respectively a minimal loop, if the following conditions are satisfied:

(a) G is connected;
(b) G has a face F whose boundary is a cycle C;
(c) there are exactly two vertices on C that have degree 2 in G (called the base vertices of the 2-region), respectively there is exactly one vertex on C that has degree 2 in G (called the base vertex of the minimal loop);
(d) every non-base vertex on C has degree 3 in G;
(e) each vertex that lies in the interior of the region $\mathbb{R}^2 - F$ has degree 4 in G.

The cycle C is called the *boundary* of the 2-region or minimal loop, respectively, while the interior of the region $\mathbb{R}^2 - F$ is called the *interior* of the 2-region, respectively minimal loop (note that in the event that all vertices of G lie on C, then F is not uniquely determined–in such a case, let F be the bounded region). In the case of a 2-region with boundary cycle C and base vertices p and q, the two paths between p to q that C determines are called the boundary paths of G.

More generally, if G is a plane graph with vertex set V, then a subgraph G' of G with vertex set $V' \subseteq V$ is said to be a 2-region (respectively, minimal loop) of G if G' is a 2-region (respectively, minimal loop) and every vertex of G that lies in the interior of G' belongs to V', and every edge of G that meets the interior of G' belongs to V'. If G' is a 2-region or minimal loop of G, then G is said to contain the 2-region or minimal loop G'.

A 2-region G is said to be *minimal* if H is a 2-region of G implies that $H = G$, and a 2-region that has no vertices in its interior is called a 2-*group*. A minimal loop with a single vertex is said to be trivial.

Note that the interior of a 2-region or minimal loop could be the unbounded region determined by the boundary of the 2-region, respectively minimal loop. Further note that a 2-group has exactly two vertices, which are multiply connected by two edges.

Proposition 3 (Proposition 8, [5]). Every non-trivial minimal loop contains a 2-region, and every 2-region contains a minimal 2-region.

Definition 7. Let D be a reduced alternating diagram of a prime alternating link L, and let v be a component crossing of D. Choose any of the four edges incident to v and construct a closed walk based at v by following a link traversal in the direction of the chosen arc until v is reached for the first time. Such a closed walk is called a *component circuit based at* v.

The proof of the next result as presented in [5] for prime alternating knots is actually valid for prime alternating links, with the modifications to the statement as shown below.

Proposition 4 (Proposition 9, [5]). Let D be a reduced alternating diagram of a prime alternating link L. Then

(a) For every component crossing v of D, and each component circuit C based at v, there is a minimal loop of D whose boundary cycle is a subwalk of C;

(b) every minimal loop of D contains a 2-region of D, and

(c) every 2-region of D contains a minimal 2-region of D.

Corollary 1. Every reduced alternating diagram of a prime alternating link different from the unknot has a minimal 2-region.

Proof. To begin with, recall that by link we mean proper link, so our link is not the unknot, nor does it consist of two or more unlinked unknots. If it contains a component crossing, then by Proposition 4, it contains a minimal 2-region. Suppose then that our diagram contains only link crossings. Choose any crossing, and select two adjacent edges incident to the crossing. Follow these edges out from the crossing. The two strands belong to different components, and since the configuration has no component crossings, each component forms a simple closed curve in the plane. Thus the two strands must meet again, and we proceed until a point of intersection of the two strands is encountered. The two paths that we have followed together form a simple closed curve in the plane, and the two paths, together with all vertices and edges in one of the regions determined by the closed curve forms a 2-region. By Proposition 4 (c), the configuration contains a minimal 2-region.   □

Note that an empty 2-region is simply a 2-subgroup.

Our next observation is that the OTS operation on an alternating link diagram has an analog for 4-regular plane graphs.

Definition 8. Let G be a 4-regular plane graph. If C is a 3-cycle in G such that no two vertices of C are multiply-connected, then C is called an *ots*-triangle in G. The 4-regular plane graph G' that results from G upon modifying an *ots*-triangle C as shown in Figure 5 is said to have been obtained from G by applying *ots* to C.

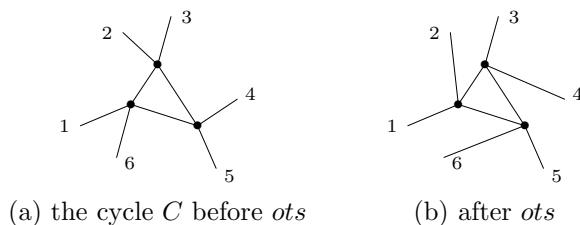(a) the cycle C before *ots*   (b) after *ots*

Figure 5: The *ots* operation on a 4-regular plane graph

Furthermore, since a 4-regular, 3-edge-connected plane graph can be considered as an reduced alternating diagram of a prime alternating link, it follows from Proposition 1 that the result of an *ots* operation on a 4-regular, 3-edge-connected plane graph is again a 4-regular, 3-edge-connected plane graph.

