Pattern-Avoiding Permutations

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Let $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ be a permutation on $\{1, 2, \ldots, m\}$. Define a **pattern** $\tilde{\sigma}$ to be the string $\sigma_1\varepsilon_1\sigma_2\varepsilon_2\cdots\varepsilon_{m-1}\sigma_m$, where each ε_j is either the dash symbol - or the empty string. For example,

1-3-2,   1-32,   132

are three distinct patterns. The first is known as a **classical pattern** (dashes in all $m-1$ slots); the third is also known as a **consecutive pattern** (no dashes in any slots). Some authors call $\tilde{\sigma}$ a "generalized pattern" and use the word "pattern" exclusively for what we call "classical patterns".

Let $\tau = \tau_1\tau_2\cdots\tau_n$ be a permutation on $\{1, 2, \ldots, n\}$, where $n \geq m$. We say that τ **contains** $\tilde{\sigma}$ if there exist $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ such that

- for each $1 \leq j \leq m-1$, if ε_j is empty, then $i_{j+1} = i_j + 1$;
- for all $1 \leq k \leq m$, $1 \leq l \leq m$, we have $\tau_{i_k} < \tau_{i_l}$ if and only if $\sigma_k < \sigma_l$.

The string $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_m}$ is called an **occurrence** of $\tilde{\sigma}$ in τ. If τ does not contain $\tilde{\sigma}$, then we say τ **avoids** $\tilde{\sigma}$ or that τ is $\tilde{\sigma}$**-avoiding**. For example,

24531 contains 1-3-2

because 253 has the same relative order as 132, but

42351 avoids 1-3-2.

As another example,

6725341 contains 4132

because 7253 has the same relative order as 4132 and consists of four consectutive elements, but

41352 avoids 4132.

As a final example,

3542716 contains 12-4-3

because 3576 has the same relative order as 1243 and its first two elements are consecutive, but

$$3542716 \text{ avoids } 12\text{-}43.$$

Define $\alpha_n(\tilde{\sigma})$ to be the number of n-symbol, $\tilde{\sigma}$-avoiding permutations. We naturally wish to understand the rate of growth of $\alpha_n(\tilde{\sigma})$ with increasing n.

0.1. Classical Patterns. The Stanley-Wilf conjecture, proved by Marcus & Tardos [1], was rephrased by Arratia [2] as follows:

$$L(\tilde{\sigma}) = \lim_{n\to\infty} \left(\alpha_n(\sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_m)\right)^{1/n}$$

exists and is finite. We have [3, 4, 5, 6, 7]

$$L(\tilde{\sigma}) = 4 \qquad \text{when } m = 3,$$

$$L(1\text{-}2\text{-}\cdots\text{-}m) = (m-1)^2 \qquad \text{for all } m \geq 2,$$

$$L(1\text{-}3\text{-}4\text{-}2) = 8,$$

$$L(1\text{-}2\text{-}4\text{-}5\text{-}3) = \left(1+\sqrt{8}\right)^2 = 9 + 4\sqrt{2}.$$

A conjecture that $L(\tilde{\sigma}) \leq (m-1)^2$ has been disproved [8]:

$$9.47 \leq L(1\text{-}3\text{-}2\text{-}4) \leq 288$$

and hence the maximum limiting value (as a function of m) remains open. Also, we wonder if $L(\tilde{\sigma})$ is always necessarily an algebraic number.

0.2. Consecutive Patterns. Elizalde & Noy [9, 10] examined the cases $m = 3$ and $m = 4$. The quantities $\alpha_n(123)$ and $\alpha_n(132)$ satisfy

$$\alpha_n(123) \sim \gamma_1 \cdot \rho_1^n \cdot n!, \qquad \alpha_n(132) \sim \gamma_2 \cdot \rho_2^n \cdot n!$$

where

$$\rho_1 = 3\sqrt{3}/(2\pi) = 0.8269933431..., \qquad \gamma_1 = \exp\left(\pi/(3\sqrt{3})\right) = 1.8305194665...,$$

$$\rho_2 = 1/\xi = 0.7839769312..., \qquad \gamma_2 = \exp(\xi^2/2) = 2.2558142944...$$

and $\xi = 1.2755477364...$ is the unique positive solution of

$$\int_0^x \exp(-t^2/2)\,dt = 1, \qquad \text{that is,} \qquad \sqrt{\frac{\pi}{2}}\,\text{erf}\left(\frac{x}{\sqrt{2}}\right) = 1.$$