We shall consider *ots* operations on 3-cycles in 2-regions and minimal loops, and for this purpose, it is convenient to introduce specialized versions of *ots* for 2-regions and minimal loops. These amount to the restrictions of *ots* to the various situations involving a 3-cycle in a 2-region or a minimal loop.

Definition 9. Let G be a 2-region or a minimal loop. An *ots*-triangle in G is a face of degree 3 none of whose boundary edges belong to the boundary of a 2-group of G.

Let O be an *ots*-triangle in G, say with boundary edges B_1, B_2 and B_3. There are three possible situations: none, exactly one, or exactly two of the edges B_1, B_2, B_3 is a boundary edge of G. We define the *ots* operation in each of these three cases. Let V denote the vertex set of G and E denote the edge set of G.

Case 1: none of B_1, B_2, B_3 is a boundary edge of C. Now each pair of these edges have exactly one endpoint in common. Since a boundary vertex has at most one edge lying in the interior of G incident to it, we see that none of these three common endpoints is a boundary vertex of G, and so the compact set $B_1 \cup B_2 \cup B_3$ is contained in the interior of G. Furthermore, each of the three common endpoints has degree 4, so at each there are two additional incident edges. Since the edge curves are smooth, there is an open neighborhood U of O whose intersection with each of these six additional edges is connected, and which does not meet any other edge curve of G. Arbitrarily choose one pair of boundary edges of O, and suppose that the edges were labelled so that B_1 and B_2 are the chosen edges. Let a denote the common endpoint of B_1 and B_2, and let e and f denote the two edges that are incident to a in addition to B_1 and B_2. Further suppose that all labelling has been done so that in a clockwise scan at a, the edges are encountered in the order B_1, e, f and B_2. Choose a point $x \neq a$ in $e \cap U$, and choose a point $y \neq a$ in $f \cap U$. Let B_3' be a smooth curve from x to y within U that does not meet any curve of G other than e at x and f at y. Let b denote the common endpoint of B_2 and B_3, and let c denote the common endpoint of B_3 and B_1. Further, let g and h denote the two edges incident to b other than B_2 and B_3, labelled in order in the clockwise direction, and let i and j denote the two edges incident to c other than B_3 and B_1, labelled in order in the clockwise direction. Let j' denote a smooth curve with endpoint x that agrees with j outside of U and within U meets no edge curve of G except for e at x, and let g' denote a smooth curve with endpoint y that agrees with g outside of U and within U meets no edge curve of G except for f at y. Let the portion of e from a to x be denoted by B_1' and denote the remaining portion of e by e'. Similarly, let the portion of f from a to y be denoted by B_2' and denote the remaining portion of f by f'. Finally, let $i' = i \cup B_1$ and $h' = h \cup B_2$. Let G' denote the plane graph with piecewise smooth edge curves whose vertex set is $(V - \{b, c\}) \cup \{x, y\}$ and edge set $(E - \{B_1, B_2, B_3, e, f, g, h, i, j\}) \cup \{B_1', B_2', B_3', e', f', g', h', i', j'\}$. Since G' agrees with G outside of U, it follows that G' is a 2-region, respectively minimal loop, said to be obtained from G by an *ots* operation (on O).

Case 2: exactly one of B_1, B_2, B_3 is a boundary edge of G. Suppose that the curves were labelled so that B_3 is the boundary curve of G. Label the vertices and edges as in Case 1, with the only differences stemming from the fact that the endpoints of B_3 are boundary vertices of G (non-base, since neither B_1 nor B_2 is a boundary edge of G), so there is no edge h or i. Carry out the construction as in Case 1 (so there is no corresponding h' or i'), and let G' denote the plane graph with piecewise smooth edge curves whose vertex set is $(V - \{a, b, c\}) \cup \{x, y\}$ and edge set $(E - \{B_1, B_2, B_3, e, f, g, i\}) \cup \{B_3', e', f', g', i'\}$. Since G' agrees with G outside of U, it follows that G' is a 2-region, respectively minimal loop, said to be

obtained from G by an *ots* operation (on O). Note that the number of vertices of G' is one less than the number of vertices of G. In this case, we say that vertex a has been *ots*-ed out of G.

Case 3: two of the edges B_1, B_2, B_3 are boundary edges of G. Suppose that the curves have been labelled so that B_3 is not a boundary edge of G. Let the endpoints of B_3 be b and c. Then b and c are non-base boundary vertices, while the common endpoint of B_1 and B_2 is a base boundary vertex. Each of b and c have one additional boundary edge incident to them. Let h be the additional boundary edge incident to b and let i be the additional boundary edge incident to c, and set $h' = h \cup B_2$ and $i' = i \cup B_1$. Let G' denote the plane graph with piecewise smooth edge curves whose vertex set is $V - \{ b, c \}$ and edge set $(E - \{ B_1, B_2, B_3, h, i \}) \cup \{ h', i' \}$. Since G' agrees with G outside of U, it follows that G' is a 2-region, respectively minimal loop, said to be obtained from G by an *ots* operation (on O).

In Cases 2 and 3, we say that the *ots*-triangle is on the boundary of G.

(a) An interior *ots*-triangle

(b) After the *ots* operation

(c) An *ots*-triangle with one boundary edge

(d) After the *ots* operation

(e) An *ots*-triangle with two boundary edges
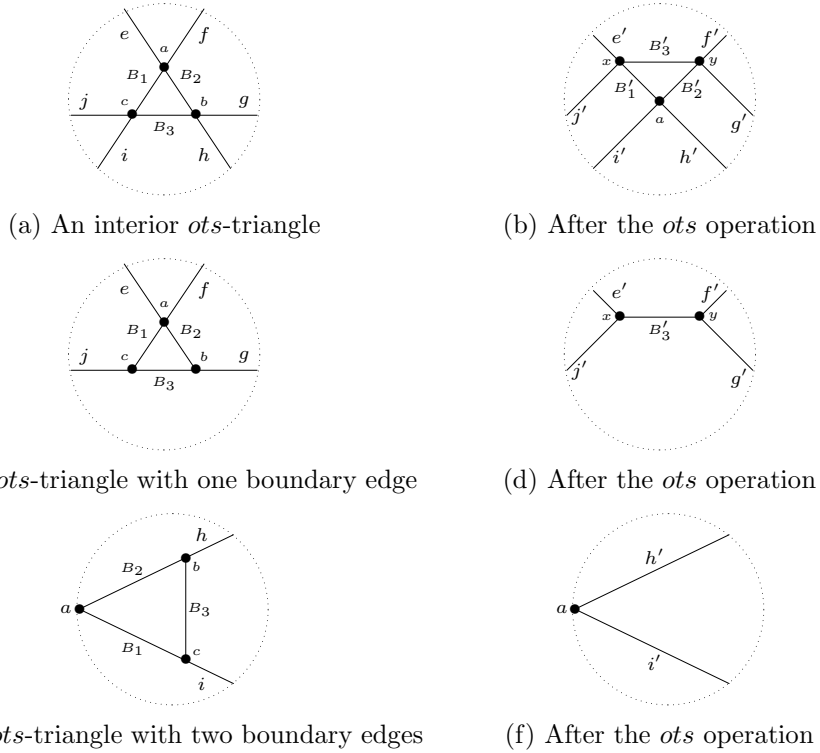

(f) After the *ots* operation

Figure 6: *ots* operations in a 2-region or a minimal loop

For example, in Figure 6 (a) and (b), an *ots*-triangle in the interior of a 2-region or minimal loop and the result of applying an *ots* operation to the *ots*-triangle are shown, while in (c) and (d), an *ots*-triangle with exactly one edge on the boundary of the 2-region or minimal loop, together with the outcome of an *ots* operation to this *ots*-triangle are shown. Finally, in (e) and (f), an *ots*-triangle with two edges

on the boundary of the 2-region or minimal loop and the effect of applying an *ots* operation to the *ots*-triangle are shown.

We remark that the plane graphs that result from each of the three possible *ots* operations that may be performed on an *ots*-triangle that is contained in the interior of a 2-region or minimal loop are isomorphic via an isotopy of the plane that fixes all points in an open neighborhood of the compact set that consists of the *ots*-triangle together with the six additional edges incident to the vertices of the *ots*-triangle.

Theorem 1 (Theorem 5, [5]). Given a minimal 2-region, there is a finite sequence of *ots* operations which will transform the minimal 2-region into an empty 2-region.

As we mentioned above, if H is a minimal 2-region of a 4-regular plane graph G, then each *ots* operation on H, as defined in Definition 9, is the restriction of a (unique) *ots* operation on G.

Definition 10. Let G be a 4-regular plane graph, and let v and w be the two vertices of a 2-(sub)group H of G, with edges e and f being the edges of the subgroup. Form a new plane graph G^* from G by the following process: continuously shrink the two edges e and f to cause v and w to become identified, forming a new vertex $\overline{vw}$ located in the interior of the empty region bounded by e and f. At the same time, permit every edge of G that is incident to either v or w to continuously extend, ultimately to have v and/or w become $\overline{vw}$ when the contraction of e and f is complete. G^* has one fewer vertices and two fewer edges than G. We say G^* has been created by *collapsing the (sub)group H*.

Proposition 5. Let H be a 2-(sub)group of a 4-regular, 3-edge-connected plane graph G, and let G^* denote the graph obtained by collapsing H. Then G^* is a 4-regular, 3-edge-connected plane graph.