The quantities $\alpha_n(1342)$, $\alpha_n(1234)$ and $\alpha_n(1243)$ satisfy

$$\alpha_n(1342) \sim \gamma_1 \cdot \rho_1^n \cdot n!, \qquad \alpha_n(1234) \sim \gamma_2 \cdot \rho_2^n \cdot n!, \qquad \alpha_n(1243) \sim \gamma_3 \cdot \rho_3^n \cdot n!$$

where

$$\rho_1 = 1/\xi = 0.9546118344..., \qquad \gamma_1 = 1.8305194...,$$
$$\rho_2 = 1/\eta = 0.9630055289..., \qquad \gamma_2 = 2.2558142...,$$
$$\rho_3 = 1/\zeta = 0.9528914198..., \qquad \gamma_3 = 1.6043282...;$$

ξ, η and ζ are the smallest positive solutions of

$$\int_0^x \exp(-t^3/6)\,dt = 1, \qquad \cos(y) - \sin(y) + \exp(-y) = 0,$$

$$3^{1/2}\int_0^z \operatorname{Ai}(-s)\,ds + \int_0^z \operatorname{Bi}(-s)\,ds = \frac{3^{1/3}\Gamma(1/3)}{\pi},$$

respectively, where Ai(t) and Bi(t) are the Airy functions [11].

0.3. Other Results. Elizalde [12, 13] proved that

$$\lim_{n\to\infty}\left(\frac{\alpha_n(\text{1-23-4})}{n!}\right)^{1/n} = 0$$

and believed that the same applies to α_n(12-34), although a proof is not yet known. Ehrenborg, Kitaev & Perry [14] gave more detailed asymptotic expansions for $\alpha_n(123)$ and $\alpha_n(132)$; a similar "translation" of combinatorics into operator eigenvalue analysis was explored in [15]. The field is wide open for research.

Let us focus on classical patterns in the following. Define $\sigma \leq \tau$ if τ contains $\tilde{\sigma}$. A **permutation class** C is a set of permutations such that, if $\tau \in C$ and $\sigma \leq \tau$, then $\sigma \in C$. Let C_n denote the permutations in C of length n. If $C = \{$all permutations$\}$, then $|C_n| = n!$; such behavior is regarded as degenerate and this case is excluded from now on. The Marcus-Tardos theorem implies that, for nondegenerate C,

$$L(C) = \limsup_{n\to\infty} |C_n|^{1/n} < \infty.$$

Consider the set R of all growth rates $L(C)$ and the derived set R' of all accumulation points of R. Vatter [16] proved that

$$\inf\{r \in R : r > 2\} = 2.0659948920...$$

which is the unique positive zero of $1 + 2x + x^2 + x^3 - x^4$, and

$$\inf \{s : s \text{ is an accumulation point of } R'\} = 2.2055694304...$$

which is the unique positive zero of $1 + 2x^2 - x^3$. Albert & Linton [17] proved that R is uncountable and thus contains transcendental numbers. Vatter [18] subsequently proved that

$$\inf \{t : R \text{ contains the interval } (t, \infty)\} \leq 2.4818728574...$$

which is the unique positive zero of $-1 - 2x - 2x^2 - 2x^4 + x^5$ and conjectured that $\leq$ can be replaced by $=$. The question of whether limsup in the definition of $L(C)$ can be replaced by lim is also unanswered.

0.4. Addendum. With regard to classical patterns, the upper bound on L(1-3-2-4) has been improved to $7 + 4\sqrt{3} < 13.93$ [19, 20]. With regard to consecutive patterns, a permutation τ is **nonoverlapping** if it contains no permutation σ such that two copies of σ overlap in more than one entry [21]. For example, $\tau = 214365$ contains both 2143 and 4365, both which follow the same pattern and overlap in two entries, hence τ is overlapping. Bóna [22] examined the probability p_n that a randomly selected n-permutation is nonoverlapping, showed that $\{p_n\}_{n=2}^{\infty}$ is strictly decreasing, and computed $\lim_{n\to\infty} p_n = 0.36409....$

From the fact that $\alpha_n(123) > \alpha_n(132)$ and $\alpha_n(1234) > \alpha_n(1342) > \alpha_n(1243)$ for suitably large n, it is natural to speculate that $\alpha_n(123\ldots m)$ is asymptotically larger than $\alpha_n(\sigma)$ for any other m-permutation σ (except $m(m-1)\ldots 21$, which is equivalent by symmetry). This conjecture is now a theorem, due to Elizalde [23].

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