Proof. G^* is a plane graph by construction, and it is immediate that G^* is 4-regular and connected. Suppose that G^* is not 3-edge-connected. Then there must exist two edges, say e_1 and e_2, whose deletion from G^* results in a disconnected graph. But then there must be distinct vertices x and y of G^* such that every path from x to y uses either e_1 or e_2. Suppose that neither x nor y is the vertex representing the collapsed group H. Then x and y are vertices of G, and since G is 3-edge-connected, there exists a path P from x to y that uses neither e_1 nor e_2 (where we consider e_1 and e_2 as edges of G, identifying an edge incident to v or w with its extended image in G^* if necessary). If P contains one of the edges of H, then replace the segment of P that consists of the edge and the two end points by the vertex that represents H to obtain a path in G^* from x to y that uses neither e_1 nor e_2. Since this is not possible, it must be that one of x or y is the vertex representing H. Without loss of generality, suppose that x is this vertex. Let v be one of the vertices of H. Since $y \neq x$, y is a vertex of G, and y is not a vertex of H. As before, since G is 3-edge-connected, there is a path P from v to y that uses neither e_1 nor e_2. Let w be the other vertex of H. If w does not appear in P, then P is a path in G^* from x to y that uses neither e_1 nor e_2. Since this is not possible, w must appear in P. But then the segment of P from w to y provides a path from x to y in G^* that uses neither e_1 nor e_2. Thus the assumption that G^* is not 3-edge-connected has led to a contradiction, and so it followes that G^* is 3-edge-connected, as required. □

Proposition 6. Let G be a 4-regular, 3-edge-connected plane graph without 2-(sub)groups. Then either G is simple or else G consists of a single vertex with two loops.

Proof. Suppose that G is not simple. By hypothesis, G has no multiply-connected vertices, so it must have a loop e at some vertex v. Since G is 4-regular, either there is a second loop based at v, or else that are two additional edges incident to v. But G is 3-edge-connected, so there can't be two additional edges incident to v, and so there is a second loop based at v. Since G is connected and 4-regular, it follows that v is the only vertex of G, and the two loops are the only edges of G. $\qquad\square$

3.1. *Condensing an alternating link diagram*

Definition 11. The *condensation* of a reduced alternating diagram D of an alternating link L is the 4-regular plane graph G that is obtained from D by replacing each group by a single crossing.

For example, the condensation of the reduced alternating diagram of the $(n, 2)$ torus link is a single vertex with two loops. This graph shall be denoted by G_0.

It was established in Proposition 5 that for any 4-regular, 3-edge-connected plane graph D, the result of collapsing a 2-(sub)group is again a 4-regular, 3-edge-connected plane graph. Since the condensation of D can be obtained by repeatedly collapsing 2-(sub)groups until none of the original 2-(sub)groups of D remain, it follows that the condensation of D is again a 4-regular, 3-edge-connected plane graph. Moreover, we may then repeat the process to find the condensation of that graph. By iterating the condensation operator, we will eventually arrive at a 4-regular, 3-edge-connected plane graph without 2-groups. By Proposition 6, such a graph is either G_0 (a single vertex with two loops), or else it is simple. Suppose that the graph is not G_0. Then it may be considered to be a reduced alternating diagram of some prime alternating link and so by Proposition 4, it contains a minimal 2-region (but no 2-group, since all 2-groups have been collapsed). By Theorem 1, there is a finite sequence of *ots* operations that will empty the minimal 2-region. The result of this is a 4-regular, 3-edge-connected plane graph with at least one 2-group. The entire process can now be repeated. Since each condensation step results in a decrease in the number of vertices, the process must eventually terminate, with G_0 as the result.

The process outlined above provides a blueprint for the transformation of an n-crossing reduced alternating diagram of a prime alternating link to the reduced alternating diagram of the $(n, 2)$ torus link, as we shall show next that each of the steps in the graph transformation described above is supported by a corresponding sequence of T and/or OTS operations on an n-crossing reduced alternating diagram. The sequence of condensations and/or *ots* operations on the graph which is the reduced alternating diagram D of a prime alternating link L thereby gives rise to a (possible longer) sequence of T and/or OTS operations on D which transforms D into the reduced alternating diagram of the $(n, 2)$ torus link.

To begin with, suppose that D is an n-crossing reduced alternating diagram of a prime alternating link L. Set $G = D$, so that G is a 4-regular, 3-edge-connected plane graph with n vertices, and carry out the reduction of G to the graph G_0

consisting of two loops on a single vertex. Suppose that at a certain point in this process, we have arrived at an n-crossing reduced alternating diagram D', and the next step in the graph transformation is a condensation. If this is the very first step in the process, then nothing need be done to D. Otherwise, it will be the case that in each 2-subgroup of the group being condensed, each of the two edges of the 2-subgroup represents one edge from each end of a group in D'. Such a situation is illustrated in Figure 7. As shown there, the act of collapsing the 2-subgroup in the graph corresponds to turning one (or both) of the two subgroups of D' to form a larger subgroup in the resulting link.

(a)   a 2-subgroup about to be collapsed as part of a condensation

(b) the collapsed 2-subgroup

(c)   the groups represented by the 2-subgroup

(d)   the group represented by the collapsed 2-subgroup
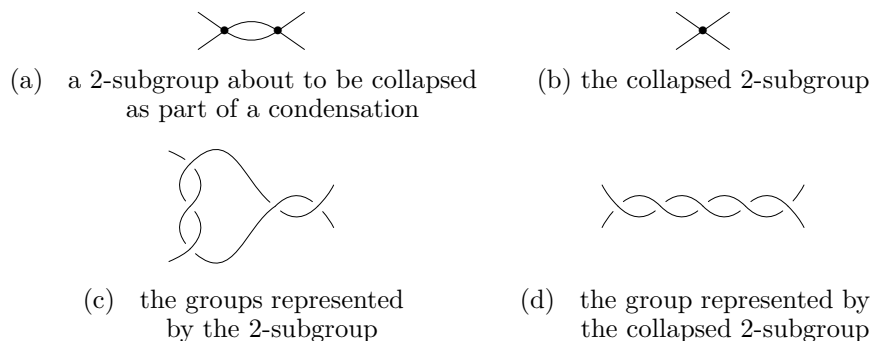
Figure 7: Collapsing may require turning a subgroup

Thus we see how to manipulate a link diagram to mirror the collapse of a 2-subgroup in the graph and hence the condensation of the link diagram. Suppose now that we are at a stage where no further condensation is possible. At this point, we have an n-crossing reduced alternating diagram D'' and its condensation, G'', and either $G'' = G_0$, and we are done, or else G'' is a simple graph on k vertices for some k with $2 \le k \le n$. Suppose that G'' is simple. Since G'' is a 4-regular, 3-edge-connected plane graph, it is a reduced alternating diagram of some prime alternating link with k crossings. By Proposition 4, G'' contains a minimal 2-region, and by Theorem 1, there is a finite sequence of *ots* operations that, when applied to G'', empties the minimal 2-region. The result is a 4-regular, 3-edge-connected plane graph, G_1, on k vertices that contains a 2-group (possibly as many as three 2-groups). It remains for us to demonstrate that each of these *ots* operations is mirrored by a sequence of T and/or OTS operations on D'', resulting in a reduced alternating diagram D_1 of a prime alternating n-crossing link whose condensation is G_1. In fact, there are many different such sequences, and we shall describe only one, which we shall refer to as $TOTS$.

3.2. $TOTS$: representing *ots* operations at the link diagram level

Suppose that L is an n-crossing prime alternating link with reduced alternating diagram D, and let G denote the condensation of D. Suppose further that G is a simple graph on k vertices for some k with $2 \le k \le n$, and that in G we have an *ots*-triangle with nodes A, B, and C representing groups in D of sizes a, b, and c, respectively, as shown in Figure 8 (i) and (ii). Note that we are using symbolism for

groups that was introduced in [5], where the major axis of the ellipse is intended to represent the two strands that are wrapping around each other to form the group. We wish to find a finite sequence of T and/or OTS operations which, when applied to D, results in a reduced alternating diagram D' of a prime alternating link L' such that the condensation of D' is the result of applying *ots* to the *ots*-triangle with nodes A, B, and C in G. Note that it may have been necessary to turn any or all of the three groups represented by A, B, and C in order that they be aligned as shown. In such a case, these turn operations would be included at the beginning of the sought-after sequence of T and/or OTS operations.

(i) an *ots*-triangle in G (ii) the configuration in D that is
                                            represented by the *ots*-triangle
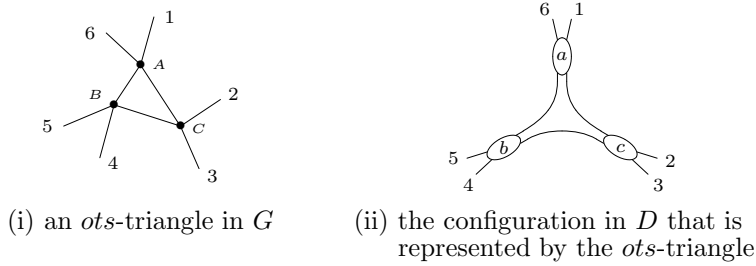
Figure 8: An *ots* triangle in G and the configuration it represents in D

We begin with the case when one of the groups labelled by A, B, and C is a loner.

Proposition 7. If the group labelled by A is a loner, then there exists a finite sequence of T and OTS operations on D that results in a reduced alternating diagram D' of a prime alternating link L' whose condensation is the result of applying *ots* in G to the *ots*-triangle with nodes A, B, a group of size b, and C, a group of size c. Specifically, if b is odd, then there exists a sequence of T and OTS operations that transforms Figure 9 (a) into Figure 9 (b), while if b is even, then there exists a sequence of T and OTS operations that transforms Figure 9 (a) into Figure 9 (c).
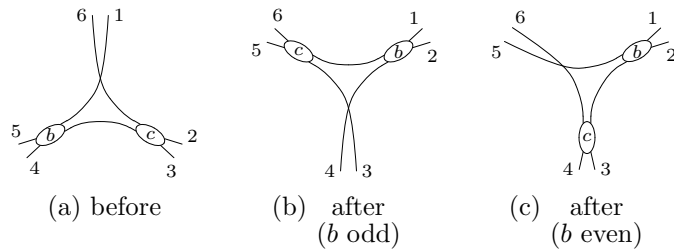

(a) before   (b)   after   (c)   after
                         (b odd)   (b even)

Figure 9: The representation of *ots* with at least one loner

Proof.   The proof will be by induction on b. If $b = 1$, then the *ots*-triangle corresponds to an OTS 6-tangle in the link, where two of the three crossings of the OTS 6-tangle are loners, while the third is an end crossing of a group of size c (it is possible that $c > 1$). The arc with endpoints the groups labelled by A and B may be OTS'ed across each crossing in the group labelled C in turn, requiring c OTS

operations in all. The result is as shown in Figure 9 (b) (with $b = 1$), which is the desired result since $b = 1$ is odd.

Suppose now that $b \geq 1$ is an integer for which the statement of the proposition is valid for all groups B of size at most b, and consider an *ots*-triangle in which vertex A represents a loner, the crossing x in Figure 10 (a), B represents a group of size $b + 1$, and C represents a group of arbitrary size c. Let B_1 represent the subgroup of size b that is obtained from the group B by separating off the end crossing (denoted by y) whose arcs are connected to groups A and C, as shown in Figure 10 (a). Then y, x and C form an *ots*-triangle with y and x single crossings. This is handled in the same way as in the base case, namely the arc with endpoints y and x is OTS'ed across the group C. The result is as shown in Figure 10 (b).
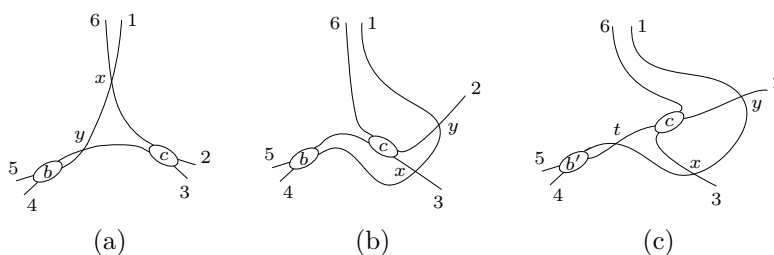
Figure 10:

Turn group C, and separate off a crossing t from the group of size b, denoting the result as a group of size $b' = b - 1$ (note that $b' = 0$ is possible in this setting). Consider the triangle with nodes the crossing t, the group of size c, and the crossing x, as shown in Figure 10 (c). OTS the arc with endpoints t and x across the group C, obtaining the result shown in Figure 11 (a).
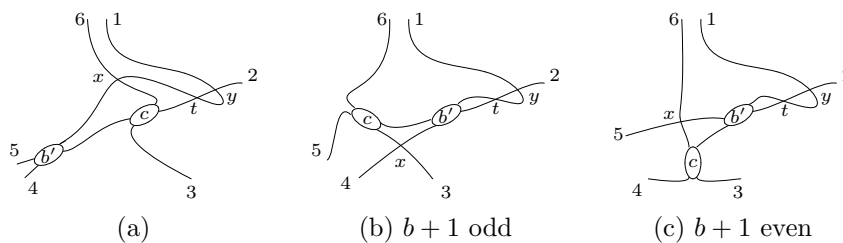

Figure 11:

After turning the group of size c in Figure 11 (a), we may apply the induction hypothesis to the ots triangle whose nodes are the crossing x together with the groups of size b' and c. We consider the two possibilities: $b + 1$ odd, and $b + 1$ even. Since $b' = b - 1$ has the same parity as $b + 1$, the result when $b + 1$ is odd is as shown in Figure 11 (b), and the result when $b + 1$ is even (even in the case $b' = 0$) is as shown in Figure 11 (c). In both cases, the group of size $b' = b - 1$ combines with the group of 2 consisting of the crossings t and y to reconstitute the group of size b, as required. The result now follows by induction. $\square$

Theorem 2. Given any reduced alternating diagram D of a link L, and any groups A, B, and C in D which form an *ots*-triangle O in the condensation G of D, there exists a finite sequence of T and OTS operations on D that results in a reduced alternating diagram D' of a prime alternating link L' whose condensation is the result of applying *ots* in G to the *ots*-triangle O.

Specifically, let a, b, and c denote the sizes of A, B, and C, respectively, which form the configuration in D shown in Figure 12, where the groups have been labelled so that if at least two of the groups have odd size, then B and C are taken to be groups of odd size.
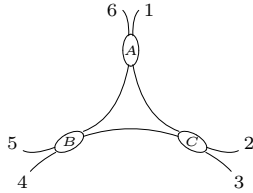
Figure 12:

(a) If b and c are both odd, then there is a finite sequence of T and OTS operations that transforms the configuration in Figure 12 to that shown in Figure 13 (a) if a is even, or Figure 13 (b) if a is odd.
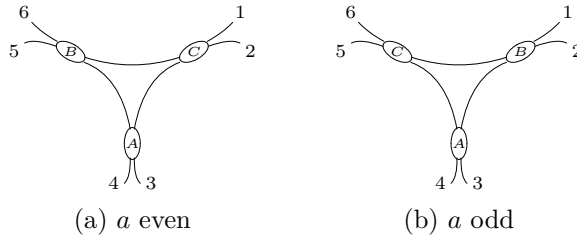

(a) a even   (b) a odd

Figure 13:

(b) If b is odd, while a and c are even, then there is a finite sequence of T and OTS operations that transforms the configuration in Figure 12 to that shown in Figure 14 (a), while if b is even and a is even, then there is a finite sequence of T and OTS operations that transforms the configuration in Figure 12 to that shown in Figure 14 (b).
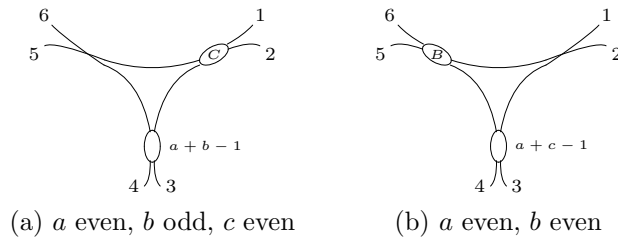

(a) a even, b odd, c even   (b) a even, b even

Figure 14:

Proof.   The proof of (a) is a straightforward induction argument (utilizing Proposition 7) on the size of the group A, where B and C are both odd size groups.

For (b), we begin with the case when b is odd, while a and c are both even. The proof will be by induction on n, where $a = 2n$. Our hypothesis is that for any odd group B of size b and even group C of size c and A a group of even size $a = 2n$, there exists a finite sequence of T and/or OTS operations that will transform the diagram shown in Figure 12 into the diagram shown in Figure 14 (a). We shall provide an argument which will establish both the base case and the inductive step. Suppose now that we have such a situation with $n \geq 1$. Apply Proposition 7 to the diagram formed by B, C and the bottom crossing of A. The result is as shown in Figure 15 (a).
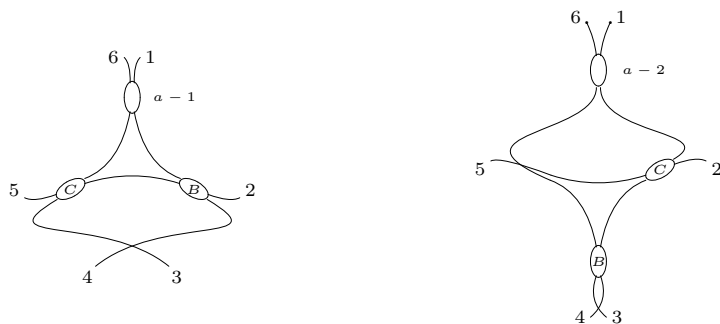
(a) the first crossing of A has been OTS'ed across B and C

(b) the second crossing of A has now been OTS'ed

Figure 15:

We now apply Proposition 7 to the diagram formed by group C, group B and the bottom crossing of the group of size $a - 1$. Since C is a group of even size, the result is as shown in Figure 15 (b). If $a - 2 = 0$, we have the proof of the base case, while if $a - 2 > 0$, then the induction hypothesis applies and the remaining $a - 2$ crossings can OTS to join the group of size $b + 1$ formed by merging B and the first OTS'ed crossing of A. The result follows now by induction.

Finally, consider the case when b is even and a is even. The proof will be by induction on n, where $a = 2n$. Our hypothesis is that for any even group B of size b, group C of size c, and group A of even size $a = 2n$, there exists a finite sequence of T and/or OTS operations that will transform the diagram shown in Figure 12 into the diagram shown in Figure 14 (b). We shall provide an argument which will establish both the base case and the inductive step. Suppose now that we have such a situation with $n \geq 1$. Apply Proposition 7 to the diagram formed by B, C and the bottom crossing of A. The result is as shown in Figure 16 (a). We may now apply the induction hypothesis (or the base case has been established in the case $a = 2$) to OTS the group of size $a - 2$ across the arc joining B and the single crossing so as to join up with the group which is the combination of C and the first OTS'ed crossing of A, thereby forming a group of size $(a - 2) + 1 - 1 + (c + 1) = a + c - 1$, as required. The result now follows by induction. $\square$
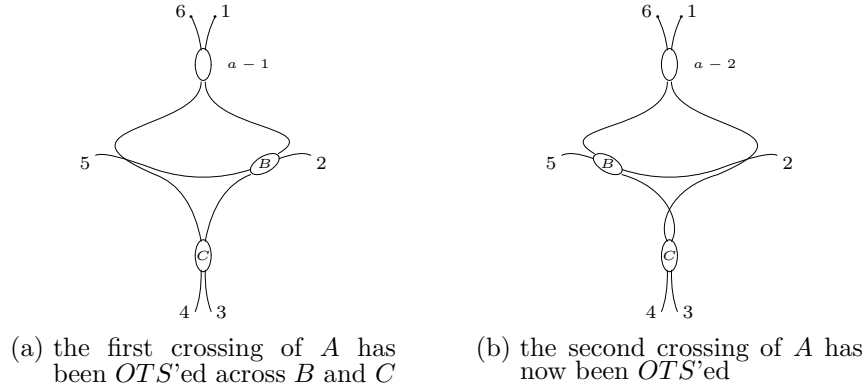
(a) the first crossing of A has been OTS'ed across B and C

(b) the second crossing of A has now been OTS'ed

Figure 16:

4. A 9-crossing example

In this final section, we give an example of a 9-crossing alternating link (actually, knot 9_{32}) and its reduction to the 9-crossing torus link. We begin with a reduced alternating diagram of knot 9_{32} and its graph, and then show one sequence of t and *ots* that will reduce the graph to the one vertex, 2 loop graph alongside the changes in the knot diagram that result from the corresponding sequence of T and OTS applications. The end result is a reduced alternating diagram of the 9-crossing torus knot.
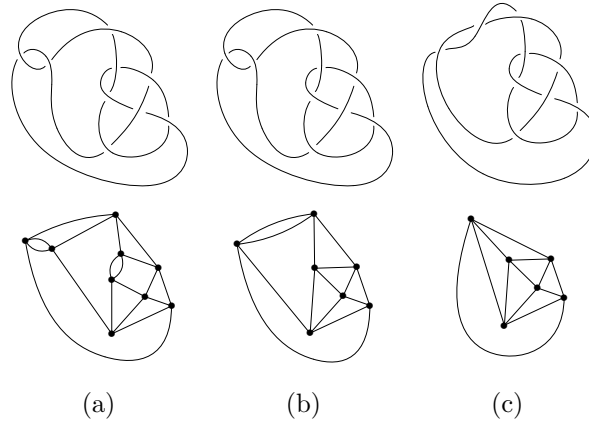

(a)   (b)   (c)

Figure 17: The initial stage in the reduction of knot 9_{32} to the 9-crossing torus knot.

In Figure 17 (a), we present a reduced alternating diagram for knot 9_{32}, with the graph of the diagram shown below it. Then in Figure 17 (b), the first round of contraction in the graph is shown. Note that there is no change in the diagram.

Then in Figure 17 (c), we see that the second round of contraction results in a 4-regular simple graph. This required that a single T operation be performed on

a negative even (2) group in the diagram of Figure 17 (b), and, as described in Proposition 2, the number of components has gone up by one, so we now have a two component link. Since we have now arrived at a 4-regular simple graph, it is necessary to use *ots* operations to empty out a min-2-region in the graph. There are eight *ots*-triangles in the graph shown in Figure 17 (c), and as it turns out, performing any one of the corresponding *ots* operations will result in a 2-group. We choose an *ots*-triangle and perform an *ots*-operation on it. The result of our choice is shown in the graph of Figure 18 (a), while in the diagram above the graph, we show the result of the T operation that must precede the two OTS operations which, taken together, correspond to the *ots* operation on the graph as shown in Figure 18 (a).
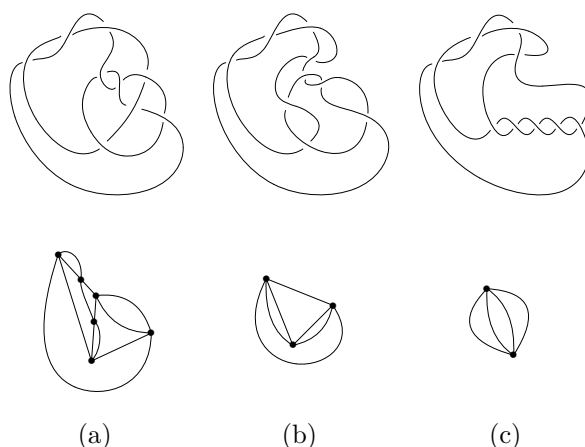

(a)   (b)   (c)

Figure 18: The second stage in the reduction of knot 9_{32} to the 9-crossing torus knot.
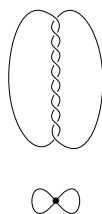

Figure 19: The final stage in the reduction of knot 9_{32} to the 9-crossing torus knot.

In the diagram shown in Figure 18 (b), the two OTS operations have been performed, and the graph shown below the diagram displays the result of the first round of contractions in the graph of Figure 18 (a). No T operations in the diagram are required for the first round of contractions.

Next, the graph in Figure 18 (c) shows the result of the second round of contractions. This round of contractions require that two applications of T be performed on the diagram shown in Figure 18 (b). One of the applications of T is to a link 2-group and so the number of components is reduced by one, resulting in a diagram of a knot.

In Figure 19, the final collapse in the graph is shown. This requires one T operation on the diagram, and the result is the 9-crossing torus knot.

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